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×2 and ×3 invariant measures and entropy

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Abstract. Let p and q be relatively prime natural numbers. Define T_0 and S_0 to be multiplication by p and q (mod 1) respectively, endomorphisms of $[0, 1)$.

Let μ be a borel measure invariant for both T_0 and S_0 and ergodic for the semigroup they generate. We show that if μ is not Lebesgue measure, then with respect to μ both T_0 and S_0 have entropy zero. Equivalently, both T_0 and S_0 are μ-almost surely invertible.

1. *Introduction*

In [F] Furstenberg showed that any closed set $S \subseteq [0, 1)$, and invariant under multiplication by a non-lacunary semigroup of integers must either be finite or all of $[0, 1)$. He has conjectured that perhaps a stronger result held, that any invariant, ergodic, borel probability measure for such a semigroup must be either atomic, or Lebesgue measure.

To be non-Lacunary is to *not* be all powers of a single integer. Under a stronger hypothesis, that the semigroup contains two relatively prime integers, Lyons [*L*] obtains a partial result. If either one of these two elements is exact as a measure-preserving endomorphism (a one-sided K-system) then μ is Lebesgue measure.

Motivated by this indication of the role entropy might play in this problem, we have obtained the following strengthening of Lyons' work. If either of the relatively prime pair has positive entropy with respect to μ, then μ is Lebesgue measure. This is equivalent to saying that if μ is not Lebesgue measure, the semigroup generated by these two elements is μ-almost surely a group.

The theorem requires us to only look at semigroups of multiplication generated by two relatively prime integers p and q. If μ is ergodic for a larger semigroup, it decomposes into ergodic components for this one. The entropies of $\times p$ and $\times q$ will be constant μ-a.e. Hence the theorem for the sub-semigroup is sufficient.

The proof is constructed as follows. First we partition $[0, 1)$ into pq intervals and use this to construct a symbolic version of the $\mathbb{N}^2$-action. This turns out to be a

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2-dimensional subshift of finite type. We lift this to its inverse limit, a $\mathbb{Z}^2$-subshift of finite type. Let T and S be the left and down shifts, corresponding to $\times p$ and $\times q$, respectively.

Any invariant and ergodic borel measure μ on $[0, 1)$ lifts to an invariant, ergodic borel measure $\hat{\mu}$ on this symbolic cover.

Using the origins of these shifts of finite type as maps of $[0, 1)$ we show that for any such $\hat{\mu}$, and any T and S invariant factor algebra $\mathscr{H}$, that

(i) $h_{\hat{\mu}}(T, \mathscr{H}) = [\log(p)/\log(q)]h_{\hat{\mu}}(S, \mathscr{H})$.

To complete the theorem we show how, under the assumption that μ is not Lebesgue measure, to construct such an algebra $\mathscr{H}$ with

(ii) $h_{\hat{\mu}}(T, \mathscr{H}) = h_{\hat{\mu}}(T)$, i.e. $(T, \mathscr{H})$ is a full entropy factor and

(iii) $(S, \mathscr{H})$ has rational, pure point spectrum, hence

$$h_{\hat{\mu}}(S, \mathscr{H}) = 0.$$

Facts (i), (ii) and (iii) of course say $h_{\hat{\mu}}(S) = h_{\hat{\mu}}(T) = 0$.

The existence of the factor $(S, \mathscr{H})$ hints tantalizingly at the possibility of proving the full conjecture. A simple corollary of the argument, though, is that $\mathscr{H}$ is trivial. Some other method is needed for the 0-entropy case.

2. *The symbolic representation*

At this point we make no restrictions on p and q other than that they are $\neq 1$. Whenever $GCD(p, q) = 1$ is needed, it will be explicitly stated. Let $T_0(x)$ and $S_0(x)$ be px mod 1 and qx mod 1 on $[0, 1)$ respectively. Let

$$V = \left\{x \in [0, 1): x = \frac{t}{p^n q^m}, t, n, m \in \mathbb{Z}\right\}, \quad \text{and} \quad V' = [0, 1)|_V.$$

We build a symbolic version of $\langle T_0, S_0 \rangle$ acting on $[0, 1)$ by partitioning it into pq disjoint invervals

$$I_j = \left[\frac{j}{pq}, \frac{j+1}{pq}\right], \quad j = 0, \ldots, pq-1.$$

These form a Markov partition for both T_0 and S_0.

In fact $T_0^{-1}(I_i)$ consists of p disjoint intervals, each contained in one of p distinct I_j's. Let $L_T(i)$ be the subscripts of these p distinct I_j's.

Similarly $S_0^{-1}(I_i)$ consists of q distinct intervals, each contained in one of q distinct I_j's. Let $L_s(i)$ be the subscripts of these q distinct I_i's.

We define $F_T(i) = \{j: i \in L_T(j)\}$ and similarly $F_S(i) = \{j: i \in L_S(j)\}$. As $T_0(I_i)$ covers exactly p of the I_j, $F_T(i)$ also consists of exactly p elements, the subscripts of these I_j. Similarly $F_S(i)$ consists of q subscripts for each i. In these definitions L stands for 'leader' and F for 'follower'.

To both T_0 and S_0 we associate $(0, 1)$-transition matrices; $M_T = [a_{i,j}]$ where $a_{i,j} = 1$ iff $j \in F_T(i)$, $M_S = [b_{i,j}]$ where $b_{i,j} = 1$ iff $j \in F_S(i)$. Let $\Sigma = (0, 1, \ldots, pq-1)$, the state space of these processes.

As $T_0S_0 = S_0T_0 = \times pq (\text{mod } 1)$ which lifts to the full shift on Σ, M_TM_S has all nonzero elements. As the sum on each row of M_TM_S is exactly pq, $M_TM_S = M_SM_T = [1]$.

If $[i_0, i_1, \ldots, i_{n-1})$ is a finite word of elements of Σ with all $a_{i_k, i_{k+1}} = 1$, then

$$\bigcap_{j=0}^{n-1} T_0^{-j}(I_{i_j}) \quad \text{is an interval } \left[\frac{t}{p^n q}, \frac{t+1}{p^n q}\right].$$

Thus to any one-sided infinite M_T-allowed word $\mathbf{i} = [i_0, i_1, \ldots]$, there corresponds a point

$$x_{\mathbf{i}} = \bigcap_{j=1}^{\infty} T_0^{-j}(I_{i_j}) \in [0, 1]$$

as long as $x \notin V$, x is $x_{\mathbf{i}}$ for a unique $\mathbf{i}$.

Similarly for any M_S-allowed word

$$\begin{bmatrix} i_{m-1} \\ \vdots \\ i_1 \\ i_0 \end{bmatrix} = i^{\uparrow}, \quad \bigcap_{j=0}^{m-1} S_0^{-j}(I_{i^j})$$

is an interval

$$\left[\frac{t}{pq^m}, \frac{t+1}{pq^m}\right]$$

and there is a natural identification of one-sided infinite M_S-allowed words and points $x \in [0, 1)$ which is 1-1 for $x \notin V$.

For our purposes $\mathbb{N} = \{0, 1, 2, \ldots\}$ and now let $Y \subseteq \Sigma^{\mathbb{N}^2}$ consist of all arrays which are M_T-allowed on rows and M_S allowed on columns. Let T be the left shift and S the down shift on such arrays.

To any point $x \in [0, 1)|_V$ there corresponds a unique point $y_x \in Y$. Just set

$$y_x(n, m) = j \quad \text{if } T_0^n S_0^m(x) \in I_j.$$

For $x \in V$ there are two such representations with $y_x(n, m) = j$ if $T_0^n S_0^m(x) \in I_j$. For $x \in k/p^r q^s$, once $n \geq r-1$, $m \geq s-1$, $T_0^n S_0^m(x)$ is on the boundary of two I_j's. Once we specify which symbol to place at one such index (n, m), all the rest are forced. If we choose the I_j to the left (right) of $T_0^n S_0^m(x)$, then we must always choose left (right) to satisfy the transition rules.

We now want to see that Y consists of precisely those arrays that arise from points $x \in [0, 1)$.

LEMMA 2.1. *For any symbols $a, b \in \Sigma$ and $i \in \mathbb{N}$, there are $y \in Y$ with $y(i, i) = a$, $y(i+1, i+1) = b$ but all such agree on $y(i+1, i)$ and $y(i, i+1)$.*

Proof. This is just our earlier remark that $M_TM_S = [1]$. □

LEMMA 2.2. *There is a natural almost 1-1 conjugation*

$$\varphi(y) = \bigcap_{i=j} T_0^{-j} S_0^{-j}[I_{y(i,j)}]$$

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between (S, T, Y) *and* $(S_0, T_0, [0, 1))$

Proof. From Lemma 2.1, the values $(y(0,0), y(1,1), \ldots)$ determine y completely. They also determine a unique

$$x = \bigcap_i T_0^{-i} S_0^{-i}(I_{y(i,i)}) \in [0, 1), \quad \text{for } x \notin v.$$

Hence $y(i, j) \in \Sigma$ is such that $T_0^i S_0^j(x) \in I_{y(i,j)}$ as this symbolic array is in Y and agrees with y on the diagonal.

We have seen φ is 1–1 off of V and precisely 2–1 on V. □

Putting the product topology on Y, φ is continuous. Let $V^* \subset Y$ be those countably many points with $\varphi(y) \in V$. V^* is both forward and backward invariant for both T and S.

COROLLARY 2.3. *Any* M_T*-allowed horizontal ray of symbols* $i_{(n,m)}$, $i_{(n+1,m)}, \ldots$, *determines all symbols* $y(j, k)$, $j \geq n$, $k \geq m$ *of any* $y \in Y$ *with* $y(j, k) = i_{(j,k)}$ *on the ray, as long as* $y \notin V^*$.

Similarly any M_S*-allowed vertical ray determines the symbols to its right in* $y \in Y$, *as long as* $y \notin V^*$.

Proof. Determining $y(j, k) = i_{(j,k)}$ determines

$$x = \varphi(T^n S^m(y)) = \bigcap_{t=0}^{\infty} T^{-t}[I_{i_{(n+t,m)}}].$$

We assume $x \notin V$ and hence $y(j, k) = \varphi^{-1}(x)(j-n, k-m)$. □

Let $\hat{Y} \subseteq \Sigma^{Z^2}$ be those doubly infinite arrays where all rows are M_T-allowed and columns are M_S-allowed.

For $\hat{y} \in \hat{Y}$, let $\hat{\varphi}(\hat{y}) \in Y$ be its restriction to the first quadrant, and $\varphi(\hat{y})$ its image in $[0, 1)$.

Letting T and S also represent the left and down shifts respectively on $\hat{Y}$,

$$T_0\hat{\varphi} = \hat{\varphi}T, \qquad S_0\hat{\varphi} = \hat{\varphi}S.$$

LEMMA 2.4. *The symbols* $\hat{y}(i, i)$, $i \in z$ *completely determine* $\hat{y}$.

Proof. See Lemma 2.1. □

At this point, and for much of our entropy work, $GCD(p, q) = 1$ is not needed. We now state a lemma which requires it, and explains its role in our work.

LEMMA 2.5. *Assume* $GCD(p, q) = 1$.

(i) *For any* $y \in Y$, y/V^*, *and* $k \in L_T(y(0, m_0))$ *there is a unique* $y^* \in Y$ *with* $y^*(0, m_0) = k$ *and* $T(y^*) = y$.

(ii) *For any* $y \in Y$, $y \notin V^*$ *and* $k \in L_S(y(n_0, 0))$ *there is a unique* $y^* \in Y$, $y^*(n_0, 0) = k$ *and* $S(y^*) = y$.

(iii) *For any* $y \in Y$ *and* $k \in \Sigma$ *there is a unique* $y^* \in Y$ *with* $y^*(0, 0) = k$ *and* $ST(y^*) = y$.

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Proof. (iii) is just Lemma 2.1. To see (i), notice any $x=\varphi(y)\in[0,1)$ has exactly p inverse images for T_0,

$$x_i=\frac{x}{p}+\frac{i}{p},\quad 0\le i<p.$$

Acting on these by S^{m_0} we obtain points

$$S_0^{m_0}(x_i)=\frac{xq^{m_0}}{p}+\frac{iq^{m_0}}{p}.$$

As $GCD(p,q)=1$, $q^{m_0}, 2q^{m_0},\dots,(p-1)q^{m_0} \pmod p$ are distinct, and hence each of these p points lies in a distinct I_j. The subscripts of these intervals are all in $L_T(y(0,m_0))$. Hence exactly one of them must be k.

Thus for some unique x_i, $S^{m_0}(x_i)\in I_k$. Set $y^*=\varphi^{-1}(x_i)$.

Statement (ii) is symmetric. □

3. *Invariant measures and entropy*:

Let $\mathcal{M}$ be the space of all T_0 and S_0 invariant borel probability measures on $[0,1)$. This is a weakly compact convex space. The extreme points, $\mathcal{M}_0$, are the ergodic measures.

LEMMA 3.1. *If $\mu\in\mathcal{M}$ and $x\in V$, $x\ne 0$ then $\mu(x)=0$.*

Proof. If $x\in V$, $x\ne0$ and $\mu(x)>0$, as there is an $S_0^mT_0^n(x)=0$, $\mu(0)>0$. But as 0 is a fixed point $\mu(0)=\mu(S_0^{-m}T_0^{-n}(0))\ge\mu(x)+\mu(0)$. This is a conflict. □

COROLLARY 3.2. *Any measure $\mu\in\mathcal{M}$ with $\mu(0)=0$ lifts to a unique T and S invariant borel probability measure on $\hat{Y}$.*

Proof. As $\mu(V)=0$, μ lifts to Y. $\hat{Y}$ is the inverse limit of Y under the action of (TS) hence μ lifts to a unique measure on $\hat{Y}$.

Let $\mathcal{M}$ be the T and S invariant borel probability measures on $\hat{Y}$, $\hat{\mathcal{M}}_0$, the ergodic ones, excluding the point mass at 0.

For a measure $\hat{\mu}\in\hat{\mathcal{M}}_0$, our work will focus on the $\hat{\mu}$-conditional expectations of finite words along the negative horizontal and vertical axes, given the symbols in the first quadrant.

Let P be the partition of $\hat{Y}$ according to symbol $\hat{y}(0,0)$.

LEMMA 3.3. *For any $\mu\in\hat{\mathcal{M}}$,*
(a) $\bigvee_{j=0}^{\infty}T^{-j}(P)=\bigvee_{j=0}^{\infty}S^{-j}(P)$ $\hat{\mu}$-a.s. *and*
(b) *if $GCD(p,q)=1$, then for any $m_0\ge0$, $n_0\ge0$*

$$S^{-m_0}T(P)\bigvee_{j=0}^{\infty}T^{-j}(P)=\bigvee_{j=-1}^{\infty}T^{-j}(P),$$

$$T^{-n_0}S(P)\bigvee_{j=0}^{\infty}S^{-j}(P)=\bigvee_{j=-1}^{\infty}S^{-j}(P),$$

$$\bigvee_{j=1}^{\infty}T^{-j}(P)\bigvee_{j=1}^{\infty}S^{-j}(P)=\bigvee_{j=0}^{\infty}T^{-j}S^{-j}(P),\ \hat{\mu}\text{-a.s.}$$

Proof. These are just reinterpretations of Lemmas 2.1, 2.2 and 2.5 as $\hat{\mu}(V^*)=0$.

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LEMMA 3.4. *If* $GCD(p,q)=1$, *then for any* $\hat\mu\in\hat{\mathcal{M}}_0$, $n_0\ge 1$, $m_0\ge 1$, *for* $\hat\mu$-a.e., $\hat y$,

$$E_{\hat\mu}\Big(\hat y(-1,m_0)\,|\,\bigvee_{j=0}^{\infty} S^{-j}(P)\Big)=a(\hat y),\quad \text{independent of } m_0, \text{ and}$$

$$E_{\hat\mu}\Big(\hat y(n_0,-1)\,|\,\bigvee_{j=0}^{\infty} T^{-j}(P)\Big)=b(\hat y),\quad \text{independent of } n_0.$$

Note: $E_{\hat\mu}$ *is the conditional expectation with respect to* $\hat\mu$.

Proof. Just notice that $\hat y(-1,m_0)$ and $\bigvee_{j=0}^{\infty} S^{-j}(P)$ together determine all other $\hat y(-1,m)$, $m\ge 0$, and $\hat y(n_0,-1)$ together with $\bigvee_{j=0}^{\infty} T^{-j}(P)$ determine all other $y(n,-1)$, $n\ge 0$. □

For $\hat\mu\in\hat{\mathcal{M}}_0$, let $h_{\hat\mu}(T)$ and $h_{\hat\mu}(S)$ be the measure-theoretic entropies of these two maps, respectively. It is important to bear in mind that S and T separately need not be ergodic. An assumption of ergodicity of either T or S greatly simplifies our work. See [K] or [I–T] for discussions of non-ergodic entropy theory.

LEMMA 3.5. *For any* $\hat\mu\in\hat{\mathcal{M}}_0$, $h_{\hat\mu}(T)=h_{\hat\mu}(T,P)$ *and* $h_{\hat\mu}(S)=h_{\hat\mu}(S,P)$.

Proof. P generates under $\langle S,T\rangle$ so,

$$\begin{aligned}h_{\hat\mu}(T)&=\lim_{n\to\infty}\int h_{\hat\mu}\Big(\bigvee_{k=-n}^{n} S^{-k}(P)\,\Big|\,\bigvee_{k=-n}^{n} S^{-k}\Big(\bigvee_{j=1}^{\infty} T^{-j}(P)\Big)\Big)\\&=\lim_{n\to\infty}\int h_{\hat\mu}\Big(S^{-n}(P)\,\Big|\,S^{-n}\Big(\bigvee_{j=1}^{\infty} T^{-j}(P)\Big)\Big)\\&=\lim_{n\to\infty}\int h_{\hat\mu}\Big(P\,\Big|\,\bigvee_{j=1}^{\infty} T^{-j}(P)\Big)=h_{\hat\mu}(T,P).\end{aligned}$$

The argument for S is symmetric. □

COROLLARY 3.6. *If* $GCD(p,q)=1$, *then for any* $\hat\mu\in\hat{\mathcal{M}}_0$,

$$E_{\hat\mu}\Big(\hat y(-1,-1)\,\Big|\,\bigvee_{i=0}^{\infty}\bigvee_{j=0}^{\infty} T^{-i}S^{-j}(P)\Big)=a(\hat y)b(\hat y)$$

(*see Lemma* 3.4).

Proof. Computing,

$$\begin{aligned}&\int h_{\hat\mu}\Big(P\,\Big|\,\bigvee_{j=1}^{\infty}\bigvee_{i=1}^{\infty} T^{-i}S^{-j}(P)\Big)\\&=\int h_{\hat\mu}\Big(P\bigvee_{j=1}^{\infty} T^{-j}(P)\,\Big|\,\bigvee_{j=1}^{\infty}\bigvee_{i=1}^{\infty} T^{-i}S^{-j}(P)\Big)\\&=\int h_{\hat\mu}\Big(T^{-1}(P)\,\Big|\,\bigvee_{j=1}^{\infty}\bigvee_{i=1}^{\infty} T^{-i}S^{-j}(P)\Big)+\int h_{\hat\mu}\Big(P\,\Big|\,\bigvee_{j=1}^{\infty}\bigvee_{i=0}^{\infty} T^{-i}S^{-j}(P)\Big)\\&=\int h_{\hat\mu}\Big(P\,\Big|\,\bigvee_{j=1}^{\infty} S^{-j}(P)\Big)+\int h_{\hat\mu}\Big(P\,\Big|\,\bigvee_{i=1}^{\infty} T^{-i}(P)\Big)=h_{\hat\mu}(T)+h_{\hat\mu}(S).\end{aligned}$$

But also as

$$GCD(p,q)=1,\ P\bigvee_{j=1}^{\infty}\bigvee_{i=1}^{\infty}T^{-i}S^{-j}(P)$$
$$=(T^{-1}(P)\vee S^{-1}(P))\vee\bigvee_{j=1}^{\infty}\bigvee_{i=1}^{\infty}T^{-i}S^{-j}(P).$$

Thus

$$\int h_{\hat\mu}\Big(P\,\Big|\bigvee_{j=1}^{\infty}\bigvee_{i=1}^{\infty}T^{-i}S^{-j}(P)\Big)$$
$$=\int h_{\hat\mu}\Big(T^{-1}(P)\vee S^{-1}(P)\,\Big|\bigvee_{j=1}^{\infty}\bigvee_{i=1}^{\infty}T^{-i}S^{-j}(P)\Big)$$
$$\le\int h_{\hat\mu}\Big(T^{-1}(P)\,\Big|\bigvee_{j=1}^{\infty}\bigvee_{i=1}^{\infty}T^{-i}S^{-j}(P)\Big)+\int h_{\hat\mu}\Big(S^{-1}(P)\,\Big|\bigvee_{j=1}^{\infty}\bigvee_{i=1}^{\infty}S^{-i}(P)\Big)$$
$$=h(T)+h(S).$$

Equality holds iff $T^{-1}(P)$ and $S^{-1}(P)$ are $\bigvee_{i=1}^{\infty}\bigvee_{j=1}^{\infty}T^{-i}(P)S^{-j}(P)$ conditionally independent. Hence they are, and the result follows. □

Let $\hat\mu\in\hat{\mathcal{M}}_0$ and A be a $\hat\mu$-complete, T and S invariant, σ-algebra. We want to relate $h_{\hat\mu}(T,A)$ and $h_{\hat\mu}(S,A)$.

THEOREM 3.7. *For $\hat\mu\in\hat{\mathcal{M}}_0$, and any T and S invariant algebra A,*

$$h_{\hat\mu}(T,A)=\frac{\log(p)}{\log(q)}h_{\hat\mu}(S,A).$$

Note: *All we assume is p and $q\neq 1$.*

Proof. As $h_{\hat\mu}(T,P)=h_{\hat\mu}(T,A)+h_{\hat\mu}(T,P|A)$ and $h_{\hat\mu}(S,P)=h_{\hat\mu}(S,A)+h_{\hat\mu}(S,P|A)$ it is sufficient to show

$$h_{\hat\mu}(T,P|A)=\frac{\log(p)}{\log(q)}h_{\hat\mu}(S,P|A).$$

Select n_i, m_i tending to ∞ with

$$\left|n_i-m_i\frac{\log(q)}{\log(p)}\right|<\frac{\log(1.1)}{\log(p)}.$$

Thus

$$0.9<\frac{p^{n_i}}{q^{m_i}}<1.1\quad\text{and}\quad\lim_{i\to\infty}\left(\frac{m_i}{n_i}\right)=\frac{\log(p)}{\log(q)}.$$

For $f\in\bigvee_{i=0}^{n}T^{-i}(P)$, $\varphi^{-1}(f)\subset[0,1)$ is an interval of length $q^{-1}p^{-n-1}$, and for any atom $g\in\bigvee_{i=0}^{m}S^{-i}(P)$, $\varphi^{-1}(g)\subset[0,1)$ is an interval of length $p^{-1}q^{-m-1}$.

Thus any atom of $\bigvee_{j=0}^{n_i}T^{-j}(P)$ is contained in the union of at most three atoms of $\bigvee_{j=0}^{m_i}S^{-j}(P)$. Thus

$$h_{\hat\mu}\Big(\bigvee_{j=0}^{n_i}T^{-j}(P)\,\Big|\bigvee_{j=0}^{m_i}S^{-j}(P)\vee A\Big)\le\log(3)$$

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and

$$h_{\hat{\mu}}\left(\bigvee_{j=0}^{n_i} S^{-j}(P)\,\middle|\,\bigvee_{j=0}^{n_i} T^{-j}(P)\vee A\right)\le \log(3).$$

So

$$\begin{aligned}\left|h_{\hat{\mu}}\left(\bigvee_{j=0}^{n_i} T^{-j}(P)\,\middle|\,A\right)-h_{\hat{\mu}}\left(\bigvee_{j=0}^{m_i} S^{-j}(P)\vee A\right)\right| \\ =\left|h_{\hat{\mu}}\left(\bigvee_{j=0}^{n_i} T^{-j}(P)\,\middle|\,\bigvee_{j=0}^{m_i} S^{-j}(P)\vee A\right)\right. \\ \left.-h_{\hat{\mu}}\left(\bigvee_{j=0}^{m_i} S^{-j}(P)\,\middle|\,\bigvee_{j=0}^{n_i} T^{-j}(P)\vee A\right)\right|<2\log(3).\end{aligned}$$

Thus

$$\left|\frac{1}{n_i}h_{\hat{\mu}}\left(\bigvee_{j=0}^{n_i} T^{-j}(P)\,\middle|\,A\right)-\left(\frac{m_i}{n_i}\right)\frac{1}{m_i}\left(\bigvee_{j=0}^{m_i} S^{-j}(P)\,\middle|\,A\right)\right|$$

converges to 0 in i. But the limit is also

$$\left|h_{\hat{\mu}}(T, P\,|\,A)-\frac{\log(p)}{\log(q)}h_{\hat{\mu}}(S, P\,|\,A)\right|. \qquad \square$$

Notice that under $GCD(p, q)=1$ we will reduce the possibilities for $h(T, P)$ and $h(T, Q)>0$ down to Lebesgue measure. Even in this case, when A is a nontrivial algebra, Theorem 9 has content.

4. *Completion of the Result*:
Fix an element $\hat{\mu}\in\hat{\mathcal{M}}_0$. We construct a factor algebra $\mathcal{H}$ by defining a sequence of probability density valued functions. $\mathcal{H}$ will be minimal σ-algebra for which they are measurable.

For a point $\hat{y}\in\hat{Y}$, $\varphi(\hat{y})=x\in[0,1)$. There are p^n points

$$x_1, x_1+\frac{1}{p^n},\dots, x_1+\frac{p^n-1}{p^n}\ (\mathrm{mod}\ 1)\quad\text{with } T_0^n\left(\frac{x_1+t}{p^n}\right)=x.$$

One of these points is $\varphi(T^{-n}(\hat{y}))$. This is the one we call $x_1=x_1(\hat{y})$.

For $\hat{\mu}$-a.e. $\hat{y}\in\hat{Y}$, for all n and t, we can compute

$$E_{\hat{\mu}}\left(T^n\left(\varphi^{-1}\left(x_1(\hat{y})+\frac{t}{p^n}\right)\right)\,\middle|\,\varphi^{-1}(x_1(\hat{y})\right)=\delta(\hat{y}, t, n).$$

This is just the $\hat{\mu}$-conditional expectation that the M_T-allowed name $(i_0, i_i,\dots,)$ of $\varphi(\hat{y})$ will extend to

$$(i_{-n}, i_{-n+1},\dots, i_0, i_1,\dots),\quad\text{the } M_T\text{-allowed name of } x_1(\hat{y})+\frac{t}{p^n}\ (\mathrm{mod}\ 1).$$

Thus for each such $\hat{y}$ and n we obtain a distribution on the points

$$\left(0, \frac{1}{p^n},\dots,\frac{p^n-1}{p^n}\right)\quad\text{given by } \delta(\hat{y}, n)\left(\frac{t}{p^n}\right)=\delta(\hat{y}, t, n).$$

This is the probability that $\varphi(\hat{y})$ extends under T^{-n} to a point rotated by t/p^n from $\varphi(T^{-n}(\hat{y}))$.

The following are easily checked.

LEMMA 4.1

(i) $\delta(\hat{y}, n-1)\left(\frac{t}{p^{n-1}}\right) = \sum_{s \bmod p^{n-1}=t} \delta(\hat{y}, n)\left(\frac{s}{p^n}\right)$ *and so* $\delta(\hat{y}, n)$ *determines* $\delta(\hat{y}, k)$ *for all* $k \leq n$.

(ii) *If* $\hat{y}_1$ *and* $\hat{y}_2$ *agree on the positive horizontal axis* (*coordinates* $(n, 0)$, $n \geq 0$), *then* $\delta(\hat{y}_1, n)$ *and* $\delta(\hat{y}_2, n)$ *differ by a translation* (mod 1) *by*

$$\varphi(T^{-n}(\hat{y}_2)) - \varphi(T^{-n}(\hat{y}_1)).$$

(iii) *If* $GCD(pq) = 1$, *then by Corollary* 3.6,

$$\delta(S(\hat{y}), n)\left(\frac{t}{p^n}\right) = \delta(\hat{y}, n)\left(\frac{qt}{p^n} \bmod 1\right).$$

□

Definition. We say a point $\hat{y}$ is symmetric if there are points $\hat{y}_1 \neq \hat{y}_2$ with

(i) $\varphi(\hat{y}_1) = \varphi(\hat{y}_2) = \varphi(\hat{y})$, *and*

(ii) *for all* $n > 0$, $m \geq 0$,

$$\delta(T^m(\hat{y}_1), n) = \delta(T^m(\hat{y}_2), n).$$

LEMMA 4.2. *If* $GCD(p, q) = 1$ *then the set of symmetric points is both* T *and* S *invariant, hence of* $\hat{\mu}$*-measure* 0 *or* 1.

Proof. For T invariance, the necessary points for $T(\hat{y})$ are $T(\hat{y}_1)$ and $T(\hat{y}_2)$.

For S invariance, as

$$\delta(T_0^m S(y_1), n)\left(\frac{t}{p^n}\right) = \delta(T^m(y_1), n)\left(\frac{qt}{p^n} \bmod 1\right)$$
$$= \delta(T_0^m S(y_2), n)\left(\frac{t}{p^n}\right),$$

$S(\hat{y}_1)$ and $S(\hat{y}_2)$ are the needed points. □

We now want to show that if $\hat{\mu}$-a.e. point is symmetric, then μ is Lebesgue measure.

LEMMA 4.3. *If*

$$a = \sum_{i=1}^{n} \frac{a_i}{p^i}, \quad a_i \in \mathbb{Z} \text{ with all } -p < a_i < p,$$

and $a_n \neq 0$ *then if* $a = u/v$ *in least terms,* $v \geq 2^n$.

Proof. For $n = 1$ this is clear. If $a = u/v$ in least terms, all prime divisors of v divide p. Thus

$$ap = \sum_{i=1}^{n-1} \frac{a_{i+1}}{p^i} = \frac{u'}{v'} \quad \text{and} \quad v'$$

has at least one fewer term in its prime decomposition than v. Hence $v \geq 2v' \geq 2^n$ by induction. □

LEMMA 4.4. *If* $\hat{y}$ *is a symmetric point, then* $\delta(\hat{y}, n)$ *converges weakly to Lebesgue measure on* $[0, 1)$.

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Proof. Let $\hat{y}_1 \neq \hat{y}_2$ be the two points of the definition and $-i_0$ the first index with

$$\hat{y}_1(-i_0, 0) \neq \hat{y}_2(-i_0, 0).$$

We know that $\delta(\hat{y}, n)$ is invariant under a shift, mod 1, by

$$a_n = \varphi(T^{-n}(\hat{y}_2)) - \varphi(T^{-n}(\hat{y}_1)) = \sum_{i=1}^{n} \frac{\hat{y}_2(-i, 0) - \hat{y}_1(-i, 0)}{p^{n-i+1}}.$$

If $n \geq i_0$, as $\hat{y}_2(-i_0, 0) \neq \hat{y}_1(-i_0, 0)$, by Lemma 4.3, a_n is a fraction, in least terms, with denominator $\geq 2^{n-i_0+1}$. Thus the group of shifts (mod 1) preserving $\delta(\hat{y}, n)$ is of order at least 2^{n-i_0+1}, and its minimal element $d_n \leq 1/2^{n-i_0+1}$. For any continuous f on $[0, 1)$, $f(0) = f(1)$, this forces

$$\lim_{n\to\infty} \int f\, d(\delta(\hat{y}, n)) = \int f\, dm$$

and hence the result. □

THEOREM 4.5. *If $\hat{\mu}$-a.e. $\hat{y} \in \hat{Y}$ is symmetric, then $\mu = m$ is Lebesgue measure.*

Proof. Let R_a represent addition of a (mod 1) on $[0, 1)$ and let $\mu = \varphi(\hat{\mu})$.

T_0-invariance of μ implies

$$\mu = \int R_{\varphi(T^{-n}(\hat{y},n))}(\delta(\hat{y}, n))\, d\hat{\mu}(\hat{y}).$$

For $\hat{\mu}$-a.e. $\hat{y}$, as $\hat{y}$ is symmetric,

$$R_{\varphi(T^{-n}(\hat{y}))}(\delta(\hat{y}, n))$$

converges weakly in n to m. Hence $\mu = m$. □

Thus what remains to be seen is that if $\hat{\mu}$-a.e. $\hat{y}$ is not symmetric, then T and S are of zero entropy.

Let $\mathcal{H}$ be the minimal T and S invariant σ-algebra for which all the functions $\delta(\hat{y}, n)$ are measurable.

Now $\delta(\hat{y}, n)$ determines $\delta(\hat{y}, k)$, $k \leq n$ and $\delta(T^{-k}(\hat{y}), n-k)$ for $k < n$ by Lemma 4.1.

If $GCD(p, q) = 1$, $\delta(\hat{y}, n)$ determines $\delta(S^l(\hat{y}), n)$ for all $l \in \mathbb{Z}$.

Hence letting $\mathcal{H}_n$ be the minimal σ-algebras for which $\delta(2n+1, T^{-n}(\hat{y}))$ is measurable, the $\mathcal{H}_n$'s are S-invariant and refine in n to $\mathcal{H}$.

LEMMA 4.6. *If $GCD(p, q) = 1$, then the action of S on $\mathcal{H}_n$ is periodic, i.e. there is a j_n and for all $A \in \mathcal{H}_n$, $S^{j_n}(A) = A$.*

Proof. Let j_n be such that

$$q^{j_n} t = t \bmod (p^{2n+1})$$

for all $0 \leq t < p^{2n+1}$. By Lemma 4.1, (iv), for $\hat{\mu}$-a.e. $\hat{y}$,

$$\delta(2n+1, S^{j_n}(T^{-n}(\hat{y})) = \delta(2n+1, (T^{-n}(\hat{y}))$$

and S^{j_n} is the identity on $\mathcal{H}^n$.

COROLLARY 4.7. *The dynamical system $(S, \mathcal{H}, \hat{\mu})$ has rational pure point spectrum on all its ergodic components and hence $h_{\hat{\mu}}(S, \mathcal{H}) = 0$.*

On each algebra $\mathcal{H}_n$, $(S, \mathcal{H}_n)$ is periodic, hence on each ergodic component is a finite rotation. Thus on the ergodic components of $(S, \mathcal{H})$, we have a rational pure point spectrum. In fact, as $\langle T, S\rangle$ acts ergodically on $\mathcal{H}$, all components have the same point spectrum, and so all components are isomorphic. □

LEMMA 4.8. *If $\hat{\mu}$-a.e. $\hat{y} \in \hat{Y}$ is not symmetric then*

$$T(P) \subseteq \mathcal{H} \vee \bigvee_{i=0}^{\infty} T^{-i}(P).$$

The map $\varphi(\hat{y})$ is $\bigvee_{i=0}^{\infty} T^{-i}(P)$ measurable. Suppose $\hat{y}$ is such that knowing $\varphi(\hat{y})$ and $\delta(T^n(\hat{y}), 2n+1)$ for all n is not enough to determine the element of $T(P)$ containing $\hat{y}$, i.e. is not enough to determine $\varphi(T^{-1}(\hat{y}))$. This means, for all $n \geq 0$ there are points $\hat{y}_1$ and $\hat{y}_2$ with

(1) $\varphi(\hat{y}_1) = \varphi(\hat{y}_2) = \varphi(\hat{y})$ and

(2) $\delta(T^n(\hat{y}_1), 2n+1) = \delta(T^n(\hat{y}_2), 2n+1)$ but

(3) $\varphi(T^{-1}(\hat{y}_1)) \neq \varphi(T^{-1}(\hat{y}_2))$.

Among the points satisfying (1), call $\hat{y}_1$ and $\hat{y}_2$ n-equivalent if they satisfy (2). These equivalence classes are closed and nested in n. There are only p choices for $\varphi(T^{-1}(\hat{y}_1))$ so we can intersect over a sequence of equivalence classes to obtain a pair of points $\hat{y}_1, \hat{y}_2$ satisfying (1), (2) and (3) for all n. This forces $\hat{y}$ to be symmetric. Hence the result. □

THEOREM 4.9. *If $GCD(p, q) = 1$ and $\hat{\mu} \in \hat{\mathcal{M}}_0$* but *$\hat{\mu} \neq \hat{m}$, then $h_{\hat{\mu}}(T, P) = h_{\hat{\mu}}(S, P) = 0$.*

Proof. As $GCD(p, q) = 1$, the symmetric points for $\hat{\mu}$ have measure 0. Thus

$$\begin{aligned} h_{\hat{\mu}}(T, P) &= h_{\hat{\mu}}(T, \mathcal{H}) + h_{\hat{\mu}}(T, P \mid \mathcal{H}) \\ &= h_{\hat{\mu}}(T, \mathcal{H}) + \int h_{\hat{\mu}}\left(P \,\middle|\, \bigvee_{i=1}^{\infty} T^{-i}(P) \vee \mathcal{H}\right) d\hat{\mu} \\ &= h_{\hat{\mu}}(T, \mathcal{H}) \quad \text{by Lemma 4.8.} \end{aligned}$$

But now

$$\begin{aligned} 0 = h_{\hat{\mu}}(S, \mathcal{H}) &= \frac{\log(q)}{\log(p)} h_{\hat{\mu}(T, \mathcal{H})} \\ &= \frac{\log(q)}{\log(p)} h_{\hat{\mu}}(T, P) = h_{\hat{\mu}}(S, P). \end{aligned}$$ □

COROLLARY 4.10. *If $\mu \in \hat{\mathcal{M}}_0$, $\mu \neq m$, then both T_0 and S_0 are μ-a.s. invertible.*

Proof. As $h_{\hat{\mu}}(T, P) = 0$, for $x \in [0, 1)$ $\varphi^{-1}(x)$ is μ-a.e. a single point.

$$T_0^{-1}(x) = \varphi T^{-1} \varphi^{-1}(x).$$ □

This completes our work. We are left with two questions. How far can this work be extended from $GCD(p, q) = 1$ toward the general non-lacunary subgroup? Are there 0-entropy non-atomic invariant measures? On the first question we can make a trivial extension.

COROLLARY 4.11. *If* $GCD(u, v) = 1$, $u, v \neq 1$ *and* $p = u^{n_1}$, v^{m_1}, $q = u^{n_2}v^{m_2}$ *where* $n_1m_2 - m_1n_2 \neq 0$, *then Theorem* 4.9 *holds for p and q.*

Proof. In the $\mathbb{Z}^2$-action generated by u and v, p and q generate a cofinite subgroup. If $\hat{\mu}$ is invariant for $\times p$ and $\times q$, averaging over a fundamental domain yields a measure $\hat{\nu}$ invariant for $\times u$ and $\times v$. The linear relations (Lemma 3.9) among entropies holds for all four maps (Theorem 3.9). Thus if $h_{\hat{\mu}} > 0$, $\hat{\nu} = \hat{m}$. But as $\hat{m}$ is ergodic for $\times p$ and $\times q$, $\hat{\mu} = \hat{m}$. □

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