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SCALAR BOUNDEDNESS OF VECTOR-VALUED FUNCTIONS

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Abstract. We provide sufficient conditions for a Banach space-valued function to be scalarly bounded, which do not require to test on the whole dual space. Some applications in vector integration are also given.

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1. Introduction. Throughout this paper X is a Banach space, X^* stands for its topological dual and (Ω, Σ, μ) is a complete probability space. Recall that a function $f : \Omega \to X$ is said to be *scalarly bounded* if there is $M > 0$ such that, for each $x^* \in X^*$, we have

$$|x^*f| \leq M\|x^*\| \quad \mu\text{-a.e.}$$

(the exceptional μ-null set depending on x^*), where $x^*f : \Omega \to \mathbb{R}$ is the composition of f with x^*. This notion plays an important role in vector integration, specially within the Pettis integral theory; see **[11, 12, 17]**. Several questions can be reduced to the scalarly bounded case, since for every scalarly measurable function $f : \Omega \to X$ (meaning that x^*f is measurable for all $x^* \in X^*$) there is a countable partition $\Omega = \bigcup E_n$ into measurable sets such that each restriction $f|_{E_n}$ is scalarly bounded (see e.g. Proposition 3.1 in [**11**]). To check whether a function $f : \Omega \to X$ is scalarly bounded, it can be helpful to have criteria involving only the family

$$Z_{f,A} := \{x^*f : x^* \in A\}$$

for some set $A \subset X^*$. For instance, if X is the dual of another Banach space Y, one may consider $A = Y \subset Y^{**} = X^*$. The aim of this paper is to provide such criteria as well as some applications in vector integration.

It turns out that the scalar boundedness of f is equivalent to the fact that Z_{f,X^*} is made up of essentially bounded functions (Theorem 1). In order to get a similar statement with Z_{f,X^*} replaced by a subfamily $Z_{f,A}$, the set $A \subset X^*$ must be w^*-thick

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(Proposition 5) and further assumptions on X are needed, like Corson's property C (Theorem 7). Furthermore, under some measurability requirements on f we obtain some applications in vector integration: We prove that f is Pettis integrable whenever it is scalarly measurable, $Z_{f,A} \subset \mathcal{L}^\infty(\mu)$ for some w^*-thick set $A \subset X^*$ and X has the μ-Pettis Integral Property (Theorem 9). Sufficient conditions for the Bochner and Birkhoff integrability with the same flavour are also given (Proposition 13 and Theorem 15).

We use standard terminology and notation, which can be found in [**3, 6, 17**]. All our linear spaces are real. The absolutely convex hull of a subset S of a linear space is denoted by aco(S). Given a set H, we write $\mathfrak{T}_p(H)$ to denote the topology on $\mathbb{R}^H$ of pointwise convergence on H. For a Banach space Y, we write B_Y (resp. S_Y) to denote its closed unit ball (resp. unit sphere) and the symbol rB_Y stands for the closed ball of radius r centered at 0. The evaluation of $y^* \in Y^*$ at $y \in Y$ is denoted either by $y^*(y)$ or $\langle y^*, y\rangle$.

2. Testing scalar boundedness. We denote by $\mathcal{B}(\mu)$ the linear space of all functions $h : \Omega \to \mathbb{R}$, for which there is $K > 0$ such that $|h| \leq K$ μ-a.e.

THEOREM 1. *A function $f : \Omega \to X$ is scalarly bounded if and only if $Z_{f,X^*} \subset \mathcal{B}(\mu)$.*

Proof. The 'only if' part is obvious. Conversely, assume that $Z_{f,X^*} \subset \mathcal{B}(\mu)$. The formula

$$\|h\|_\infty := \inf\{M > 0 : \ |h| \leq M \ \mu\text{-a.e.}\}$$

defines a semi-norm on $\mathcal{B}(\mu)$. It is standard to check that the quotient space $B(\mu)$ obtained from $\mathcal{B}(\mu)$ by identifying μ-a.e. equal functions is a Banach space.

Let $T : X^* \to B(\mu)$ be the linear mapping that sends each $x^* \in X^*$ to the equivalence class of x^*f. We claim that T has a closed graph. Indeed, let (x_n^*) be a sequence in X^* such that $\|x_n^*\| \to 0$ and $(T(x_n^*))$ converges to some $h \in B(\mu)$ in the norm topology of $B(\mu)$. Since $x_n^* \to 0$ in the w^*-topology, we have $x_n^*f \to 0$ pointwise and therefore $h = 0$. This shows that T has a closed graph. By the Closed Graph Theorem, T is continuous. Now, for each $x^* \in X^*$ we have

$$|x^*f| \leq \|T(x^*)\|_{B(\mu)} \leq \|T\|\|x^*\| \quad \mu\text{-a.e.}$$

Hence, f is scalarly bounded. □

In order to present an alternative proof of Theorem 1 we need the following lemma (which will also be used later).

LEMMA 2. *A function $f : \Omega \to X$ is scalarly bounded if and only if there is $M > 0$ such that the set $\{x^* \in X^* : |x^*f| \leq M \ \mu\text{-a.e.}\}$ contains a ball.*

Proof. The 'only if' part is obvious. Let us check the 'if' part. Let $x_0^* \in X^*$ and $\delta > 0$ be such that

$$x_0^* + \delta B_{X^*} \subset \{x^* \in X^* : |x^*f| \leq M \ \mu\text{-a.e.}\}. \tag{1}$$

Take any $x^* \in B_{X^*}$ and set $y^* := x_0^* + \delta x^*$, so that (1) yields $|y^*f| \leq M$ μ-a.e. Also, (1) implies that $|x_0^*f| \leq M$ μ-a.e. It follows that $|x^*f| \leq 2M/\delta$ μ-a.e. As $x^* \in B_{X^*}$ is arbitrary, f is scalarly bounded. □

Another proof of Theorem 1. Suppose $Z_{f,X^*} \subset \mathcal{B}(\mu)$. For each $n \in \mathbb{N}$, set

$$A_n := \{x^* \in B_{X^*} : |x^*f| \le n \ \mu\text{-a.e.}\}$$

and observe that A_n is norm-closed. Indeed, if (x_k^*) is a sequence in A_n, which converges in norm to some $x^* \in B_{X^*}$, then we have $x_k^* \to x^*$ in the w^*-topology. Hence, $x_k^*f \to x^*f$ pointwise and so $|x^*f| \le n$ μ-a.e.

Since $B_{X^*} = \bigcup_{n\in\mathbb{N}} A_n$, an appeal to Baire's Category Theorem ensures the existence of $n \in \mathbb{N}$, $x_0^* \in B_{X^*}$ and $\delta > 0$ such that the open ball

$$B := \{x^* \in X^* : \|x^* - x_0^*\| < \delta\}$$

satisfies $B \cap B_{X^*} \subset A_n$. Since $B \cap B_{X^*}$ is non-empty, the same holds for the intersection of B and the open unit ball, hence A_n contains a ball. The scalar boundedness of f follows from Lemma 2. □

We now study the scalar boundedness of a function $f : \Omega \to X$ via the family

$$Z_{f,A} = \{x^*f : x^* \in A\},$$

where $A \subset X^*$. The next example shows that, in general, the inclusion $Z_{f,A} \subset \mathcal{B}(\mu)$ does not imply that f is scalarly bounded even if A is assumed to be a boundary. Recall that a set $A \subset B_{X^*}$ is said to be a *boundary* if for each $x \in X$ there is some $x^* \in A$ such that $\|x\| = x^*(x)$.

EXAMPLE 3. For each $n \in \mathbb{N}$, let $e_n^* \in c_0^* = \ell^1$ be the n-th coordinate projection. Suppose μ is atomless and fix a sequence (E_n) of pairwise disjoint elements of Σ with $\mu(E_n) > 0$. Define

$$f : \Omega \to c_0, \quad f(t) := \big(n1_{E_n}(t)\big)_{n\in\mathbb{N}},$$

where 1_{E_n} stands for the characteristic function of E_n. Then $e_n^*f = n1_{E_n} \in \mathcal{L}^\infty(\mu)$ for every $n \in \mathbb{N}$, but f is not scalarly bounded.

It turns out that the previous example is a particular case of a general phenomenon, see Proposition 5. We first need to introduce some terminology. Recall that a set $C \subset X^*$ is said to be *w^*-non-norming* if

$$\inf_{x\in S_X} \sup_{x^*\in C} |x^*(x)| = 0,$$

which is equivalent to saying (via the Hahn–Banach theorem) that $\overline{\text{aco}}^{w^*}(C)$ does not contain any ball. A subset of X^* is called *w^*-thin* (resp. *w^*-thick*) if it can be written as a countable increasing union of w^*-non-norming sets (resp. if it is not w^*-thin). For instance, if $X = Y^*$ for another Banach space Y, then $Y \subset X^*$ is w^*-thick. On the other hand, if X does not contain subspaces isomorphic to c_0, then the following subsets of X^* are w^*-thick: any boundary [**7**], the set of norm-attaining functionals [**7**] and the set of w^*-exposed points of B_{X^*} whenever X is separable [**8**]. For more information on w^*-thin and w^*-thick sets in Banach spaces, we refer the reader to [**1, 13**] and the references therein.

REMARK 4. Let $f : \Omega \to X$ be a essentially separable-valued function. Then f is scalarly bounded (if and) only if it is essentially bounded.

Proof. Let $Y \subset X$ be a separable closed subspace such that $f(E) \subset Y$ for some $E \in \Sigma$ with $\mu(E) = 1$. Since B_{Y^*} is w^*-separable, there is a sequence (y_n^*) in B_{Y^*} such that $\|y\| = \sup_{n\in\mathbb{N}} |y_n^*(y)|$ for every $y \in Y$. Pick $x_n^* \in B_{X^*}$ extending y_n^* for each $n \in \mathbb{N}$. Then $\|f(t)\| = \sup_{n\in\mathbb{N}} |x_n^* f(t)|$ for every $t \in E$. From this equality it follows that f is essentially bounded whenever it is scalarly bounded. □

PROPOSITION 5. *Let $A \subset X^*$ be w^*-thin and suppose μ is atomless. Then there is a strongly measurable function $f : \Omega \to X$ such that $Z_{f,A} \subset \mathcal{L}^\infty(\mu)$ but f is not scalarly bounded.*

Proof. Let (C_j) be a sequence of pairwise disjoint measurable sets with $\mu(C_j) > 0$ for all $j \in \mathbb{N}$. Since A is w^*-thin, we can write $A = \bigcup_{j\in\mathbb{N}} A_j$, where each A_j is w^*-non-norming and $A_j \subset A_{j+1}$. For each $j \in \mathbb{N}$, choose $x_j \in X$ such that

$$\|x_j\| = j \quad \text{and} \quad \sup_{x^*\in A_j} |x^*(x_j)| \leq 1.$$

Define $f : \Omega \to X$ by $f(t) := x_j$ whenever $t \in C_j$, $j \in \mathbb{N}$ and $f(t) := 0$ if $t \notin \bigcup_{j\in\mathbb{N}} C_j$. Then f is strongly measurable and fails to be essentially bounded, hence it is not scalarly bounded (Remark 4). Take any $x^* \in A$. Then there is some $j_0 \in \mathbb{N}$ such that $x^* \in A_j$ for all $j \geq j_0$. Hence, $|x^*(x_j)| \leq 1$ for all $j \geq j_0$ and so $|x^*(f(t))| \leq 1$ for every $t \in \bigcup_{j\geq j_0} C_j$. Since $x^* f$ takes only finitely many values in $\Omega \setminus \bigcup_{j\geq j_0} C_j$, it follows that $x^* f \in \mathcal{L}^\infty(\mu)$. This shows that $Z_{f,A} \subset \mathcal{L}^\infty(\mu)$. □

Recall that a Banach space is said to have Corson's *property C* if every family of convex closed subsets with empty intersection contains a countable subfamily with empty intersection. All weakly Lindelöf (e.g. weakly compactly generated) Banach spaces enjoy property C. For detailed information on this property, we refer the reader to Chapter 12 in [**6**] and the references therein.

Within the wide class of Banach spaces with property C, testing on w^*-thick sets is enough to check scalar boundedness (see Theorem 7). We first need a lemma, which will be used several times in the sequel.

LEMMA 6. *Let $f : \Omega \to X$ be a function and, for each $n \in \mathbb{N}$, define*

$$C_n := \{x^* \in nB_{X^*} : |x^* f| \leq n \ \mu\text{-a.e.}\}.$$

Suppose there is a w^*-thick set $A \subset X^*$ such that $Z_{f,A} \subset \mathcal{B}(\mu)$ and the inclusion

$$\overline{\mathrm{aco}}^{w^*}(A \cap C_n) \subset C_n \tag{2}$$

holds for every $n \in \mathbb{N}$. Then f is scalarly bounded.

Proof. Since $C_n \subset C_{n+1}$ for every $n \in \mathbb{N}$ and $A = \bigcup_{n\in\mathbb{N}}(A \cap C_n)$, the w^*-thickness of A ensures the existence of some $n \in \mathbb{N}$ such that $\overline{\mathrm{aco}}^{w^*}(A \cap C_n)$ contains a ball, say B. By (2), we have $B \subset C_n$ and an appeal to Lemma 2 finishes the proof. □

THEOREM 7. *Suppose X has property C. Let $f : \Omega \to X$ be a function for which there is a w^*-thick set $A \subset X^*$ such that $Z_{f,A} \subset \mathcal{B}(\mu)$. Then f is scalarly bounded.*

Proof. By Lemma 6, in order to prove that f is scalarly bounded, it suffices to check that $\overline{\mathrm{aco}}^{w^*}(A \cap C_n) \subset C_n$ for every $n \in \mathbb{N}$, where

$$C_n := \{x^* \in nB_{X^*} : |x^* f| \leq n \ \mu\text{-a.e.}\}.$$

To this end, take any $x^* \in \overline{\mathrm{aco}}^{w^*}(A \cap C_n)$. Since $A \cap C_n$ is bounded and X has property C, there is a countable set $D \subset A \cap C_n$ such that $x^* \in \overline{\mathrm{aco}}^{w^*}(D)$, see for example Theorem 12.41 in [**6**]. Let $\mathrm{aco}_{\mathbb{Q}}(D) \subset X^*$ be the set made up of all finite linear combinations of the form $\sum_i \alpha_i y_i^*$, where $y_i^* \in D$, $\alpha_i \in \mathbb{Q}$ and $\sum_i |\alpha_i| \leq 1$. Since $\mathrm{aco}_{\mathbb{Q}}(D) \subset C_n$ and $\mathrm{aco}_{\mathbb{Q}}(D)$ is countable, we can find $S \in \Sigma$ with $\mu(S) = 1$ such that

$$|y^* f(t)| \leq n \quad \text{for every } t \in S \text{ and every } y^* \in \mathrm{aco}_{\mathbb{Q}}(D).$$

Observe that $x^* \in \overline{\mathrm{aco}}^{w^*}(D) = \overline{\mathrm{aco}_{\mathbb{Q}}}^{w^*}(D)$, hence $x^* f$ belongs to the $\mathfrak{T}_p(\Omega)$-closure of $Z_{f,\mathrm{aco}_{\mathbb{Q}}(D)}$ in $\mathbb{R}^\Omega$. It follows that $|x^* f(t)| \leq n$ for every $t \in S$, hence $x^* \in C_n$. This shows that $\overline{\mathrm{aco}}^{w^*}(A \cap C_n) \subset C_n$, as required. □

The previous result can fail for arbitrary Banach spaces, as we next show. We write λ to denote the Lebesgue measure on $[0, 1]$ and the symbol $\mathfrak{c}$ stands for the cardinality of the continuum.

Example 8. There is a function $f : [0, 1] \to \ell^1(\mathfrak{c})$ such that $Z_{f,c_0(\mathfrak{c})} \subset \mathcal{L}^\infty(\lambda)$, but f is not scalarly bounded.

Proof. We have $\ell^1(\mathfrak{c}) = c_0(\mathfrak{c})^*$ and the set $c_0(\mathfrak{c}) \subset \ell^1(\mathfrak{c})^* = \ell^\infty(\mathfrak{c})$ is w^*-thick. For each $\alpha < \mathfrak{c}$, let $e_\alpha \in \ell^1(\mathfrak{c})$ be defined by $e_\alpha(\beta) := \delta_{\alpha,\beta}$ (the Kronecker symbol). Let $\phi : [0, 1] \to \mathfrak{c}$ be any one-to-one function and let $h : [0, 1] \to \mathbb{R}$ be any function such that $h \notin \mathcal{B}(\lambda)$. Set

$$f : [0, 1] \to \ell^1(\mathfrak{c}), \quad f(t) := h(t) e_{\phi(t)}.$$

Let us check that f satisfies the required properties. Consider $\xi \in \ell^1(\mathfrak{c})^*$ defined by $\xi(x) := \sum_{\alpha<\mathfrak{c}} x(\alpha)$ for all $x \in \ell^1(\mathfrak{c})$. Since $\langle \xi, f(t) \rangle = h(t)$ for every $t \in [0, 1]$, the function f is not scalarly bounded. On the other hand, the family $Z_{f,c_0(\mathfrak{c})}$ is made up of functions vanishing λ-a.e. Indeed, fix $y \in c_0(\mathfrak{c})$ and consider the countable set $\mathrm{supp}(y) := \{\alpha < \mathfrak{c} : y(\alpha) \neq 0\}$. Then $\phi^{-1}(\mathrm{supp}(y))$ is also countable (because ϕ is one-to-one) and so λ-null. For each $t \in [0, 1] \setminus \phi^{-1}(\mathrm{supp}(y))$ we have

$$\langle y, f(t) \rangle = h(t) y(\phi(t)) = 0,$$

hence the composition yf vanishes λ-a.e., as claimed. □

3. Application to the Pettis and Bochner integrals. Previously we have shown that in several cases one can deduce the scalar boundedness of a function $f : \Omega \to X$ from the inclusion $Z_{f,A} \subset \mathcal{B}(\mu)$ for some w^*-thick set $A \subset X^*$. In this section we shall apply those results to study the Pettis and Bochner integrability of f via the family $Z_{f,A}$.

Recall that X is said to have the *μ-Pettis Integral Property* (shortly *μ-PIP*) if each scalarly bounded and scalarly measurable function from Ω to X is Pettis integrable. The space X has the *PIP* if it has the μ-PIP for every probability μ. It is known that every Banach space with property C also has the PIP ((Theorem 5-2-4) in [**17**]). For further examples and more information on the PIP, see [**5, 11, 12, 17**].

Theorem 9. *Suppose X has the μ-PIP. Let $f : \Omega \to X$ be a scalarly measurable function for which there is a w^*-thick set $A \subset X^*$ such that $Z_{f,A} \subset \mathcal{L}^\infty(\mu)$. Then f is scalarly bounded and Pettis integrable.*

Proof. We only have to prove that f is scalarly bounded. By Lemma 6, it suffices to check that $\overline{\mathrm{aco}}^{w^*}(A \cap C_n) \subset C_n$ for each $n \in \mathbb{N}$, where

$$C_n := \{x^* \in nB_{X^*} : |x^*f| \leq n \ \mu\text{-a.e.}\}.$$

Fix $n \in \mathbb{N}$ and take any $x^* \in \overline{\mathrm{aco}}^{w^*}(A \cap C_n)$. As we mentioned in the Introduction, the scalar measurability of f ensures that there is a countable partition $\Omega = \bigcup E_m$ into measurable sets such that each restriction $f|_{E_m}$ is scalarly bounded. Fix $m \in \mathbb{N}$ and write μ_{E_m} to denote the restriction of μ to the trace of Σ on E_m. Since X has the μ-PIP, the function $f|_{E_m}$ is Pettis integrable. Then, the mapping

$$I_m : nB_{X^*} \to L^1(\mu_{E_m}), \quad I_m(y^*) := y^*f|_{E_m}$$

is w^*-w-continuous, see for example Chapter 4 in **[17]**. Since $x^* \in \overline{\mathrm{aco}}^{w^*}(A \cap C_n)$, we have

$$I_m(x^*) \in I_m\big(\overline{\mathrm{aco}}^{w^*}(A \cap C_n)\big) \subset \overline{I_m\big(\mathrm{aco}(A \cap C_n)\big)}^{w} = \overline{I_m\big(\mathrm{aco}(A \cap C_n)\big)}^{\|\cdot\|_1}$$

because $I_m(\mathrm{aco}(A \cap C_n)) = \mathrm{aco}(I_m(A \cap C_n))$ is convex. Hence, there is a sequence (y_k^*) in $\mathrm{aco}(A \cap C_n)$ such that $\|y_k^*f|_{E_m} - x^*f|_{E_m}\|_1 \to 0$ and, by passing to a further subsequence, we can assume that $y_k^*f \to x^*f$ μ-a.e. on E_m. Bearing in mind that $\mathrm{aco}(A \cap C_n) \subset C_n$, it follows that $|x^*f| \leq n$ μ-a.e. on E_m. As $m \in \mathbb{N}$ is arbitrary, we conclude that $|x^*f| \leq n$ μ-a.e., so $x^* \in C_n$ and the proof is over. □

Let $\Gamma \subset X^*$ be a set separating the points of X and consider the (locally convex Hausdorff) topology $\sigma(X, \Gamma)$ on X of pointwise convergence on Γ. Then a function $f : \Omega \to X$ is Baire$(X, \sigma(X, \Gamma))$-measurable if (and only if) the family $Z_{f,\Gamma}$ is made up of measurable functions **[4]** (cf. Theorem 2-2-4 in **[17]**). Therefore, if the equality

$$\mathrm{Baire}(X, \sigma(X, \Gamma)) = \mathrm{Baire}(X, w)$$

holds, then the scalar measurability of a function $f : \Omega \to X$ is equivalent to the measurability of the elements of $Z_{f,\Gamma}$. This fact allows us to improve the criterion of Theorem 9 in some cases, as follows.

Corollary 10. *Suppose X^* is w^*-angelic. Let $f : \Omega \to X$ be a function for which there is a w^*-thick set $A \subset X^*$ such that $Z_{f,A} \subset \mathcal{L}^\infty(\mu)$. Then f is scalarly bounded and Pettis integrable.*

Proof. The w^*-thickness of A implies that A separates the points of X and so the angelicity of (X^*, w^*) yields $\mathrm{Baire}(X, \sigma(X, A)) = \mathrm{Baire}(X, w)$, see **[10]**. According to the comments preceding the corollary, we conclude that f is scalarly measurable. Since X has the PIP (see e.g. **[5]**), an appeal to Theorem 9 finishes the proof. □

Corollary 11. *Suppose $X = Y^*$ for another Banach space Y and suppose X has property C. Let $f : \Omega \to X$ be a function such that $Z_{f,Y} \subset \mathcal{L}^\infty(\mu)$. Then f is scalarly bounded and Pettis integrable.*

Proof. We have $\mathrm{Baire}(X, w^*) = \mathrm{Baire}(X, w)$ because X has property C, see Corollary 3.10 in **[16]**. Hence, f is scalarly measurable. The result now follows from Theorem 9 bearing in mind that $Y \subset X^*$ is w^*-thick and that every Banach space with property C has the PIP ((Theorem 5-2-4) in **[17]**). □

It is known that $\ell^1(\kappa)$ has the PIP if κ is not a real-valued measurable cardinal, while it fails property C whenever κ is uncountable, see [**5**]. In view of Example 8 above, in Corollary 11 the assumption on X cannot be weakened to 'X has the PIP', at least under the assumption that $\mathfrak{c}$ is not real-valued measurable.

The following auxiliary lemma is needed to prove the criterion of Bochner integrability isolated in Proposition 13 below.

LEMMA 12. *Let Y be a closed subspace of X, let $R : X^* \to Y^*$ be the restriction operator and let $A \subset X^*$ be a w^*-thick set. Then $R(A)$ is w^*-thick.*

Proof. Let (B_n) be an increasing sequence of sets such that $R(A) = \bigcup_{n\in\mathbb{N}} B_n$. Then $(A \cap R^{-1}(B_n))$ is an increasing sequence such that $A = \bigcup_{n\in\mathbb{N}} A \cap R^{-1}(B_n)$. Since A is w^*-thick, there is some $n \in \mathbb{N}$ such that $A_n := \overline{\text{aco}}^{w^*}(A \cap R^{-1}(B_n))$ contains a ball. By the Open Mapping Theorem, $R(A_n)$ contains a ball as well. Since R is linear and w^*-w^*-continuous, we also have

$$R(A_n) \subset \overline{\text{aco}}^{w^*}(R(A \cap R^{-1}(B_n))) \subset \overline{\text{aco}}^{w^*}(B_n).$$

Therefore $\overline{\text{aco}}^{w^*}(B_n)$ contains a ball. This shows that $R(A)$ is w^*-thick. □

PROPOSITION 13. *Let $f : \Omega \to X$ be an essentially separable-valued function for which there is a w^*-thick set $A \subset X^*$ such that $Z_{f,A} \subset \mathcal{L}^\infty(\mu)$. Then f is essentially bounded and Bochner integrable.*

Proof. Assume without loss of generality that $f(\Omega) \subset Y$ for some separable closed subspace $Y \subset X$. Let $R : X^* \to Y^*$ be the restriction operator so that $R(A) \subset Y^*$ is w^*-thick (Lemma 12). Since Y^* is w^*-angelic, we can apply Corollary 10 to f (as a Y-valued function) and the w^*-thick set $R(A)$ to conclude that f is scalarly bounded and scalarly measurable. By Remark 4, f is essentially bounded. On the other hand, f is strongly measurable by Pettis' Measurability Theorem (see e.g. Theorem 2 on p. 42 in [**3**]). It follows that f is Bochner integrable. □

4. Application to the Birkhoff integral. We finish the paper with some applications to the Birkhoff integral theory. Recall that a function $f : \Omega \to X$ is said to be *Birkhoff integrable*, with integral $x \in X$, if for every $\varepsilon > 0$ there is a countable partition $\Omega = \bigcup E_m$ into measurable sets such that for any choice of points $t_m \in E_m$, the series $\sum_m \mu(E_m)f(t_m)$ converges unconditionally in X and $\| \sum_m \mu(E_m)f(t_m) - x\| \leq \varepsilon$. This notion of integrability lies strictly between Bochner and Pettis integrability and has interesting features, see for example [**2, 9**]. For instance, it can be characterised via the Bourgain property of certain families $Z_{f,A}$ where $A \subset X^*$, see [**2, 15**]. Following [**14**], we say that a family $\mathcal{H} \subset \mathbb{R}^\Omega$ has the *Bourgain property* if for every $\varepsilon > 0$ and every $E \in \Sigma$ with $\mu(E) > 0$ there are $E_1, \ldots, E_n \in \Sigma$, $E_i \subset E$ with $\mu(E_i) > 0$ such that for each $h \in \mathcal{H}$ there is at least one E_i on which the oscillation of h is smaller than ε.

LEMMA 14. *Let $\mathcal{F} \subset \mathbb{R}^\Omega$ be a family such that*

(i) *$\mathcal{F}$ is pointwise bounded;*
(ii) *$\mathcal{F}$ has the Bourgain property;*
(iii) *there is $M > 0$ such that for each $f \in \mathcal{F}$ we have $|f| \leq M$ μ-a.e.*

If $g \in \mathbb{R}^\Omega$ belongs to the $\mathfrak{T}_p(\Omega)$-closure of aco$(\mathcal{F})$*, then $|g| \leq M$ μ-a.e.*

Proof. The function g is measurable by (i) and (ii), see Proposition 4.1 in [**15**]. We now argue by contradiction. Suppose $\mu(E) > 0$, where

$$E := \{t \in \Omega : |g(t)| > M\} \in \Sigma.$$

Since $\mathcal{F}$ is pointwise bounded, we can write $\Omega = \bigcup_{k\in\mathbb{N}} D_k$, where

$$D_k := \{t \in \Omega : |f(t)| \leq k \text{ for all } f \in \mathcal{F}\}.$$

Since $\mu(E) > 0$, there is some $k_0 \in \mathbb{N}$ such that

$$\mu^*(E \cap D_{k_0}) > 0. \tag{3}$$

Consider the complete finite measure space $(\Omega_0, \Sigma_0, \mu_0)$, where $\Omega_0 := E \cap D_{k_0}$, $\Sigma_0 := \{C \cap \Omega_0 : C \in \Sigma\}$ and $\mu_0(S) := \mu^*(S)$ for all $S \in \Sigma_0$. It is easy to check that the family of restrictions $\mathcal{F}|_{\Omega_0} := \{f|_{\Omega_0} : f \in \mathcal{F}\}$ has the Bourgain property with respect to μ_0. Bearing in mind that $\mathcal{F}|_{\Omega_0}$ is uniformly bounded, we infer that $\mathrm{aco}(\mathcal{F}|_{\Omega_0})$ also has the Bourgain property with respect to μ_0, see Proposition 2.2 in [**15**]. Since $g|_{\Omega_0}$ belongs to the $\mathfrak{T}_p(\Omega_0)$-closure of $\mathrm{aco}(\mathcal{F}|_{\Omega_0})$ in $\mathbb{R}^{\Omega_0}$, there is a sequence (g_n) in $\mathrm{aco}(\mathcal{F})$ such that $g_n|_{\Omega_0} \to g|_{\Omega_0}$ μ_0-a.e. (Theorem 11 in [**14**]). By (iii) we have $|g_n| \leq M$ μ-a.e. for every $n \in \mathbb{N}$, hence $|g|_{\Omega_0}| \leq M$ μ_0-a.e., that is there is $C \in \Sigma$ such that $\mu^*(C \cap \Omega_0) = \mu^*(\Omega_0)$ and $E \cap (C \cap \Omega_0) = \emptyset$. Since $\Omega_0 \subset E$, we get $C \cap \Omega_0 = \emptyset$ and so $\mu^*(\Omega_0) = 0$, which contradicts (3) and finishes the proof of the lemma. □

THEOREM 15. *Let $f : \Omega \to X$ be a function such that there is a w^*-thick set $A \subset X^*$ such that $Z_{f,A} \subset \mathcal{L}^\infty(\mu)$ and $Z_{f,A}$ has the Bourgain property. Then f is scalarly bounded. Moreover, if A is convex, then f is Birkhoff integrable.*

Proof. Fix $n \in \mathbb{N}$ and set $C_n := \{x^* \in nB_{X^*} : |x^*f| \leq n\ \mu\text{-a.e.}\}$. Clearly, the family

$$\mathcal{F} := \{y^*f : y^* \in A \cap C_n\} \subset \mathbb{R}^\Omega$$

fulfills the requirements of Lemma 14. If $x^* \in \overline{\mathrm{aco}(A \cap C_n)}^{w^*}$, then x^*f belongs to the $\mathfrak{T}_p(\Omega)$-closure of $\mathrm{aco}(\mathcal{F})$ and Lemma 14 ensures that $|x^*f| \leq n$ μ-a.e. Therefore,

$$\overline{\mathrm{aco}(A \cap C_n)}^{w^*} \subset C_n.$$

As $n \in \mathbb{N}$ is arbitrary, Lemma 6 tells us that f is scalarly bounded. Moreover, the proof of Lemma 6 shows that there exist $n \in \mathbb{N}$, $x_0^* \in X^*$ and $\delta > 0$ such that

$$B := x_0^* + \delta B_{X^*} \subset \overline{\mathrm{aco}(A \cap C_n)}^{w^*}. \tag{4}$$

Now assume that A is convex. Set $D := \mathrm{aco}(A \cap C_n)$ and

$$E := \{\theta x^* : x^* \in A,\ |\theta| \leq 1\}.$$

Since $Z_{f,A}$ has the Bourgain property, the same holds for $Z_{f,E}$, as can be easily checked. By the convexity of A, we have $E - E \supset \mathrm{aco}(A) \supset D$, hence $Z_{f,D}$ has the Bourgain property. The fact that $Z_{f,\overline{D}^{w^*}}$ is contained in the $\mathfrak{T}_p(\Omega)$-closure of $Z_{f,D}$ ensures that $Z_{f,\overline{D}^{w^*}}$ has the Bourgain property as well. By (4), the family $Z_{f,B}$ has the Bourgain property. Now, it is not difficult to check that the family

$$Z_{f,B_{X^*}} = \left\{ \frac{1}{\delta}\left(-x_0^* f + h\right) : h \in Z_{f,B} \right\}$$

also has the Bourgain property. Since f is scalarly bounded, an appeal to Theorem 3.5 in [**2**] establishes that f is Birkhoff integrable. □

It was proved in Corollary 2.5 in [**2**] that the Birkhoff integrability of a *bounded* function f taking values in a dual Banach space Y^* is equivalent to the Bourgain property of the family Z_{f,B_Y}. The following corollary extends that result.

COROLLARY 16. *Suppose $X = Y^*$ for another Banach space Y. Let $f : \Omega \to X$ be a function such that $Z_{f,B_Y} \subset \mathcal{L}^\infty(\mu)$ and Z_{f,B_Y} has the Bourgain property. Then f is scalarly bounded and Birkhoff integrable.*

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