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Maximal Weight Composition Factors for Weyl Modules

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Abstract. Fix an irreducible (finite) root system R and a choice of positive roots. For any algebraically closed field k consider the almost simple, simply connected algebraic group G_k over k with root system k. One associates with any dominant weight λ for R two G_k-modules with highest weight λ, the Weyl module $V(\lambda)_k$ and its simple quotient $L(\lambda)_k$. Let λ and μ be dominant weights with $\mu < \lambda$ such that μ is maximal with this property. Garibaldi, Guralnick, and Nakano have asked under which condition there exists k such that $L(\mu)_k$ is a composition factor of $V(\lambda)_k$, and they exhibit an example in type E_8 where this is not the case. The purpose of this paper is to to show that their example is the only one. It contains two proofs for this fact: one that uses a classification of the possible pairs (λ, μ), and another that relies only on the classification of root systems.

1 Introduction

For general background on representations of Lie algebras and of algebraic groups we refer the reader to [6, 9].

1.1 Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over the complex numbers. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, denote by R the corresponding root system, choose a system R^+ of positive roots, and denote the set of simple roots by S. We write $\alpha^\vee$ for the coroot associated with a root $\alpha \in R$. Denote the Weyl group of R by W and the reflection with respect to some $\alpha \in R$ by s_α.

We denote by $X \subset \mathfrak{h}^*$ the set of integral weights and by $X_+ \subset X$ the set of dominant weights. We write $\leq$ for the usual partial order relation on X where $\mu \leq \lambda$ if and only if $\lambda - \mu \in \sum_{\alpha\in S} \mathbf{N}\alpha$. For any $\lambda \in X_+$ let $V(\lambda)$ be a simple $\mathfrak{g}$-module with highest weight λ. For any $\mu \in X$ let $V(\lambda)_\mu$ denote the corresponding weight space of $V(\lambda)$.

1.2 For any prime number p, fix an algebraically closed field k of characteristic p. Then let G_k denote the almost simple, simply connected algebraic group over k with root system R. We can then identify X with the group of characters of a maximal torus in G_k. For each $\lambda \in X_+$ let $V(\lambda)_k$ be a Weyl module with highest weight λ. Its radical $\operatorname{rad} V(\lambda)_k$ is a maximal submodule, and the quotient $L(\lambda)_k = V(\lambda)_k / \operatorname{rad} V(\lambda)_k$ is a simple module with highest weight λ. Both $V(\lambda)_k$ and $L(\lambda)_k$ are direct sums of weight spaces that we denote by $V(\lambda)_{k,\mu}$ and $L(\lambda)_{k,\mu}$, respectively.

1.3 Garibaldi, Guralnick, and Nakano prove the following result in [5].

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Theorem ([5]) *Let λ be a dominant weight. Then $V(\lambda)_k$ is simple for all k if and only we are in one of the following cases:*

(i) $\lambda = 0$;
(ii) *λ is minuscule;*
(iii) *the root system R has type E_8 and λ is the unique dominant root.*

We use minuscule here in the sense of [2, Déf. 1 in Chap. VIII, § 7]; see also [1, Exerc. 24 in Chap. VI, § 1].

Actually, the result in [5] is stronger: for all other λ one can find a field k such that $V(\lambda)_k$ is not simple and $\operatorname{Char} k \le 2 \operatorname{rk} \mathfrak{g} + 1$.

1.4 In reply to a final question in [5] we are going to prove the following theorem.

Theorem *Let λ be a dominant weight. Let $\mu < \lambda$ be a dominant weight that is maximal among the dominant weights less than λ. Exclude the case where the root system R has type E_8 and λ is the unique dominant root. Then there exists a field k such that $L(\mu)_k$ is a composition factor of $V(\lambda)_k$.*

Note that Theorem 1.4 together with information about the E_8 case implies Theorem 1.3 as stated here, *i.e.,* without the explicit bound on $\operatorname{Char} k$. (The simplicity of $V(\lambda)_k$ for all k in the cases (i)–(iii) is well known.)

1.5 Theorem 1.4 can be generalised to more general reductive groups. One should fix a root datum and set G_k equal to the connected reductive group over k with this root datum. In the simplest case one gets a direct product $G_k = G_{0,k} \times G_{1,k} \times \cdots G_{r,k}$, where $G_{0,k}$ is a torus and where each $G_{i,k}$ with $i > 0$ is an almost simple, simply connected group. In this case a dominant weight is a tuple $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_r)$ where each λ_i is a dominant weight for $G_{i,k}$ (any weight for $i = 0$). Another dominant weight, $\mu = (\mu_0, \mu_1, \dots, \mu_r)$, is maximal among the dominant weights less than λ if and only if there exists $i > 0$ such that μ_i is maximal among the dominant weights for $G_{i,k}$ less than λ_i and if $\mu_j = \lambda_j$ for all $j \neq i$ (including $j = 0$). If so, then $L(\mu)_k$ is a composition factor of $V(\lambda)_k$ if and only if $L(\mu_i)_k$ is a composition factor for $G_{i,k}$ of $V(\lambda_i)_k$, since both Weyl modules and simple modules are tensor products of the corresponding modules for each $G_{j,k}$. Now Theorem 1.4 says that there does not exist k with $L(\mu)_k$ a composition factor of $V(\lambda)_k$ if and only if $G_{i,k}$ has type E_8 and $\mu_i = 0$ and λ_i the dominant root.

For arbitrary root data one can find a central covering $G'_k \to G_k$ such that G' has a direct product decomposition as above. One then uses that Weyl modules (resp. simple modules) for G_k lift to Weyl modules (resp. simple modules) for G'_k.

2 Maximal Dominant Weights

2.1 The first proof of Theorem 1.4 will involve induction on the rank as well as a description of the maximal dominant weights less than a given one. We shall see (in Subsection 2.2) that most of the time these maximal weights arise by subtracting a simple root from the top weight.

For any subset $I \subset S$, set $R_I = R \cap \mathbf{Z}I$; this is a root system for a suitable Levi factor of $\mathfrak{g}$. One can choose $R_I^+ = R^+ \cap R_I$ as the set of positive roots; then I is the set of simple roots for this choice. We can identify the set X_I of integral weights for R_I with a lattice in $\sum_{\alpha\in I} \mathbf{Q}\alpha$. For any $\lambda \in X$, denote by $\lambda_I \in X_I$ the weight with $\langle \lambda_I, \alpha^\vee \rangle = \langle \lambda, \alpha^\vee \rangle$ for all $\alpha \in I$. If $\lambda, \mu \in X$ satisfy $\lambda - \mu = \sum_{\alpha\in I} m_\alpha \alpha$ for some $m_\alpha \in \mathbf{Z}$, then $\lambda_I - \mu_I = \sum_{\alpha\in I} m_\alpha \alpha = \lambda - \mu$.

Lemma *Let $\lambda, \mu \in X$ with λ dominant and $\mu < \lambda$. Suppose that $\lambda - \mu = \sum_{\alpha\in I} m_\alpha \alpha$ for some $m_\alpha \in \mathbf{Z}$, $m_\alpha \geq 0$.*

(i) *The weight μ is dominant if and only if μ_I is dominant for the root system R_I.*
(ii) *The weight μ is dominant and maximal among the dominant weights less than λ if and only if μ_I is dominant for the root system R_I and maximal among the weights less than λ_I and dominant for the root system R_I.*
(iii) *If the weight μ is dominant and maximal among the dominant weights less than λ, then the subset $\{\alpha \in I \mid m_\alpha > 0\}$ of S is connected in the Coxeter graph.*

Proof We have

$$\langle \mu, \beta^\vee \rangle = \langle \lambda, \beta^\vee \rangle - \sum_{\alpha\in I} m_\alpha \langle \alpha, \beta^\vee \rangle \geq \langle \lambda, \beta^\vee \rangle \geq 0$$

for all $\beta \in S \smallsetminus I$; this implies (i).

Any weight in X_I between λ_I and μ_I has the form $\lambda_I - \sum_{\alpha\in I} m'_\alpha \alpha$ with $m'_\alpha \in \mathbf{Z}$, $0 \leq m'_\alpha \leq m_\alpha$ for all $\alpha \in I$. It is then equal to ν_I where $\nu = \lambda - \sum_{\alpha\in I} m'_\alpha \alpha$. We know by (i) that ν_I is dominant for R_I if and only if ν is dominant. This implies (ii).

In order to see (iii) we assume that $m_\alpha > 0$ for all $\alpha \in I$. Suppose that $I = J \cup K$ is the disjoint union of two nonempty subsets J and K with $\langle \alpha, \beta^\vee \rangle = 0$ for all $\alpha \in J$ and $\beta \in K$. Then $\mu' = \lambda - \sum_{\alpha\in J} m_\alpha \alpha$ satisfies $\mu < \mu' < \lambda$ and $\langle \mu', \alpha^\vee \rangle = \langle \mu, \alpha^\vee \rangle \geq 0$ for all $\alpha \in J$ and $\langle \mu', \beta^\vee \rangle = \langle \lambda, \beta^\vee \rangle \geq 0$ for all $\beta \in K$ as well as $\langle \mu', \gamma^\vee \rangle \geq \langle \lambda, \gamma^\vee \rangle \geq 0$ for all $\gamma \in S \smallsetminus I$. So μ' is dominant, and μ is not maximal. ∎

2.2 Lemma 2.1 reduces the classification of the maximal dominant weights less than a given dominant weight to the following result. Here we use the numbering of the simple roots from [1, Planches I–IX] and write $\varpi_i = \varpi_{\alpha_i}$. Our convention below is that all roots are short when they all have the same length.

Proposition *Let $\lambda, \mu \in X_+$ such that $\mu = \lambda - \sum_{\alpha\in S} m_\alpha \alpha$ with $m_\alpha \in \mathbf{Z}$, $m_\alpha > 0$ for all $\alpha \in S$. Then μ is maximal among the dominant weights less than λ if and only if the pair (λ, μ) occurs in the following list:*

(I) *The root system has type A_1; we have $\langle \lambda, \alpha_1^\vee \rangle \geq 2$ and $\mu = \lambda - \alpha_1$.*
(II) *λ is the unique short root that is dominant and $\mu = 0$.*
(III) *The root system has type B_n, $n \geq 2$; we have $\lambda = \varpi_1 + \varpi_n$ and $\mu = \varpi_n$.*
(IV) *The root system has type G_2; we have $\lambda = \varpi_2$ and $\mu = \varpi_1$.*
(V) *The root system has type G_2; we have $\lambda = \varpi_1 + \varpi_2$ and $\mu = 2\varpi_1$.*

Proof If the root system has type A_1, then the claim is obvious. Let us exclude this case from now on.

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In Case (II), 0 is the only dominant weight less than λ, hence maximal, and of course the only maximal one.

In Cases (III)–(V), we have $\mu = \lambda - \sum_{\alpha\in S}\alpha$. In order to check maximality, one has to show for any non-empty proper subset I of S that $\lambda - \sum_{\alpha\in I}\alpha$ is not dominant. This is easily done in the G_2-cases. In the B_n-case we would otherwise get a dominant weight $\nu_I < \lambda_I$, which is impossible as λ_I is either minuscule or equal to 0. The equality $\lambda - \mu = \sum_{\alpha\in S}\alpha$ in Cases (III)–(V) implies also that μ is the only maximal dominant weight $\mu' < \lambda$ such that $\lambda - \mu'$ has support in all simple roots.

We now have to prove that our list is complete. So let $\lambda, \mu \in X_+$ such that $\mu = \lambda - \sum_{\alpha\in S} m_\alpha \alpha$ with $m_\alpha \in \mathbf{Z}$, $m_\alpha > 0$ for all $\alpha \in S$ and such that μ is maximal among the dominant weights less than λ. This implies that $\lambda - \sum_{\alpha\in I}\alpha$ is not dominant for any non-empty proper subset I of S. Write $\lambda = \sum_{\alpha\in S} n_\alpha \varpi_\alpha$. For each $\alpha \in S$ we know by assumption that $\lambda - \alpha$ is not dominant; this implies $n_\alpha \le 1$. We next want to show that the cases (II) for type A_n, (III), and (V) are the only ones where there is more than one $\alpha \in S$ with $n_\alpha = 1$.

Write $n_i = n_{\alpha_i}$. Let us first look at R of type A_n. If $n_i = 1$ for at least two distinct indices i, then we can find indices $1 \le i < j \le n$ such that $n_i = n_j = 1$ and $n_l = 0$ for all l with $i < l < j$. Then $\lambda - \sum_{m=i}^{j}\alpha_m$ is dominant. Our assumption now implies $i = 1$ and $j = n$, so $\lambda = \varpi_1 + \varpi_n$ is the dominant root; we are in Case (II).

We can apply the construction from the preceding paragraph for arbitrary R to any subset I of S such that R_I has type A. Since the subset is proper, we see that there is at most one $\alpha \in I$ with $n_\alpha = 1$.

In the case where all roots have the same length, any two simple roots belong to a subsystem R_I of type A. So for types D_n and E_n we are reduced to the case where λ is a fundamental weight.

If there are two root lengths in R, then we have to look at $\lambda = \varpi_\alpha + \varpi_\beta$ with α long and β short. For R of type B_n, $n \ge 2$, and for R of type C_n, $n \ge 3$, we have to deal with $\lambda = \varpi_i + \varpi_n$ with $i < n$. For R of type B_n and $i = 1$ we are in Case (III). For R of type B_n and $i > 1$ we observe that $\varpi_i - \sum_{j=i}^{n}\alpha_i = \varpi_{i-1}$. This rules out not only $\lambda = \varpi_i + \varpi_n$, but also $\lambda = \varpi_i$.

In the C_n case we have $\varpi_n - (\alpha_{n-1} + \alpha_n) = \varpi_{n-2}$. This rules out not only $\lambda = \varpi_i + \varpi_n$, but also $\lambda = \varpi_n$.

For R of type F_4 one has $\varpi_1 - (\alpha_1 + \alpha_2 + \alpha_3) = \varpi_4$ and $\varpi_2 - (\alpha_2 + \alpha_3) = \varpi_1 + \varpi_4$. This implies $n_1 = n_2 = 0$ and rules out not only $\lambda = \varpi_\alpha + \varpi_\beta$ as above, but also $\lambda \in \{\varpi_1, \varpi_2\}$.

Now the only candidates left for λ are fundamental weights. Of course, any minuscule ϖ_α cannot lead to examples, since there are no dominant weights less than ϖ_α. This takes care of type A_n. In type B_n this excludes ϖ_n. We have already taken care of the ϖ_i with $1 < i < n$. And ϖ_1 leads to Case (II).

Let us look at types C_n and D_n. Exclude the minuscule fundamental weights, *i.e.*, ϖ_1 for C_n and ϖ_1, ϖ_{n-1}, ϖ_n for D_n. In the remaining cases the dominant weights less than ϖ_n are all ϖ_{i-2j} with $0 < j \le i/2$ where we set $\varpi_0 = 0$. (This can be seen by realising $V(\varpi_n)$ as a submodule of the i-th exterior power of the natural representation of $\mathfrak{g}$.) The only maximal dominant weight is ϖ_{i-2}, and $\varpi_i - \varpi_{i-2}$ is a linear combination of the α_h with $h > i - 2$. So we have total support only for $\lambda = \varpi_2$, which is Case (II) for our root system.

In type F_4 we have already excluded ϖ_1 and ϖ_2. For $\lambda = \varpi_4$ we are in Case (II). The only dominant weights less than ϖ_3 are $0 < \varpi_4 < \varpi_1$ (*cf.* the tables in [10] or [2, Ch. VIII, §9, Exerc. 16]). As $\varpi_3 - \varpi_1 = \alpha_1 + 2\alpha_2 + \alpha_3$, we can rule out $\lambda = \varpi_3$.

We are left with the fundamental weights for R of type E_6, E_7, E_8. Here one checks the claim by inspection. There is in [3] a list of all weights for any $V(\varpi_i)$, actually for all root systems. For $\mathfrak{g}$ of type E_8 there is one correction to [3] in [4]. One checks that the only contribution to our list in type E_8 comes from ϖ_8, which is the dominant root. By using Lemma 2.1 one can get the result for E_6 and E_7 from the calculations in type E_8. This concludes the proof of the proposition. ■

2.3 We use the notation $[M:E]$ to denote the multiplicity of a simple module E as a composition factor of a module M. (It will be clear from the context, what type of modules we consider.)

Note: If $\mu \in X_+$ is maximal among the dominant weights less than some $\lambda \in X_+$, then $L(\lambda)_k$ and $L(\mu)_k$ are the only possible composition factors of $V(\lambda)_k$ that can have weight μ. This implies

$$[V(\lambda)_k : L(\mu)_k] = \dim V(\lambda)_{k,\mu} - \dim L(\lambda)_{k,\mu} = \dim V(\lambda)_\mu - \dim L(\lambda)_{k,\mu}. \tag{2.1}$$

2.4 Proof of Theorem 1.4

Consider λ and μ as in the theorem. We want to use induction on the rank of $\mathfrak{g}$. Write $\lambda - \mu = \sum_{\alpha \in S} m_\alpha \alpha$ and set $I = \{\alpha \in S \mid m_\alpha > 0\}$. If $I \neq S$ then we consider the analogue $G_{I,k}$ to G_k for the root system R_I. Denote by $V_I(\lambda_I)_k$ and $L_I(\mu_I)_k$ the analogues to $V(\lambda)_k$ and $L(\lambda)_k$ for $G_{I,k}$. One then has

$$[V(\lambda)_k : L(\mu)_k] = [V_I(\lambda_I)_k : L_I(\mu_I)_k];$$

see [9, II.5.21(2)]. Lemma 2.1 implies that we can apply induction on the right-hand side, since R_I cannot have type E_8; we thus get some k with $[V_I(\lambda_I)_k : L_I(\mu_I)_k] > 0$, hence also with $[V(\lambda)_k : L(\mu)_k] > 0$.

So we can assume that $I = S$, which means that we are in one of Cases (I)–(V) in Proposition 2.2. In most of these cases one can find a field k with $[V(\lambda)_k : L(\mu)_k] > 0$ in [5]. For example, Case (I) is treated there in the final section "Further Directions", Case (II) appears at the end of Section 1 under the heading "Quasi-minuscule representations", and Case (III) is the topic of [5, Section 5]. Case (IV) is mentioned at the end of [5, Section 4]. In Case (V), one can apply the results on G_2 in [8]: If we take k with $\operatorname{Char} k = 7$, then λ belongs to the interior of the "third" dominant alcove and μ is its mirror image in the "second" dominant alcove, which implies $[V(\lambda)_k : L(\mu)_k] = 1$. ■

3 Alternative Proof and Multiplicities

In this section we give an elementary proof for Theorem 1.4 that does not rely on the classification in Subsection 2.2. It generalises the method used in [7, pp. 19–20], for λ a dominant short root in the case of two root lengths. To start with, we recall the classical construction of the Weyl modules $V(\lambda)_k$. A reference for the following

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subsections is [9, Chapter II.8], in particular II.8.3 and II.8.17. At the end we show how our method yields a unified approach to the computation of $[V(\lambda)_k : L(\mu)_k]$ in the cases from Proposition 2.2.

3.1 For any root α let $H_\alpha \in \mathfrak{h}$ be the element with $\lambda(H_\alpha) = \langle \lambda, \alpha^\vee \rangle$ for all $\lambda \in \mathfrak{h}^*$. We choose a Chevalley system $(X_\alpha \mid \alpha \in R)$ of root vectors satisfying the classical assumption that $[X_\alpha, X_{-\alpha}] = H_\alpha$ for all α. (A different sign convention is used in [2].)

Denote by $U_{\mathbf{Z}}$ the $\mathbf{Z}$-subalgebra of the enveloping algebra of $\mathfrak{g}$ generated by all $X_\alpha^r/r!$ with $\alpha \in R$ and $r \in \mathbf{N}$, and by $U_{\mathbf{Z}}^-$ the $\mathbf{Z}$-subalgebra generated by all $X_{-\alpha}^r/r!$ with $\alpha \in R^+$ and $r \in \mathbf{N}$. One now constructs for each $\lambda \in X_+$ a $\mathbf{Z}$-lattice $V(\lambda)_{\mathbf{Z}}$ in $V(\lambda)$ by choosing a highest weight vector v_λ in $V(\lambda)$ and by setting

$$V(\lambda)_{\mathbf{Z}} = U_{\mathbf{Z}} v_\lambda = U_{\mathbf{Z}}^- v_\lambda.$$

This is a free $\mathbf{Z}$-module of finite rank; any $\mathbf{Z}$-basis for $V(\lambda)_{\mathbf{Z}}$ is also a $\mathbf{C}$-basis for $V(\lambda)$. Furthermore $V(\lambda)_{\mathbf{Z}}$ is the direct sum of its weight spaces $V(\lambda)_{\mathbf{Z},\mu} = V(\lambda)_{\mathbf{Z}} \cap V(\lambda)_\mu$.

Denote by $(\cdot\,,\cdot)$ the contravariant form on $V(\lambda)$ normalised such that $(v_\lambda, v_\lambda) = 1$. This is a symmetric bilinear form on $V(\lambda)$ satisfying

$$(X_\alpha v, v') = (v, X_{-\alpha} v') \qquad \text{for all } \alpha \in R \text{ and all } v, v' \in V(\lambda).$$

This form takes integer values on $V(\lambda)_{\mathbf{Z}} \times V(\lambda)_{\mathbf{Z}}$; distinct weight spaces are orthogonal with respect to this form. It is positive definite on the real span of $V(\lambda)_{\mathbf{Z}}$ (because it is invariant under a compact real form of $\mathfrak{g}$).

3.2 Now the Weyl module with highest weight λ for G_k can be constructed as $V(\lambda)_k = V(\lambda)_{\mathbf{Z}} \otimes_{\mathbf{Z}} k$. The ($\mathbf{Z}$-valued) contravariant form on $V(\lambda)_{\mathbf{Z}}$ yields a k-bilinear form $(\cdot\,,\cdot)_k$ on $V(\lambda)_k$. Then the unique maximal submodule rad $V(\lambda)_k$ of $V(\lambda)_k$ is equal to the radical of the form $(\cdot\,,\cdot)_k$. In particular, $V(\lambda)_k$ is simple if and only if $(\cdot\,,\cdot)_k$ is non-degenerate on $V(\lambda)_k$.

By the orthogonality of distinct weight spaces we have for any weight μ of $V(\lambda)$: the μ-weight space of the simple G_k-module with highest weight λ is the quotient of $V(\lambda)_{\mathbf{Z},\mu} \otimes k$ by the radical of the form $(\cdot\,,\cdot)_k$ restricted to this space. Therefore, the dimension of $L(\lambda)_{k,\mu}$ is equal to the rank after reduction modulo Char k of the Gram matrix for a $\mathbf{Z}$-basis of $V(\lambda)_{\mathbf{Z},\mu}$. So we have $\dim L(\lambda)_{k,\mu} < \dim V(\lambda)_\mu$ if and only if Char k divides the determinant of this Gram matrix.

If $\mu \in X_+$ is maximal among the dominant weights less than λ, then (2.1) implies that $[V(\lambda)_k : L(\mu)_k] > 0$ if and only if Char k divides the determinant of the Gram matrix for a $\mathbf{Z}$-basis of $V(\lambda)_{\mathbf{Z},\mu}$.

3.3 For any weight $\nu \in W\lambda$ the weight space $V(\lambda)_\nu$ has dimension 1, so we can choose a basis vector v_ν (unique up to sign) such that $V(\lambda)_{\mathbf{Z},\nu} = \mathbf{Z} v_\nu$. We then have $(v_\nu, v_\nu) = 1$ for all $\nu \in W\lambda$. This holds, *e.g.*, since also $L(\lambda)_{k,\nu}$ has dimension 1, so $(\cdot\,,\cdot)_k$ is non-degenerate on $V(\lambda)_{k,\nu}$; this implies that the image of (v_ν, v_ν) in k is non-zero. As this holds for all k ,this means that no prime number divides (v_ν, v_ν); now the claim follows from the positivity of the form.

3.4 Let $\nu \in W\lambda$ and $\alpha \in S$ with $r = \langle \nu, \alpha^\vee \rangle > 0$. We then have $\nu - r\alpha = s_\alpha(\nu) \in W\lambda$. Any $\nu + m\alpha$ with $m > 0$ is not a weight of $V(\lambda)$. It follows that $X_\alpha^m v_\nu = 0$ for any $m > 0$, and we get (by using a standard commutator formula, see [6, Lemma 26.2])

$$\Big(\frac{X_{-\alpha}^r}{r!} v_\nu, \frac{X_{-\alpha}^r}{r!} v_\nu \Big) = \Big(v_\nu, \frac{X_\alpha^r}{r!} \frac{X_{-\alpha}^r}{r!} v_\nu \Big) = \Big(v_\nu, \binom{H_\alpha}{r} v_\nu \Big) = \binom{\langle \nu, \alpha^\vee \rangle}{r} = 1.$$

Since $(X_{-\alpha}^r/r!)\, v_\nu$ belongs to $V(\lambda)_{\mathbf{Z},\nu - r\alpha} = \mathbf{Z} v_{\nu - r\alpha}$, this yields the first claim in

(3.1) $$\frac{X_{-\alpha}^{\langle \nu,\alpha^\vee\rangle}}{\langle \nu, \alpha^\vee \rangle!} v_\nu = \pm v_{\nu - \langle \nu, \alpha^\vee \rangle \alpha} \qquad \text{and} \qquad \frac{X_{\alpha}^{\langle \nu,\alpha^\vee\rangle}}{\langle \nu, \alpha^\vee \rangle!} v_{\nu - \langle \nu, \alpha^\vee \rangle \alpha} = \pm v_\nu.$$

The second one follows symmetrically; one can also use that $(X_\alpha^r/r!)\,(X_{-\alpha}^r/r!)\, v_\nu = v_\nu$.

3.5 The $\mathbf{Z}$-algebra $U_{\mathbf{Z}}^-$ is generated already by all $X_{-\alpha}^r/r!$ with $\alpha \in S$ and $r \in \mathbf{N}$. (This was observed by Verma, *cf.* [7, Satz I.7].) It follows that

$$V(\lambda)_{\mathbf{Z},\mu} = \sum_{\alpha \in S} \sum_{r > 0} \frac{X_{-\alpha}^r}{r!} V(\lambda)_{\mathbf{Z},\mu + r\alpha}$$

for any weight $\mu < \lambda$.

3.6 We now return to the situation where μ is maximal among the dominant weights less than λ.

Lemma *Let $\nu \in X$ with $\mu < \nu$. Then ν is a weight of $V(\lambda)$ if and only if $\nu \in W\lambda$.*

Proof If ν is a weight of $V(\lambda)$, then so is the unique dominant weight $\nu^+ \in W\nu$. Now $\mu < \nu \le \nu^+ \le \lambda$ and the maximality of μ imply $\lambda = \nu^+$, hence $\nu \in W\lambda$. The other direction is obvious. ■

3.7 Set

$$S_0 = \{ \alpha \in S \mid \mu + \alpha \in W\lambda \}.$$

Set $z_\alpha = X_{-\alpha} v_{\mu+\alpha} \in V(\lambda)_{\mathbf{Z},\mu}$ for all $\alpha \in S_0$. Note that we can replace $v_{\mu+\alpha}$ by $-v_{\mu+\alpha}$ if we so wish; hence we can replace z_α by $-z_\alpha$.

Lemma *The $\mathbf{Z}$-module $V(\lambda)_{\mathbf{Z},\mu}$ is spanned by all z_α with $\alpha \in S_0$. We have $(z_\alpha, z_\alpha) = \langle \mu, \alpha^\vee \rangle + 2$ for all $\alpha \in S_0$.*

Proof Let $\alpha \in S$; suppose that $\mu + r\alpha$ is a weight of $V(\lambda)$ for some $r > 0$. The α-string of all weights of $V(\lambda)$ of the form $\mu + s\alpha$ with $s \in \mathbf{Z}$ contains μ and does not admit any holes. So $\mu + \alpha$ has to be a weight of $V(\lambda)$, and thus $\alpha \in S_0$ by Lemma 3.6.

Let $\alpha \in S_0$. Then $\mu + \alpha \in W\lambda$ is an extremal weight of $V(\lambda)$, hence at the top or the bottom of its α-string. Since $\mu = (\mu + \alpha) - \alpha$ is a weight, $\mu + \alpha$ has to be at the top; so no $\mu + r\alpha$ with $r > 1$ is a weight of $V(\lambda)$. Now Subsection 3.5 implies

$$V(\lambda)_{\mathbf{Z},\mu} = \sum_{\alpha \in S_0} X_{-\alpha} V(\lambda)_{\mu+\alpha} = \sum_{\alpha \in S_0} \mathbf{Z} z_\alpha.$$

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Furthermore, $V(\lambda)_{\mu+2\alpha} = 0$ yields $X_\alpha v_{\mu+\alpha} = 0$ for all $\alpha \in S_0$, hence

$$(z_\alpha, z_\alpha) = (X_{-\alpha} v_{\mu+\alpha}, X_{-\alpha} v_{\mu+\alpha}) = (v_{\mu+\alpha}, X_\alpha X_{-\alpha} v_{\mu+\alpha}) = (v_{\mu+\alpha}, H_\alpha v_{\mu+\alpha}) = \langle \mu + \alpha, \alpha^\vee \rangle = \langle \mu, \alpha^\vee \rangle + 2. \qquad \blacksquare$$

3.8 We are going to prove the following proposition.

Proposition *All z_α with $\alpha \in S_0$ form a $\mathbf{Z}$-basis for $V(\lambda)_{\mathbf{Z},\mu}$. The determinant of the Gram matrix of all (z_α, z_β) is a positive integer; it is equal to 1 only if R has type E_8 with $S = S_0$ and $\mu = 0$.*

Note that this proposition implies Theorem 1.4: One uses the facts from Subsection 3.2 and notes that $\mu = 0$ and $S = S_0$ imply $\lambda \in W\alpha$ for all $\alpha \in S$, hence that λ is the (unique) dominant root.

3.9 Let $(\cdot \mid \cdot)$ be a positive definite bilinear form on $\sum_{\alpha\in R} \mathbf{R}\,\alpha = \sum_{\nu\in X} \mathbf{R}\,\nu$ that is invariant under the Weyl group W. We then have $\langle \nu, \alpha^\vee \rangle = 2\,(\nu \mid \alpha)/(\alpha \mid \alpha)$ for all $\nu \in X$ and $\alpha \in R$.

Lemma *Let $\alpha, \beta \in S_0$ with $\alpha \neq \beta$. Then we have $\mu + \alpha + \beta \in W\lambda$ if and only if $(\alpha \mid \beta) < 0$ and $\langle \mu + \alpha, \beta^\vee \rangle = -1 = \langle \mu + \beta, \alpha^\vee \rangle$.*

Proof If $\langle \mu+\alpha, \beta^\vee \rangle = -1$, then $\mu+\alpha+\beta = s_\beta(\mu+\alpha) \in W\lambda$. This yields one direction.

Suppose on the other hand that $\mu+\alpha+\beta \in W\lambda$. The invariance of $(\cdot \mid \cdot)$ under W implies

$$(\lambda \mid \lambda) = (\mu + \alpha \mid \mu + \alpha) = (\mu + \beta \mid \mu + \beta) = (\mu + \alpha + \beta \mid \mu + \alpha + \beta).$$

Now

$$(\mu + \alpha + \beta \mid \mu + \alpha + \beta) = (\mu + \alpha \mid \mu + \alpha) + (\beta \mid \beta) + 2\,(\mu + \alpha \mid \beta)$$

yields $(\beta \mid \beta) + 2\,(\mu+\alpha \mid \beta) = 0$, hence $1 + \langle \mu+\alpha, \beta^\vee \rangle = 0$. We thus get $\langle \mu+\alpha, \beta^\vee \rangle = -1$ and by symmetry also $\langle \mu + \beta, \alpha^\vee \rangle = -1$. Furthermore, $\mu \in X_+$ implies $\langle \mu, \beta^\vee \rangle \geq 0$, hence $\langle \alpha, \beta^\vee \rangle < 0$. $\blacksquare$

3.10

Lemma (i) *Let $\alpha, \beta \in S_0$ with $\alpha \neq \beta$. If $\mu + \alpha + \beta \notin W\lambda$, then $(z_\alpha, z_\beta) = 0$. If $\mu + \alpha + \beta \in W\lambda$, then $(z_\alpha, z_\beta) = \pm 1$.*

(ii) *We can choose the elements z_α $(\alpha \in S_0)$ such that $(z_\alpha, z_\beta) = -1$ for all $\alpha, \beta \in S_0$ with $\alpha \neq \beta$ and $\mu + \alpha + \beta \in W\lambda$.*

Proof (i) We have

$$(3.2)\quad (z_\alpha, z_\beta) = (X_{-\alpha} v_{\mu+\alpha}, X_{-\beta} v_{\mu+\beta}) = (v_{\mu+\alpha}, X_\alpha X_{-\beta} v_{\mu+\beta}) = (v_{\mu+\alpha}, X_{-\beta} X_\alpha v_{\mu+\beta}).$$

If $\mu + \alpha + \beta \notin W\lambda$, then $\mu + \alpha + \beta$ is not a weight of $V(\lambda)$, hence $X_\alpha v_{\mu+\beta} = 0$ and $(z_\alpha, z_\beta) = 0$.

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If $\mu + \alpha + \beta \in W\lambda$, then Lemma 3.9 implies $\langle \mu + \alpha + \beta, \alpha^\vee \rangle = 1 = \langle \mu + \alpha + \beta, \beta^\vee \rangle$. Now (3.1) yields $X_\alpha v_{\mu+\beta} = \pm v_{\mu+\alpha+\beta}$ and $X_{-\beta} v_{\mu+\alpha+\beta} = \pm v_{\mu+\alpha}$. Plugging this into (3.2) shows $(z_\alpha, z_\beta) = \pm(v_{\mu+\alpha}, v_{\mu+\alpha}) = \pm 1$.

(ii) Recall from Subsection 3.7 that we can replace any z_α by $-z_\alpha$ if we so wish. Since the Coxeter graph of S is a tree, we can choose a numbering $\alpha_1, \alpha_2, \dots, \alpha_n$ of S such that for each i there is at most one $j < i$ with $(\alpha_j \mid \alpha_i) < 0$. We now inductively modify all z_{α_i} with $\alpha_i \in S_0$ as follows: If there is no $j < i$ with $\alpha_j \in S_0$ and $\mu + \alpha_i + \alpha_j \in W\lambda$, then we do not change z_{α_i}. If there is a $j < i$ with $\alpha_j \in S_0$ and $\mu + \alpha_i + \alpha_j \in W\lambda$, then $(\alpha_j \mid \alpha_i) < 0$ by Lemma 3.9, so j is unique by our choice of numbering. We have $(z_{\alpha_j}, z_{\alpha_i}) = \pm 1$ by (i), and now replace z_{α_i} by $\mp z_{\alpha_i}$ so to get $(z_{\alpha_j}, z_{\alpha_i}) = -1$. (Note that these sign changes do not change the determinant of the Gram matrix in 3.8 or the basis property there.) ■

3.11 We shall need an auxiliary result. If $A = (a_{ij}) \in M_n(\mathbf{R})$ is a symmetric $(n \times n)$-matrix with real entries, then we denote by q_A the quadratic form on $\mathbf{R}^n$ given by $q_A(x_1, x_2, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$.

Lemma *Let $A, B \in M_n(\mathbf{R})$ be symmetric matrices, $A = (a_{ij})$ and $B = (b_{ij})$, such that $a_{ij} = b_{ij}$ whenever $i \neq j$ and $b_{ii} \geq a_{ii}$ for all i. If q_A is positive definite, then so is q_B, and we have* $\det B \geq \det A$ *with equality only for $B = A$.*

Proof Assume that q_A is positive definite. We have for all $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$,

$$q_B(x) - q_A(x) = \sum_{i=1}^n (b_{ii} - a_{ii})\, x_i^2 \geq 0.$$

So also q_B is positive definite.

In order to prove the claim on the determinants we can assume $n > 1$; we want to use induction on the number of i, $1 \leq i \leq n$, with $b_{ii} > a_{ii}$. Suppose that $B \neq A$ and fix an index i with $b_{ii} > a_{ii}$. For any $t \in \mathbf{R}$ denote by $A(t) \in M_n(\mathbf{R})$ the symmetric matrix with (i,i)-entry equal to t and all other entries equal to the corresponding entry in A. Note that $A(a_{ii}) = A$.

There exists a constant c such that $\det A(t) = A_{ii} t + c$ where A_{ii} is the (i,i)-minor of A. We have $A_{ii} > 0$, since q_A is positive definite. Therefore, $\det A(t)$ is an increasing function of t; we get $\det A(b_{ii}) > \det A(a_{ii}) = \det A$.

The first part of the proof shows that $q_{A(t)}$ is positive definite for $t \geq a_{ii}$. Therefore, the pair $(A(b_{ii}), B)$ satisfies the same assumptions as (A, B). Now induction yields $\det B \geq \det A(b_{ii}) > \det A$. ■

3.12 Proof of Proposition 3.8

We want to apply Lemma 3.11 with B equal to the matrix of all (z_α, z_β) where we work with an arbitrary numbering of S_0 and where we assume that Lemma 3.10(ii) holds. We set A equal to the matrix we get from B by replacing all diagonal entries by 2. Since μ is dominant, we have $\langle \mu, \alpha^\vee \rangle \geq 0$ for all $\alpha \in S_0$, hence $(z_\alpha, z_\alpha) = \langle \mu, \alpha^\vee \rangle + 2 \geq 2$. So the general assumptions of Lemma 3.11 are satisfied.

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We can regard A as the Cartan matrix associated with a Dynkin diagram with vertices S_0 where two vertices $\alpha \neq \beta$ are joined by an edge if and only if $(z_\alpha, z_\beta) = -1$, and in that case they are joined be a single edge. We know by Lemmas 3.9 and 3.10 that $(z_\alpha, z_\beta) = -1$ implies $(\alpha \mid \beta) < 0$. So we get the new Dynkin diagram from the one of R by removing the vertices not in S and by removing some edges; in particular, a double or triple edge is either removed or replaced by a single edge. It follows that each connected component of the new Dynkin diagram is of type A, D, or E. Furthermore, we can get a component of type E_8 only if R has type E_8 and $S = S_0$.

Now $\det A$ is the product of the determinants of the Cartan matrices of the connected components of the new diagram. Each factor is a positive integer; it is equal to 1 only if the component has type E_8. It follows that $\det B \geq \det A$ is a positive integer. We get $\det B = 1$ only if $B = A$ and if all components of the new diagram have type E_8. The latter condition means that R has type E_8 and $S = S_0$, while $B = A$ implies $\langle \mu, \alpha^\vee \rangle = 0$ for all $\alpha \in S_0$, hence (combined with $S = S_0$) that $\mu = 0$.

Note finally that since $\det B > 0$, the Gram matrix of all (z_α, z_β), is non-zero, the z_α have to be linearly independent; hence by Lemma 3.7 they form a $\mathbf{Z}$-basis for $V(\lambda)_{\mathbf{Z},\mu}$.

3.13 The preceding subsection completes the proof of Theorem 1.4 that relies only on the classification of Dynkin diagrams. The method used here turns out also to yield a uniform approach to the multiplicities in Cases (I)–(V) from Proposition 2.2.

Note first that

$$S_0 = \begin{cases} S & \text{in Cases (I), (III), and (V),} \\ \{\text{ all short simple roots }\} & \text{in Cases (II) and (IV).} \end{cases}$$

One checks this by inspection.

Assuming that we are in one of these cases, we can improve on Lemma 3.9.

Lemma *Let $\alpha, \beta \in S_0$ with $(\alpha \mid \beta) < 0$. Then $\mu + \alpha + \beta \in W\lambda$.*

Proof We have to do this by inspection in our five cases. We can switch α and β and thus assume that $\langle \alpha, \beta^\vee \rangle = -1$. According to Lemma 3.9 we have to check that $\langle \mu, \beta^\vee \rangle = 0$. This is obvious in Case (II). In Cases (III) and (V), β will be a long root and the long roots indeed satisfy $\langle \mu, \beta^\vee \rangle = 0$. We have $|S_0| = 1$ in Cases (I) and (IV), so they do not arise here. ■

Remark It follows that we can refine Lemma 3.10 and get for $\alpha, \beta \in S_0$ with $(\alpha \mid \beta) < 0$,

$$(z_\alpha, z_\beta) = \begin{cases} -1 & \text{if } \langle \beta, \alpha^\vee \rangle < 0, \\ 0 & \text{if } \langle \beta, \alpha^\vee \rangle = 0, \end{cases}$$

having made the same normalisation as in Lemma 3.10(ii).

3.14

Proposition *In the cases from Proposition 2.2 the determinant of the Gram matrix of all (z_α, z_β) is given as follows:*

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(I) $\langle\lambda, \alpha_1^\vee\rangle$;
(II) *for R of type* $A_n/B_n/C_n/D_n/E_6/E_7/E_8/F_4/G_2$*, it is* $n+1/2/n/4/3/2/1/3/2$;
(III) $2n+1$;
(IV) 3;
(V) 7.

Proof In Cases (I) and (IV), we have $|S_0| = 1$, say $S_0 = \{\alpha\}$, so the determinant is equal to $\langle\mu, \alpha^\vee\rangle + 2$ which yields the claim in these cases.

In Case (II), if R has type A_n, D_n, or E_n, then our Gram matrix is the Cartan matrix of R, so the determinant is equal to the (known) index of connection of R. If R has type B_n, C_n, F_4, or G_2, then the Gram matrix is equal to the Cartan matrix of a root system of type A_m with $m = |S_0|$, hence has determinant $m+1$. And the value of m is 1, $n-1$, 2, 1 in these cases.

In Case (III), one gets the Gram matrix from the Cartan matrix for the root system A_n by replacing the last diagonal 2 by 3. Expanding the determinant after the last row, one gets $3n - (n-1) = 2n+1$.

In Case (V), the matrix is

$$\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}. \qquad \blacksquare$$

3.15

Proposition *Let* (λ, μ) *be one of the pairs from Proposition 2.2. We have* $[V(\lambda)_k : L(\mu)_k] > 0$ *if and only if* Char k *divides the determinant corresponding to* (λ, μ)*; if so, then we have* $[V(\lambda)_k : L(\mu)_k] = 1$ *except in Case* (II) *for R of type* D_n *with n even, where* $[V(\varpi_2)_k : L(0)_k] = 2$ *when* $\mathrm{Char}(k) = 2$.

Proof The first claim follows from the general discussion in Subsection 3.2. Set $m = |S_0| = \dim V(\lambda)_\mu$. Suppose that $p = \mathrm{Char}\, k$ divides the determinant. By (2.1) we have to show that $\dim L(\lambda)_{k,\mu} = m-1$, hence that the rank of the Gram matrix reduced modulo p is equal to $m-1$. This is obvious when $m = 1$, so suppose now that $m > 1$.

If we exclude for the moment types D_n and E_n, then we can find a numbering $\alpha_1, \dots, \alpha_m$ of S_0 such that $(\alpha_i \mid \alpha_j) < 0$ if and only if $|j-i| = 1$. Then for each $i < m$ the i-th row has $(i+1)$-st entry -1, and all entries to the right are 0. This shows that the first $m-1$ rows in the Gram matrix are linearly independent modulo any prime, hence that the rank is at least $m-1$.

In the D_n-case one checks that the rank of the Cartan matrix modulo 2 is equal to the rank of the embedded Cartan matrix of type A_{n-1}. In type E_6 one notes that the determinant of the Cartan matrix is coprime with the determinant of the embedded Cartan matrix of type D_5. And in type E_7 one uses the embedded Cartan matrix of type E_6. $\blacksquare$

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