GROWTH OF POLYNOMIALS WHOSE ZEROS ARE
WITHIN OR OUTSIDE A CIRCLE

ABDUL AZIZ

Let $P(z)$ be a polynomial of degree $n$ which does not vanish in the disk $|z| < K$. For $K = 1$, it is known that

$$\max_{|z| = r < 1} |P(z)| \geq (\frac{1+r}{2})^n \max_{|z| = 1} |P(z)|,$$

and

$$\max_{|z| = R > 1} |P(z)| \leq (\frac{R+1}{2})^n \max_{|z| = 1} |P(z)|.$$

In this paper we consider the two cases $K \geq 1$ and $K < 1$, and present certain generalizations of these results.

If $P(z)$ is a polynomial of degree $n$, then [7, p.346] or [6, Vol.I, p.137 Problem III 269]

(1) $$\max_{|z| = R > 1} |P(z)| \leq R^n \max_{|z| = 1} |P(z)|.$$

Here equality holds if and only if $P(z) = az^n$.

It was shown by Ankeny and Rivlin [4] that if $P(z) \neq 0$ in $|z| < 1$, then (1) can be replaced by

Received 2 April 1986. We thank Professor Q.I. Rahman for his useful suggestions.
Inequality (2) is sharp and equality holds for $P(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$. 

By the maximum modulus principle
\[
\max_{|z| = \frac{1}{r} > 1} |z^n P(1/z)| \geq \max_{|z| = 1} |z^n P(1/z)| = \max_{|z| = 1} |P(z)|,
\]
and so
\[
\max_{|z| = r < 1} |P(z)| \geq r^n \max_{|z| = 1} |P(z)|,
\]
where equality holds if and only if $P(z) = \alpha z^n$.

If $P(z) \neq 0$ in $|z| < 1$, then [8] the stronger inequality
\[
\max_{|z| = r > 1} |P(z)| \geq \left(\frac{r+1}{r-1}\right)^n \max_{|z| = 1} |P(z)|
\]
holds. Here equality is attained if $P(z) = \alpha (z - \beta)^n$, $|\beta| = 1$.

In this paper we obtain certain generalizations of inequalities (2) and (4). We prove.

**THEOREM 1.** If $P(z)$ is a polynomial of degree $n$ such that $P(z) \neq 0$ in $|z| < K$ where $K \geq 1$, then
\[
\max_{|z| = r < 1} |P(z)| \geq (\frac{r+K}{1+K})^n \max_{|z| = 1} |P(z)|.
\]
Here equality holds if $P(z) = (z + K)^n$.

Applying Theorem 1 to the polynomial $z^n P(1/z)$, we obtain

**THEOREM 1'.** If $P(z)$ is a polynomial of degree $n$ which has all its zeros in the disk $|z| \leq k$ where $k \leq 1$, then
\[
\max_{|z| = r > 1} |P(z)| \geq (\frac{r+k}{1+k})^n \max_{|z| = 1} |P(z)|.
\]
The result is sharp and in (6) equality holds for $P(z) = (z + k)^n$.

**THEOREM 2.** If $P(z)$ is a polynomial of degree $n$ such that $P(z) \neq 0$ in $|z| < k$ where $k \leq 1$, then
Growth of Polynomials

(7) \[ \max_{|z|=r} |P(z)| \geq \left( \frac{r+k}{1+k} \right)^n \max_{|z|=1} |P(z)| \quad \text{if} \quad 0 < r < k^2. \]

The estimate is sharp with equality in (7) for \( P(z) = (z + k)^n \).

This result when applied to \( z^n P(1/z) \) gives:

THEOREM 2'. If \( P(z) \) is a polynomial of degree \( n \) which has all its zeros in the disk \( |z| < K \) where \( K \geq 1 \), then

(8) \[ \max_{|z|=R} |P(z)| \geq \left( \frac{R+k}{1+k} \right)^n \max_{|z|=1} |P(z)| \quad \text{if} \quad R \geq K^2. \]

The result is sharp with equality in (8) for \( P(z) = (z + K)^n \).

The precise estimate for \( \max_{|z|=r} |P(z)| \) in Theorem 2 for \( k^2 < r < 1 \) and the corresponding estimate for \( \max_{|z|=r} |P(z)| \) in Theorem 2' for \( 1 < R < K^2 \) does not seem to be easily obtainable. It was shown by Aziz and Mohammad [2] that if \( P(z) \) is a polynomial of degree \( n \) which does not vanish in the disk \( |z| < K \) where \( K \geq 1 \), then

\[ \max_{|z|=R} |P(z)| \leq \left( \frac{R+k}{1+k} \right)^n \max_{|z|=1} |P(z)| \quad \text{for} \quad 1 \leq R \leq K^2 \]

and in addition, if \( P(z) \) has non-negative coefficients or if \( P(K^2 Rz) \) and \( P(Rz) \) become maximum at the same point on \( |z| = 1, R > 1 \), then

(9) \[ \max_{|z|=R} |P(z)| \leq \left( \frac{R+k}{1+k} \right)^n \max_{|z|=1} |P(z)| \quad \text{for} \quad R > K^2 \]

We take this opportunity to point out that the statement of the inequality (5) of Theorem 2 of [2] should read as the statement of the inequality (9) above, as the proof given for the first part of Theorem 2 in [2] covers only the above mentioned class of polynomials and so the general case is still open.

However, we have a considerable evidence in favour of the following

CONJECTURE. If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in the disk \( |z| < k \), then

\[ \max_{|z|=r} |P(z)| \geq \left( \frac{r^n + k^n}{1 + k^n} \right) \max_{|z|=1} |P(z)| \quad \text{for} \quad k^2 < r < 1, k < 1 \]

and

\[ 
\]
\[ \max_{|z|=R} |P(z)| \leq \frac{R^n + k^n}{1 + k^n} \max_{|z|=1} |P(z)| \quad \text{for } R > k^2, \ k > 1. \]

Here we prove the following generalisations of (2).

**THEOREM 3.** If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in the disk \( |z| < K \) where \( K > 1 \), then

\[ (10) \quad \max_{|z|=R} |P(z)| < \left( \frac{R^2 + 1}{2} \right) \max_{|z|=1} |P(z)| \]

provided \( |P'(K^2 z)| \) and \( |P'(z)| \) become maximum at the same point on \( |z| = 1 \).

The result is best possible with equality in (10) for \( P(z) = z^n + k^n \).

The next result is an interesting generalisation of the inequality (2).

**THEOREM 4.** If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in the disk \( |z| < K \) where \( K > 1 \), then

\[ (11) \quad \max_{|z|=R>1} |P(z)| < \left( \frac{R^n + 1}{2} \right) \max_{|z|=1} |P(z)| - \left( \frac{R^n - 1}{2} \right) \min_{|z|=1} |P(z)|. \]

The result is best possible and equality in (11) holds for the polynomial \( P(z) = az^n + \beta k^n, \ |a| = |\beta| = 1, \ k > 1 \).

As an application of Theorem 4, we establish

**THEOREM 5.** If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in the disk \( |z| < k, \ k < 1 \), then for \( 0 \leq r \leq k \) we have

\[ (12) \quad (1 + r^n) \max_{|z|=r} |P(z)| - (1 - r^n) \min_{|z|=r} |P(z)| \geq 2r^n \max_{|z|=1} |P(z)|. \]

The result is best possible and equality in (12) holds for the polynomial \( P(z) = az^n + \beta k^n \) where \( |a| = |\beta| = 1 \) and \( k < 1 \).

For the proofs of these theorems, we need the following lemmas.

**LEMMA 1.** If \( P(z) \) is a polynomial of degree \( n \), then on \( |z|=1 \),

\[ |P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|, \]

where \( Q(z) = z^n P(1/z) \).
This is a special case of a result due to Govil and Rahman [5, Lemma 10] (see also [3]).

**Lemma 2** [1,2]. If \( P(z) \) is a polynomial of degree \( n \), then for all \( R > 1 \) and \( 0 < \theta < 2\pi \)

\[
|P(re^{i\theta})| + |Q(Re^{i\theta})| \leq (R^n + 1) \frac{\max |P(z)|}{|z|=1},
\]

where \( Q(z) = z^n P(1/z) \).

**Proofs of the theorems.**

**Proof of Theorem 1.** Since all the zeros of \( P(z) \) lie in \( |z| \geq K \), \( K \geq 1 \), we write

\[
P(z) = C \prod_{j=1}^{n} (z - R_j e^{i\theta_j}) \quad \text{where} \quad R_j \geq K, \quad j = 1, 2, \ldots, n.
\]

Therefore, for \( 0 \leq \theta < 2\pi \) and \( r < 1 \), we have clearly

\[
|P(re^{i\theta})/P(e^{i\theta})| = \prod_{j=1}^{n} \left| \frac{r e^{i\theta} - R_j e^{i\theta_j}}{e^{i\theta} - R_j e^{i\theta_j}} \right| \geq \prod_{j=1}^{n} \left| \frac{(r^2 + R_j^2 - 2rR_j \cos(\theta - \theta_j))/(1 + R_j^2 - 2R_j \cos(\theta - \theta_j))}{(r + R_j)/(1 + R_j)} \right|^{1/2} \geq \prod_{j=1}^{n} \left( \frac{r + K}{1 + K} \right)^n.
\]

This implies

\[
|P(re^{i\theta})| \geq \left( \frac{r + K}{1 + K} \right)^n |P(e^{i\theta})| \quad \text{for} \quad r \leq 1, \quad 0 \leq \theta < 2\pi.
\]

Hence

\[
\max_{|z|=1} |P(z)| \geq \left( \frac{r + K}{1 + K} \right)^n \max_{|z|=1} |P(z)|
\]

and the proof of Theorem 1 is complete.
Proof of Theorem 2. Since the polynomial $P(z)$ has all its zeros in $|z| \geq k$ where $k \leq 1$, we write as before
\[ P(z) = C \prod_{j=1}^{n} (z - R_j e^{i\theta_j}) \text{ where } R_j \geq k, \quad j=1,2,\ldots, n. \]
Then clearly for $r \leq k^2$ and $0 \leq \theta < 2\pi$, we have
\[ \left| \frac{P(re^{i\theta})}{P(e^{i\theta})} \right| = \prod_{j=1}^{n} \left| \frac{re^{i\theta} - R_j e^{i\theta_j}}{e^{i\theta} - R_j e^{i\theta_j}} \right| \geq \prod_{j=1}^{n} \frac{(r + R_j)/(1 + R_j)}{(1 + k)^n}. \]
This gives
\[ |P(re^{i\theta})| \geq \left( \frac{r + k}{1 + k} \right)^n |P(e^{i\theta})| \text{ for } r \leq k^2 \text{ and } 0 \leq \theta < 2\pi. \]
Hence
\[ \max_{|z|=r} |P(z)| \geq \left( \frac{r + k}{1 + k} \right)^n \max_{|z|=1} |P(z)| \text{ for } 0 \leq r \leq k^2, \]
which proves inequality (7).

Proof of Theorem 3. It is clearly sufficient to consider the case $K > 1$. Since $P(z)$ has all its zeros in $|z| \geq K > 1$, it follows that the polynomial $H(z) = P(Kz)$ has all its zeros in $|z| \geq 1$. If now $Q(z) = z^n P(1/z^2)$, then the polynomial
\[ G(z) = z^n H(1/z^2) = z^n P(K/z^2) = K^n Q(z/K) \]
has all its zeros in $|z| \leq 1$. Moreover $|H(z)| = |G(z)|$ for $|z| = 1$. Hence $G(z)/H(z)$ is analytic on and inside the unit circle and on the boundary $|G(z)/H(z)| = 1$. By the maximum modulus principle it follows that $|G(z)| \leq |H(z)|$ for $|z| \leq 1$. Replacing $z$ by $1/z^2$ and nothing that $z^n G(1/z^2) = H(z)$, we conclude that $|H(z)| \leq |G(z)|$ for $|z| \geq 1$. Hence in particular $|H(Kz)| \leq |G(Kz)|$ for $|z| \geq 1$. Equivalently
\[ |P(K^2 z)| \leq K^n |Q(z)| \text{ for } |z| \geq 1. \]
Since all the zeros of $Q(z)$ lie in $|z| \leq \frac{1}{K} < 1$, therefore, if $\alpha$ is a complex number such that $|\alpha| > 1$, then Rouche's theorem, the
polynomial $P(K^2z) - \alpha K^n Q(z)$ has all its zeros in $|z| < 1$. By the Gauss-Lucas theorem, the polynomial $K^2 P'(K^2z) - \alpha K^n Q'(z)$ does not vanish in $|z| \geq 1$. This implies that

$$K^2 |P'(K^2z)| \leq K^n |Q'(z)| \quad \text{for} \quad |z| \geq 1,$$

which gives with the help of Lemma 1

$$K^2 |P'(K^2z)| + K^n |P'(z)| \leq nK^n \max_{|z|=1} |P(z)| \quad \text{for} \quad |z| = 1.$$

This, by hypothesis, implies that

$$K^2 \max_{|z|=1} |P'(K^2z)| + K^n \max_{|z|=1} |P'(z)| \leq nK^n \max_{|z|=1} |P(z)|.$$

Now $P'(z)$ is a polynomial of degree $(n-1)$ and $K > 1$, therefore, by (1), it follows that

$$\max_{|z|=1} |P'(K^2z)| = \max_{|z|=K^2} |P'(z)| \leq K^2(n-1) \max_{|z|=1} |P'(z)|.$$

Using this in (14) we obtain

$$(1 + K^n) K^2 \max_{|z|=1} |P'(K^2z)| \leq nK^{2n} \max_{|z|=1} |P(z)|.$$

Applying (1) again to the polynomial $P'(K^2z)$, we obtain for all $r \geq 1$ and $0 \leq \theta < 2\pi$

$$K^2 |P'(K^2re^{i\theta})| \leq \frac{nK^{2n} n^{-1}}{1 + K^n} \max_{|z|=1} |P(z)|.$$

Now for each $\theta$, $0 \leq \theta < 2\pi$ and $R > 1$, we have

$$P(K^2Re^{i\theta}) - P(K^2 e^{i\theta}) = \int_1^R K^2 e^{i\theta} P'(K^2 re^{i\theta}) \, dr.$$

This gives with the help of (15)

$$\left| P(K^2Re^{i\theta}) - P(K^2 e^{i\theta}) \right| \leq \int_1^R K^2 |P'(K^2 re^{i\theta})| \, dr \leq \frac{nK^{2n} n^{-1}}{1 + K^n} \max_{|z|=1} |P(z)|$$

$$= \frac{nK^{2n}(R^n - 1)}{1 + K^n} \max_{|z|=1} |P(z)|.$$
Since by (13)
\[ |P(K^2 e^{i\theta})| \leq K^n |Q(e^{i\theta})| = K^n |P(e^{i\theta})| , \]
it follows from (16) that for each \( \theta, 0 \leq \theta < 2\pi \) and \( R > 1 \)
\[ |P(K^2 R e^{i\theta})| \leq \left\{ \frac{K^n (R^n - 1) + K^n}{1 + K^n} \right\} \max_{|z|=1} |P(z)| \]
\[ = \frac{K^n R^n + K^n}{1 + K^n} \max_{|z|=1} |P(z)| . \]

This gives
\[ \max_{|z|=R \geq K^2} |P(z)| \leq \frac{R^n + K^n}{1 + K^n} \max_{|z|=1} |P(z)| , \]
which is the desired result.

Proof of Theorem 4. Let \( m = \min_{|z|=1} |P(z)| = \min_{|z|=1} |Q(z)| \) where
\( Q(z) = z^n P(1/z) \), then \( m \leq |Q(z)| \) for \( |z|=1 \). Since \( P(z) \) has all its zeros in \( |z| \geq K \geq 1 \), therefore, all the zeros of \( Q(z) \) lie in \( |z| \leq 1 \). Hence by Rouché's theorem, it follows that for every complex number \( a \) with \( |a| < 1 \), the polynomial \( F(z) = Q(z) - \alpha m z^n \) of degree \( n \) has all its zeros in \( |z| \leq 1 \) (note that, this is true even if \( m = 0 \)). So that the polynomial
\( \tilde{G}(z) = z^n P(1/z) - \alpha m z^n = \tilde{Q}(z) - \tilde{a} \tilde{m} z^n \)
has all its zeros in \( |z| \geq 1 \) and \( |\tilde{G}(z)| = |F(z)| \) for \( |z|=1 \). Thus the function \( F(z)/G(z) \) is analytic in \( |z| \leq 1 \) and \( |F(z)/G(z)| = 1 \) for \( |z|=1 \). It now follows as in the proof of Theorem 3 that
\[ |G(z)| \leq |F(z)| \text{ for } |z| \geq 1 . \]
Equivalently
\[ |P(z) - \tilde{a} m z^n| \leq |Q(z) - \alpha m z^n| \text{ for } |z| \geq 1 . \]
Taking in particular \( z = \text{Re} e^{i\theta} \) where \( R \geq 1 \) and \( 0 \leq \theta < 2\pi \), we get
\[ (17) \quad |P(\text{Re} e^{i\theta}) - \tilde{a} m R^n e^{in\theta}| \leq |Q(\text{Re} e^{i\theta}) - \alpha m| \]
for every \( \alpha \) with \( |\alpha| < 1 \). Choosing argument of \( \alpha \) in (17) such that
Growth of Polynomials 255

\[ |P(Re^{i\theta}) - \alpha mR^n e^{in\theta}| = |P(Re^{i\theta})| + |\alpha|mR^n, \]

we obtain

\[ |P(Re^{i\theta})| + |\alpha|mR^n \leq |Q(Re^{i\theta})| + |\alpha|m, \]
or

\[(18) \quad |P(Re^{i\theta})| + |\alpha|m(R^n - 1) \leq |Q(Re^{i\theta})| \]

for all \( R \geq 1, 0 \leq \theta < 2\pi \) and for every \( \alpha \) with \( |\alpha| < 1 \). Letting \( |\alpha| \to 1 \) in (18) we get

\[ |P(Re^{i\theta})| + m(R^n - 1) \leq |Q(Re^{i\theta})| \]

for all \( R \geq 1 \) and \( 0 \leq \theta < 2\pi \). This gives with the help of Lemma 2 that

\[(19) \quad 2|P(Re^{i\theta})| + m(R^n - 1) \leq (R^n + 1) \max_{|z|=1} |P(z)| \]

for all \( R \geq 1 \) and \( 0 \leq \theta < 2\pi \). From (19) we finally obtain

\[ \max_{|z|=R \geq 1} |P(z)| \leq \left( \frac{R^n + 1}{2} \right) \max_{|z|=1} |P(z)| - \left( \frac{R^n - 1}{2} \right) \min_{|z|=1} |P(z)|, \]

which is (11) and Theorem 4 is completely proved.

Proof of Theorem 5. Since all the zeros of \( P(z) \) lie in \( |z| \geq K, K \leq 1 \), therefore, for \( 0 < r \leq K \), the polynomial \( P(rz) \) has all its zeros in \( |z| \geq \frac{K}{r} \geq 1 \). Applying Theorem 4 to the polynomial \( P(rz) \), we obtain

\[ \max_{|z|=R \geq 1} |P(rz)| \leq \left( \frac{R^n + 1}{2} \right) \max_{|z|=1} |P(rz)| - \left( \frac{R^n - 1}{2} \right) \min_{|z|=1} |P(rz)|. \]

Equivalently

\[ \max_{|z|=1} |P(Rrz)| \leq \left( \frac{R^n + 1}{2} \right) \max_{|z|=r} |P(z)| - \left( \frac{R^n - 1}{2} \right) \min_{|z|=r} |P(z)|. \]

Taking \( R = 1/r \), then for \( 0 < r \leq K \), we obtain

\[ \left( \frac{1 + r^n}{2r^n} \right) \max_{|z|=r} |P(z)| - \left( \frac{1 - r^n}{2r^n} \right) \min_{|z|=r} |P(z)| \geq \max_{|z|=1} |P(z)|, \]

which is equivalent to (12) and Theorem 5 is proved.
References


Post-Graduate Department of Mathematics
University of Kashmir
Hazratbal Srinagar- 190006
Kashmir India.