TWO RESULTS CONCERNING THE ZEROS OF FUNCTIONS WITH FINITE DIRICHLET INTEGRAL

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A function f, analytic in the unit disk, is said to have *finite Dirichlet integral* if

$$(1) \qquad ||f||_D{}^2 = \frac{1}{\pi} \int\int_{|z|<1} |f'(z)|^2 r\, dr d\theta < \infty .$$

Geometrically, this is equivalent to f mapping the disk onto a Riemann surface of finite area. The class of Dirichlet integrable functions will be denoted by $\mathscr{D}$. The condition above can be restated in terms of Taylor coefficients; if $f(z) = \sum a_n z^n$, then $f \in \mathscr{D}$ if and only if $\sum n|a_n|^2 < \infty$. Thus, $\mathscr{D}$ is contained in the Hardy class H^2.

In particular, every such function has boundary values

$$f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

almost everywhere and $\log |f(e^{i\theta})| \in L^1(d\theta)$.

The zeros z_n of a function $f \in \mathscr{D}$ must satisfy the Blaschke condition

$$\sum (1 - |z_n|) < \infty,$$

and $f(z) = B(z)F(z)$, where $F(z)$ has no zeros and

$$B(z) = z^m \prod \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z}$$

is the *Blaschke product* with zeros z_n; see (**5**).

In earlier studies by Carleson (**3**) and by Shapiro and Shields (**6**), several results concerning the possible sets of zeros of functions of $\mathscr{D}$ were established. In particular, it was proved that if a sequence converges to the boundary fast enough, i.e., if

$$\sum \left(\frac{1}{\log (1 - |z_n|)} \right) > -\infty,$$

then there is a function of $\mathscr{D}$ which vanishes at those points. On the other hand, sequences were constructed which satisfy the Blaschke condition, and in fact $\sum (1 - |z_n|)^\epsilon < \infty$ for every $\epsilon > 0$, but on which no non-zero function of $\mathscr{D}$ can vanish.

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In these counterexamples, however, every point of the circumference is a limit point of the sequence, and it is natural to ask how "thin" the set of limit points of zeros may be. In Theorem 1, a sequence is constructed which converges to 1 and which satisfies the Blaschke condition, but on which no non-zero function of $\mathscr{D}$ may vanish. This example is due to Professor Lennart Carleson and appears with his permission. His construction is presented in a modified form due to A. L. Shields, P. L. Duren, and the author.

On the positive side, Theorem 2 shows that if a sequence satisfies the Blaschke condition and all its points lie within a curve making a finite degree of contact with the unit circle at 1, then the points of that sequence are the zeros of a function of $\mathscr{D}$. Thus, one can infer that a Blaschke sequence is the set of zeros of a function of $\mathscr{D}$ solely from its geometric configuration.

THEOREM 1. *There exists a sequence* $\{z_n\}$, $|z_n| < 1$, *which satisfies the Blaschke condition and which converges to* 1, *but on which no non-zero function with a finite Dirichlet integral can vanish.*

Proof. The expression in (1) has been shown (**4**) to satisfy

$$||f||_D^2 \geqq \frac{1}{2\pi} \int_0^{2\pi} \left(|f(e^{it})|^2 \sum \frac{1 - |z_n|^2}{|e^{it} - z_n|^2}\right) dt,$$

where the points z_n are the zeros of f. Applying the geometric-arithmetic mean inequality, we see that

$$||f||_D^2 \geqq \exp\left[\frac{1}{2\pi} \int \log\left(|f(e^{it})|^2 \sum \frac{1 - |z_n|^2}{|e^{it} - z_n|^2}\right) dt\right].$$

Since any such function has log-integrable boundary values,

$$\int \log\left(\sum \frac{1 - |z_n|^2}{|e^{it} - z_n|^2}\right) dt < \infty, \tag{2}$$

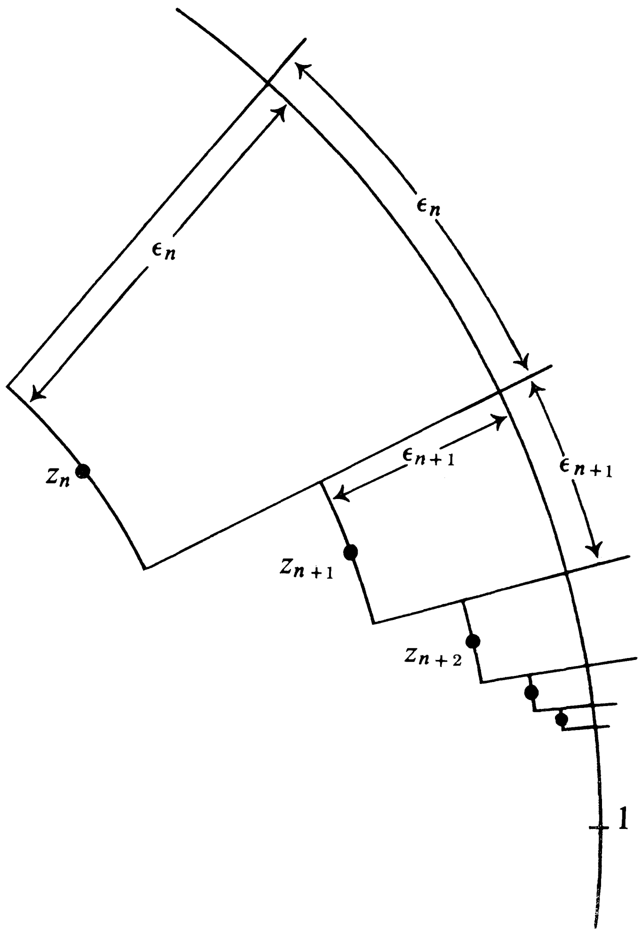
if the points z_n are the zeros of a function satisfying (1).

The construction begins with the choice of a sequence $\{\epsilon_n\}$, $0 < \epsilon_n < 1$, for which $\sum \epsilon_n \leqq 2\pi$, but such that $\sum \epsilon_n \log \epsilon_n = -\infty$. For example, $\{(n(\log n)^2)^{-1}\}$, suitably normalized, is such a sequence. Choose open, disjoint arcs I_n on the circumference, of lengths ϵ_n, converging to 1 (see the figure). Let $r_n = 1 - \epsilon_n$. On each circle of radius r_n, place a point z_n whose signum lies at the centre of I_n. Then the condition for the convergence of the Blaschke product with zeros z_n, $\sum (1 - r_n) = \sum \epsilon_n < \infty$ is satisfied. Notice that $|e^{it} - z_n| < 2\epsilon_n$ for $e^{it} \in I_n$. Thus,

$$\frac{1 - |z_n|^2}{|e^{it} - z_n|^2} > \frac{(1 + r_n)\epsilon_n}{4\epsilon_n^2} > \frac{1}{4\epsilon_n}$$

in this interval. Hence, if

$$F_n(t) = \frac{1 - |z_n|^2}{|e^{it} - z_n|^2},$$



CONSTRUCTION OF THEOREM 1

we see that

$$\begin{aligned}\int \log\left(\sum_{n=1}^{\infty} F_n(t)\right) dt &\geqq \sum_{k=1}^{\infty} \int_{I_k} \log\left(\sum_{n=1}^{\infty} F_n(t)\right) dt \\ &> \sum_{k=1}^{\infty} \int_{I_k} \log F_k(t)\, dt \\ &> -\sum_{k=1}^{\infty} \int_{I_k} \log 4\epsilon_k\, dt \\ &= -\sum_{k=1}^{\infty} \epsilon_k \log 4\epsilon_k.\end{aligned}$$

However, this series diverges by hypothesis, contradicting (2).

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The following special case of a theorem of Beurling (**1**, p. 13) and Carleson (**2**) will be required for Theorem 2. A *Carleson set* is a closed set of measure zero with complementary arcs of lengths ϵ_n, which satisfies

$$\sum \epsilon_n \log \epsilon_n > -\infty.$$

THEOREM. *If f is an analytic function with a bounded derivative, then $\{e^{i\theta}: f(e^{i\theta}) = 0\}$ is a Carleson set. Conversely, given a Carleson set E and a positive integer m, there is a function g, which vanishes on E, which is outer in the sense of Beurling, and for which $g^{(m)}(z)$ is bounded in the disk.*

In the example of Theorem 1, the points z_n tend to 1 "very tangentially". Theorem 2 implies that if the Blaschke condition is satisfied and the z_n tend to 1 not "too tangentially", then there is a non-zero function f which is analytic in the disk, whose derivative is continuous in the closed disk, and which vanishes at the points z_n. A subset S of the disk is said to have *finite degree of contact k* at a subset E of the circle if there is a constant $M > 0$ such that

$$\operatorname{dist}(w, E)^k \leqq M(1 - |w|)$$

for all $w \in S$.

THEOREM 2. *Let $\{z_n\}$ be a sequence of points of the unit disk whose limit points lie in a Carleson set E and which satisfies the Blaschke condition. Then, if the set of points z_n has a finite degree of contact at E, there is an analytic function f whose derivative is continuous in the closed disk and whose zeros are the points z_n.*

Proof. By the Carleson theorem, there is an outer function g with continuous derivative, which vanishes on E. By integration,

$$|g(z)| \leqq N \operatorname{dist}(z, E)$$

for some constant $N > 0$. By dividing g by N, we may assume that $N = 1$.

Assume that $\operatorname{dist}(z_n, E)^k \leqq M(1 - |z_n|)$ for all n, where we may assume that k is greater than 1. Let B be the Blaschke product

$$B(z) = \prod \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z},$$

and let

$$f(z) = g(z)^{2k+1} B(z).$$

Then

$$\begin{aligned} f'(z) &= (2k + 1)g'(z)g(z)^{2k}B(z) + g(z)^{2k+1}B'(z) \\ &= (2k + 1)g'(z)g(z)^{2k}B(z) + g(z)^{2k+1}\sum B_n(z) \frac{\bar{z}_n}{|z_n|} \frac{1 - |z_n|^2}{(1 - \bar{z}_n z)^2}, \end{aligned}$$

where

$$B_n(z) = \frac{z_n}{|z_n|} \frac{1 - \bar{z}_n z}{z_n - z} B(z).$$

Since B and B_n are analytic on the closed disk except at points of E (**5**, p. 68), if $g(z)^{2k}/(1 - \bar{z}_n z)^2$ is bounded for z in the unit disk independently of n, then

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f' is continuous in the closed disk. By the maximum modulus principle, this will be the case if there is a constant C such that

$$|g(e^{i\theta})|^k \leqq C|e^{i\theta} - z_n| \tag{3}$$

for all n. Let $C = 2^k(M + 1)$. Then

$$\begin{aligned} |g(e^{i\theta})|^k &\leqq \operatorname{dist}(e^{i\theta}, E)^k \\ &\leqq 2^{k-1}[|e^{i\theta} - z_n|^k + \operatorname{dist}(z_n, E)^k] \\ &\leqq 2^{k-1}[|e^{i\theta} - z_n|^k + M(1 - |z_n|)]. \end{aligned}$$

If $|e^{i\theta} - z_n| \leqq 1$, then $|e^{i\theta} - z_n|^k \leqq |e^{i\theta} - z_n|$, and using $1 - |z_n| \leqq |e^{i\theta} - z_n|$, we see that

$$|g(e^{i\theta})|^k \leqq 2^{k-1}(M + 1)|e^{i\theta} - z_n|.$$

If $|e^{i\theta} - z_n| > 1$, then

$$|g(e^{i\theta})|^k \leqq 2^k < 2^k(M + 1)|e^{i\theta} - z_n|.$$

Thus, in any case, (3) holds, and f is the desired function.

A generalization of this construction yields a function with any given number of continuous derivatives, with the same hypotheses on the points z_n. Since a function $f(z) = \sum a_n z^n$ with bounded derivative satisfies $\sum n^2|a_n|^2 < \infty$, such a function has finite Dirichlet integral.

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