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GROWTH OF POLYNOMIALS WHOSE ZEROS ARE WITHIN OR OUTSIDE A CIRCLE

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Let $P(z)$ be a polynomial of degree n which does not vanish in the disk $|z| < K$. For $K = 1$, it is known that

$$\max_{|z|=r<1} |P(z)| \geq \left(\frac{1+r}{2}\right)^n \max_{|z|=1} |P(z)| , \quad \text{and}$$

$$\max_{|z|=R>1} |P(z)| \leq \left(\frac{R^n+1}{2}\right) \max_{|z|=1} |P(z)| .$$

In this paper we consider the two cases $K \geq 1$ and $K < 1$, and present certain generalizations of these results.

If $P(z)$ is a polynomial of degree n, then [7, p.346] or [6, Vol.I, p.137 Problem III 269]

$$\max_{|z|=R>1} |P(z)| \leq R^n \max_{|z|=1} |P(z)| . \tag{1}$$

Here equality holds if and only if $P(z) = \alpha z^n$.

It was shown by Ankeny and Rivlin [4] that if $P(z) \neq 0$ in $|z| < 1$, then (1) can be replaced by

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(2) $$\max_{|z|=R>1} |P(z)| \le \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| .$$

Inequality (2) is sharp and equality holds for $P(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$.

By the maximum modulus principle

$$\max_{|z| = \frac{1}{r} > 1} |z^n P(1/z)| \ge \max_{|z|=1} |z^n P(1/z)| = \max_{|z|=1} |P(z)| ,$$

and so

(3) $$\max_{|z| = r<1} |P(z)| \ge r^n \max_{|z|=1} |P(z)| ,$$

where equality holds if and only if $P(z) = \alpha z^n$.

If $P(z) \ne 0$ in $|z| < 1$, then [8] the stronger inequality

(4) $$\max_{|z| = r>1} |P(z)| \ge \left(\frac{1+r}{2}\right)^n \max_{|z|=1} |P(z)|$$

holds. Here equality is attained if $P(z) = \alpha(z - \beta)^n$, $|\beta| = 1$.

In this paper we obtain certain generalizations of inequalities (2) and (4). We prove.

THEOREM 1. *If $P(z)$ is a polynomial of degree n such that $P(z) \ne 0$ in $|z| < K$ where $K \ge 1$, then*

(5) $$\max_{|z|=r<1} |P(z)| \ge \left(\frac{r+K}{1+K}\right)^n \max_{|z|=1} |P(z)| .$$

Here equality holds if $P(z) = (z + K)^n$.

Applying Theorem 1 to the polynomial $z^n P(1/z)$, we obtain

THEOREM 1'. *If $P(z)$ is a polynomial of degree n which has all its zeros in the disk $|z| \le k$ where $k \le 1$, then*

(6) $$\max_{|z|=R>1} |P(z)| \ge \left(\frac{R+k}{1+k}\right)^n \max_{|z|=1} |P(z)| .$$

The result is sharp and in (6) equality holds for $P(z) = (z + k)^n$.

THEOREM 2. *If $P(z)$ is a polynomial of degree n such that $P(z) \ne 0$ in $|z| < k$ where $k \le 1$, then*

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(7) $$\max_{|z|=r} |P(z)| \geq \left(\frac{r+k}{1+k}\right)^n \max_{|z|=1} |P(z)| \quad \text{if} \quad 0 \leq r \leq k^2 .$$

The estimate is sharp with equality in (7) for $P(z) = (z + k)^n$.

This result when applied to $z^n P(1/z)$ gives:

THEOREM 2'. *If* $P(z)$ *is a polynomial of degree* n *which has all its zeros in the disk* $|z| \leq K$ *where* $K \geq 1$ *, then*

(8) $$\max_{|z|=R} |P(z)| \geq \left(\frac{R+K}{1+K}\right)^n \max_{|z|=1} |P(z)| \quad \text{if} \quad R \geq K^2 .$$

The result is sharp with equality in (8) for $P(z) = (z + K)^n$.

The precise estimate for $\max_{|z|=r} |P(z)|$ in Theorem 2 for $k^2 < r < 1$ and the corresponding estimate for $\max_{|z|=R} |P(z)|$ in Theorem 2' for $1 < R < K^2$ does not seem to be easily obtainable. It was shown by Aziz and Mohammad [2] that if $P(z)$ is a polynomial of degree n which does not vanish in the disk $|z| < K$ where $K \geq 1$, then

$$\max_{|z|=R} |P(z)| \leq \left(\frac{R+K}{1+K}\right)^n \max_{|z|=1} |P(z)| \quad \text{for} \quad 1 \leq R \leq K^2$$

and in addition, if $P(z)$ has non-negative coefficients or if $P(K^2Rz)$ and $P(Rz)$ become maximum at the same point on $|z| = 1$, $R > 1$, then

(9) $$\max_{|z|=R} |P(z)| \leq \frac{R^n+K^n}{1+K^n} \max_{|z|=1} |P(z)| \quad \text{for} \quad R > K^2$$

We take this opportunity to point out that the statement of the inequality (5) of Theorem 2 of [2] should read as the statement of the inequality (9) above, as the proof given for the first part of Theorem 2 in [2] covers only the above mentioned class of polynomials and so the general case is still open.

However, we have a considerable evidence in favour of the following

CONJECTURE. *If* $P(z)$ *is a polynomial of degree* n *which does not vanish in the disk* $|z| < k$ *, then*

$$\max_{|z|=r} |P(z)| \geq \frac{r^n + k^n}{1 + k^n} \max_{|z|=1} |P(z)| \quad \text{for} \quad k^2 < r < 1,\ k < 1$$

and

$$\max_{|z|=R} |P(z)| \le \frac{R^n + k^n}{1 + k^n} \max_{|z|=1} |P(z)| \quad \text{for } R > k^2, \ k > 1 .$$

Here we prove the following generalisations of (2).

THEOREM 3. *If $P(z)$ is a polynomial of degree n which does not vanish in the disk $|z| < K$ where $K \ge 1$, then*

$$(10) \qquad \max_{|z|=R} |P(z)| \le \frac{R^n + K^n}{1 + K^n} \max_{|z|=1} |P(z)| \quad \text{for } R \ge K^2 ,$$

provided $|P'(K^2 z)|$ and $|P'(z)|$ become maximum at the same point on $|z| = 1$.
The result is best possible with equality in (10) for $P(z) = z^n + K^n$.

The next result is an interesting generalisation of the inequality (2).

THEOREM 4. *If $P(z)$ is a polynomial of degree n which does not vanish in the disk $|z| < K$ where $K \ge 1$, then*

$$(11) \qquad \max_{|z|=R>1} |P(z)| \le \left(\frac{R^n+1}{2}\right) \max_{|z|=1} |P(z)| - \left(\frac{R^n-1}{2}\right) \min_{|z|=1} |P(z)| .$$

The result is best possible and equality in (11) holds for the polynomial $P(z) = \alpha z^n + \beta K^n$, $|\alpha| = |\beta| = 1$, $K \ge 1$.

As an application of Theorem 4, we establish

THEOREM 5. *If $P(z)$ is a polynomial of degree n which does not vanish in the disk $|z| < k$, $k \le 1$, then for $0 \le r \le k$ we have*

$$(12) \qquad (1+r^n) \max_{|z|=r} |P(z)| - (1-r^n) \min_{|z|=r} |P(z)| \ge 2r^n \max_{|z|=1} |P(z)| .$$

The result is best possible and equality in (12) holds for the polynomial $P(z) = \alpha z^n + \beta k^n$ where $|\alpha| = |\beta| = 1$ and $k \le 1$.

For the proofs of these theorems, we need the following lemmas.

LEMMA 1. *If $P(z)$ is a polynomial of degree n, then on $|z| = 1$,*

$$|P'(z)| + |Q'(z)| \le n \max_{|z|=1} |P(z)| ,$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$.

This is a special case of a result due to Govil and Rahman [5, Lemma 10] (see also [3]).

LEMMA 2 [1,2]. *If* $P(z)$ *is a polynomial of degree* n *, then for all* $R \geq 1$ *and* $0 \leq \theta < 2\pi$

$$|P(re^{i\theta})| + |Q(Re^{i\theta})| \leq (R^n + 1) \max_{|z|=1} |P(z)| ,$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$.

Proofs of the theorems.

Proof of Theorem 1. Since all the zeros of $P(z)$ lie in $|z| \geq K$, $K \geq 1$, we write

$$P(z) = C \prod_{j=1}^{n} (z - R_j e^{i\theta_j}) \quad \text{where} \quad R_j \geq K,\ j = 1, 2, \ldots, n.$$

Therefore, for $0 \leq \theta < 2\pi$ and $r \leq 1$, we have clearly

$$\begin{aligned}
\left| P(re^{i\theta})/P(e^{i\theta}) \right| &= \prod_{j=1}^{n} \left| (re^{i\theta} - R_j e^{i\theta_j})/(e^{i\theta} - R_j e^{i\theta_j}) \right| \\
&= \prod_{j=1}^{n} \left| (re^{i(\theta-\theta_j)} - R_j)/(e^{i(\theta-\theta_j)} - R_j) \right| \\
&= \prod_{j=1}^{n} \left\{ (r^2 + R_j^2 - 2rR_j\cos(\theta-\theta_j))/(1 + R_j^2 - 2R_j\cos(\theta-\theta_j)) \right\}^{\frac{1}{2}} \\
&\geq \prod_{j=1}^{n} (r + R_j)/(1 + R_j) \\
&\geq \prod_{j=1}^{n} (r + K)/(1 + K) = (r + K)^n/(1 + K)^n .
\end{aligned}$$

This implies

$$|P(re^{i\theta})| \geq \left(\frac{r + K}{1 + K}\right)^n |P(e^{i\theta})| \quad \text{for} \quad r \leq 1 ,\ 0 \leq \theta < 2\pi .$$

Hence

$$\max_{|z|=r<1} |P(z)| \geq \left(\frac{r + K}{1 + K}\right)^n \max_{|z|=1} |P(z)|$$

and the proof of Theorem 1 is complete.

Proof of Theorem 2. Since the polynomial $P(z)$ has all its zeros in $|z| \geq k$ where $k \leq 1$, we write as before

$$P(z) = C \prod_{j=1}^{n} (z - R_j e^{i\theta_j}) \quad \text{where} \quad R_j \geq k,\ j=1,2,\ \ldots,\ n.$$

Then clearly for $r \leq k^2$ and $0 \leq \theta < 2\pi$, we have

$$\left|P(re^{i\theta})/P(e^{i\theta})\right| = \prod_{j=1}^{n} \left|(re^{i\theta} - R_j e^{i\theta_j})/(e^{i\theta} - R_j e^{i\theta_j})\right|$$

$$\geq \prod_{j=1}^{n} (r + R_j)/(1 + R_j) \geq (r+k)^n/(1 + k)^n .$$

This gives

$$|P(re^{i\theta})| \geq \left(\frac{r + k}{1 + k}\right)^n |P(e^{i\theta})| \quad \text{for} \quad r \leq k^2 \quad \text{and} \quad 0 \leq \theta < 2\pi .$$

Hence

$$\operatorname*{Max}_{|z|=r} |P(z)| \geq \frac{(r + k)^n}{(1 + k)^n} \operatorname*{Max}_{|z|=1} |P(z)| \quad \text{for} \quad 0 \leq r \leq k^2 ,$$

which proves inequality (7).

Proof. of Theorem 3. It is clearly sufficient to consider the case $K > 1$. Since $P(z)$ has all its zeros in $|z| \geq K > 1$, it follows that the polynomial $H(z) = P(Kz)$ has all its zeros in $|z| \geq 1$. If now $Q(z) = z^n \overline{P(1/\bar{z})}$, then the polynomial

$$G(z) = z^n \overline{H(1/\bar{z})} = z^n \overline{P(K/\bar{z})} = K^n Q(z/K)$$

has all its zeros in $|z| \leq 1$. Moreover $|H(z)| = |G(z)|$ for $|z| = 1$. Hence $G(z)/H(z)$ is analytic on and inside the unit circle and on the boundary $|G(z)/H(z)| = 1$. By the maximum modulus principle it follows that $|G(z)| \leq |H(z)|$ for $|z| \leq 1$. Replacing z by $1/\bar{z}$ and nothing that $z^n \overline{G(1/\bar{z})} = H(z)$, we conclude that $|H(z)| \leq |G(z)|$ for $|z| \geq 1$. Hence in particular $|H(Kz)| \leq |G(Kz)|$ for $|z| \geq 1$. Equivalently

$$(13) \qquad |P(K^2 z)| \leq K^n |Q(z)| \quad \text{for} \quad |z| \geq 1 .$$

Since all the zeros of $Q(z)$ lie in $|z| \leq \frac{1}{K} < 1$, therefore, if α is a complex number such that $|\alpha| > 1$, then Rouché's theorem, the polynomial $P(K^2z) - \alpha K^nQ(z)$ has all its zeros in $|z| < 1$. By the Gauss-Lucas theorem, the polynomial $K^2P'(K^2z) - \alpha K^nQ'(z)$ does not vanish in $|z| \geq 1$. This implies that

$$K^2|P'(K^2z)| \leq K^n|Q'(z)| \quad \text{for} \quad |z| \geq 1,$$

which gives with the help of Lemma 1

$$K^2|P'(K^2z)| + K^n|P'(z)| \leq nK^n \max_{|z|=1} |P(z)| \quad \text{for} \quad |z| = 1.$$

This, by hypothesis, implies that

$$(14) \qquad K^2 \max_{|z|=1} |P'(K^2z)| + K^n \max_{|z|=1} |P'(z)| \leq nK^n \max_{|z|=1} |P(z)|.$$

Now $P'(z)$ is a polynomial of degree $(n-1)$ and $K > 1$, therefore, by (1), it follows that

$$\max_{|z|=1} |P'(K^2z)| = \max_{|z|=K^2} |P'(z)| \leq K^{2(n-1)} \max_{|z|=1} |P'(z)|.$$

Using this in (14) we obtain

$$(1 + K^n)K^2 \max_{|z|=1} |P'(K^2z)| \leq nK^{2n} \max_{|z|=1} |P(z)|.$$

Applying (1) again to the polynomial $P'(K^2z)$, we obtain for all $r \geq 1$ and $0 \leq \theta < 2\pi$

$$(15) \qquad K^2|P'(K^2re^{i\theta})| \leq \frac{nK^{2n}r^{n-1}}{1 + K^n} \max_{|z|=1} |P(z)|.$$

Now for each θ, $0 \leq \theta < 2\pi$ and $R > 1$, we have

$$P(K^2Re^{i\theta}) - P(K^2e^{i\theta}) = \int_1^R K^2e^{i\theta}P'(K^2re^{i\theta})\, dr.$$

This gives with the help of (15)

$$\left|P(K^2Re^{i\theta}) - P(K^2e^{i\theta})\right| \leq \int_1^R K^2|P'(K^2re^{i\theta})|\, dr$$

$$\leq \frac{K^{2n}}{1 + K^n} \left\{ \int_1^R nr^{n-1}\, dr \right\} \max_{|z|=1} |P(z)|$$

$$(16) \qquad = \frac{K^{2n}(R^n - 1)}{1 + K^n} \max_{|z|=1} |P(z)|.$$

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Since by (13)

$$|P(K^2e^{i\theta})| \le K^n|Q(e^{i\theta})| = K^n|P(e^{i\theta})| ,$$

it follows from (16) that for each θ, $0 \le \theta < 2\pi$ and $R > 1$

$$|P(K^2Re^{i\theta})| \le \left\{ \frac{K^{2n}(R^n - 1)}{1 + K^n} + K^n \right\} \max_{|z|=1} |P(z)|$$

$$= \frac{K^{2n}R^n + K^n}{1 + K^n} \max_{|z|=1} |P(z)| .$$

This gives

$$\max_{|z|=R\ge K^2} |P(z)| \le \frac{R^n + K^n}{1 + K^n} \max_{|z|=1} |P(z)| ,$$

which is the desired result.

Proof of Theorem 4. Let $m = \min_{|z|=1} |P(z)| = \min_{|z|=1} |Q(z)|$ where $Q(z) = z^n \overline{P(1/\bar{z})}$, then $m \le |Q(z)|$ for $|z|=1$. Since $P(z)$ has all its zeros in $|z| \ge K \ge 1$, therefore, all the zeros of $Q(z)$ lie in $|z| \le 1$. Hence by Rouché's theorem, it follows that for every complex number α with $|\alpha| < 1$, the polynomial $F(z) = Q(z) - \alpha m$ of degree n has all its zeros in $|z| \le 1$ (note that, this is true even if $m = 0$). So that the polynomial

$$G(z) = z^n \overline{F(1/\bar{z})} = z^n \overline{Q(1/\bar{z})} - \bar{\alpha} m z^n = P(z) - \bar{\alpha}\, m z^n$$

has all its zeros in $|z| \ge 1$ and $|G(z)| = |F(z)|$ for $|z| = 1$. Thus the function $F(z)/G(z)$ is analytic in $|z| \le 1$ and $|F(z)/G(z)| = 1$ for $|z|=1$. It now follows as in the proof of Theorem 3 that

$$|G(z)| \le |F(z)| \quad \text{for} \quad |z| \ge 1 .$$

Equivalently

$$|P(z) - \bar{\alpha}\, m z^n| \le |Q(z) - \alpha m| \quad \text{for} \quad |z| \ge 1 .$$

Taking in particular $z = Re^{i\theta}$ where $R \ge 1$ and $0 \le \theta < 2\pi$, we get

$$|P(re^{i\theta}) - \bar{\alpha}\, mR^n e^{in\theta}| \le |Q(Re^{i\theta}) - \alpha m| \tag{17}$$

for every α with $|\alpha| < 1$. Choosing argument of α in (17) such that

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$$|P(Re^{i\theta}) - \bar{\alpha}\, mR^n e^{in\theta}| = |P(Re^{i\theta})| + |\alpha| mR^n ,$$

we obtain

$$|P(Re^{i\theta})| + |\alpha| mR^n \le |Q(Re^{i\theta})| + |\alpha| m ,$$

or

$$|P(Re^{i\theta})| + |\alpha| m(R^n - 1) \le |Q(Re^{i\theta})| \tag{18}$$

for all $R \ge 1$, $0 \le \theta < 2\pi$ and for every α with $|\alpha| < 1$. Letting $|\alpha| \to 1$ in (18) we get

$$|P(Re^{i\theta})| + m(R^n - 1) \le |Q(Re^{i\theta})|$$

for all $R \ge 1$ and $0 \le \theta < 2\pi$. This gives with the help of Lemma 2 that

$$2|P(Re^{i\theta})| + m(R^n - 1) \le (R^n + 1) \max_{|z|=1} |P(z)| \tag{19}$$

for all $R \ge 1$ and $0 \le \theta < 2\pi$. From (19) we finally obtain

$$\max_{|z|=R>1} |P(z)| \le \left(\frac{R^n + 1}{2}\right) \max_{|z|=1} |P(z)| - \left(\frac{R^n - 1}{2}\right) \min_{|z|=1} |P(z)| ,$$

which is (11) and Theorem 4 is completely proved.

Proof of Theorem 5. Since all the zeros of $P(z)$ lie in $|z| \ge K$, $K \le 1$, therefore, for $0 < r \le K$, the polynomial $P(rz)$ has all its zeros in $|z| \ge \frac{K}{r} \ge 1$. Applying Theorem 4 to the polynomial $P(rz)$, we obtain

$$\max_{|z|=R\ge 1} |P(rz)| \le \left(\frac{R^n + 1}{2}\right) \max_{|z|=1} |P(rz)| - \left(\frac{R^n - 1}{2}\right) \min_{|z|=1} |P(rz)| .$$

Equivalently

$$\max_{|z|=1} |P(Rrz)| \le \left(\frac{R^n + 1}{2}\right) \max_{|z|=r} |P(z)| - \left(\frac{R^n - 1}{2}\right) \min_{|z|=r} |P(z)| .$$

Taking $R = 1/r$, then for $0 < r \le K$, we obtain

$$\left(\frac{1 + r^n}{2r^n}\right) \max_{|z|=r} |P(z)| - \left(\frac{1 - r^n}{2r^n}\right) \min_{|z|=r} |P(z)| \ge \max_{|z|=1} |P(z)| ,$$

which is equivalent to (12) and Theorem 5 is proved.

References

[1] Abdul Aziz, "Inequalities for the derivative of a polynomial", *Amer. Math. Soc.* 89 (1983), 259-266.

[2] Abdul Aziz and Q.G. Mohammad, "Growth of polynomials with zeros outside a circle", *Proc. Amer. Math. Soc.* 81 (1981), 549-553.

[3] Abdul Aziz and Q.G. Mohammad, "Simple proof of a Theorem of Erdös and Lax", *Proc. Amer. Math. Soc.* 80 (1980), 119-122.

[4] N.C. Ankeny and T.J. Rivlin, "On a theorem of S. Bernstein", *Pacific J. Math.* 5 (1955), 849-852.

[5] N.K. Govil and Q.I. Rahman, "Functions of exponential type not vanishing in a half-plane and related polynomials", *Trans. Amer. Math. Soc.* 137 (1969), 501-517.

[6] G. Pólyá and G. Szegö, *Aufgaben und Lehrsatze aus der Analysis*, Springer-Verlag, Berlin, 1925.

[7] M. Riesz, "Über einen Satz des Herrn Serge Bernstein", *Acta Math.* 40 (1916), 337-347.

[8] T. J. Rivlin, "On the maximum modulus of polynomials", *Amer. Math. Monthly* 67 (1960), 251-253.

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