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MONOCHROMATIC SOLUTIONS TO EQUATIONS WITH UNIT FRACTIONS

TOM C. BROWN AND VOIJTECH RÖDL

Our main result is that if $G(x_1, \cdots, x_n) = 0$ is a system of homogeneous equations such that for every partition of the positive integers into finitely many classes there are distinct $y_1, \cdots, y_n$ in one class such that $G(y_1, \cdots, y_n) = 0$, then, for every partition of the positive integers into finitely many classes there are distinct $z_1, \cdots, z_n$ in one class such that

$$G\left(\frac{1}{z_1}, \cdots, \frac{1}{z_n}\right) = 0.$$

In particular, we show that if the positive integers are split into r classes, then for every $n \geqslant 2$ there are distinct positive integers $x_1, x_1, \cdots, x_n$ in one class such that

$$\frac{1}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.$$

We also show that if $[1, n^6 - (n^2 - n)^2]$ is partitioned into two classes, then some class contains $x_0, x_1, \ldots, x_n$ such that

$$\frac{1}{x_0} = \frac{1}{x_1} + \ldots + \frac{1}{x_n}.$$

(Here, $x_0, x_2, \ldots, x_n$ are not necessarily distinct.)

1. INTRODUCTION

In their monograph [1], Erdös and Graham list a large number of questions concerned with equations with unit fractions. In fact, a whole chapter is devoted to this topic. One of their questions, still open, is the following.

In the positive integers, let

$$H_m = \left\{ \{x_1, \cdots, x_m\} : \sum_{k=1}^{m} 1/x_k = 1, 0 < x_1 < \cdots < x_m \right\},$$

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and let H denote the union of all the H_m, $m \geqslant 1$. Now arbitrarily split the positive integers into r classes. It is true that some element of H is contained entirely in one class?

In this note we show (Corollary 2.4 below) that if one does not require *all* the x_k's to be distinct, but only *many* of the x_k's to be distinct, then the answer to the corresponding question is yes. More precisely, we show that if the positive integers are split into r classes, then for every n there exist $m \geqslant n$ and $x_1, \cdots, x_m$ (not necessarily distinct) in one class such that $|\{x_1, \cdots, x_m\}| \geqslant n$ and $\sum_{k=1}^{m} 1/x_k = 1$.

We actually show (Corollary 2.3 below) something stronger, namely that if the positive integers are split into r classes, then for every $n \geqslant 2$ there are *distinct* positive integers $x_0, x_1, \ldots, x_n$ in one class such that

$$\frac{1}{x_0} = \frac{1}{x_1} + \cdots \frac{1}{x_n}.$$

(The preceding result then follows by taking x_0 copies of each of $x_1, \cdots, x_n$.)

Our main result (Theorem 2.1) is that if $G(x_1, \cdots, x_n) = 0$ is a system of homogeneous equations such that for every partition of the positive integers into finitely many classes there are distinct $y_1, \cdots, y_n$ in one class such that $G(y_1, \cdots, y_n) = 0$, then, for every partition of the positive integers into finitely many classes there are distinct $z_1, \cdots, z_n$ in one class such that

$$G\left(\frac{1}{z_1}, \cdots, \frac{1}{z_n}\right) = 0.$$

We also prove (Theorem 2.5) the following quantitative result. Let $f(n)$ be the smallest N such that if $[1, N]$ is partitioned into *two* classes, then some class contains $x_0, x_1, \ldots, x_n$ such that $1/x_0 = 1/x_1 + \cdots + 1/x_n$. (Here, $x_0, x_1, \ldots, x_n$ are not necessarily distinct.) Then

$$f(n) \leqslant n^6 - \left(n^2 - n\right)^2.$$

2. Results

From now on we shall use the terminology of *colourings* rather than *partitions*. That is, instead of "partition into r classes" we say "r-colouring," and instead of "there are distinct $y_1, \cdots, y_n$ in one class such that $G(y_1, \cdots, y_n) = 0$" we say "there is a monochromatic solution of $G(y_1, \cdots, y_n) = 0$ in distinct $y_1, \cdots, y_n$".

Theorem 2.1. *Let $G(x_1, \cdots, x_n) = 0$ be a system of homogeneous equations such that for every finite colouring of the positive integers there is a monochromatic solution of $G(y_1, \cdots, y_n) = 0$ in distinct $y_1, \cdots, y_n$. Then, for every finite colouring* *of the positive integers there is a monochromatic solution of* $G(1/z_1, \cdots, 1/z_n) = 0$ *in distinct* $z_1, \cdots, z_n$

PROOF: Let r be given, and consider a system $G(x_1, \cdots, x_n) = 0$ of homogeneous equations such that for every r-colouring of the positive integers there is a monochromatic solution of $G(y_1, \cdots, y_n) = 0$ in distinct $y_1, \cdots, y_n$. By a standard compactness argument, there exists a positive integer T such that if $[1, T]$ is r-roloured, there is a monochromatic solution to $G(y_1, \cdots, y_n) = 0$ in distinct $y_1, \cdots, y_n$.

Let S denote the least common multiple of $1, 2, \cdots, T$. We show that for every r-colouring of $[1, S]$ there is a monochromatic solution of $G(1/z_1, \cdots, 1/z_n) = 0$ in distinct $z_1, \cdots, z_n$.

To do this, let

$$c\colon [1, S] \to [1, r]$$

be an arbitrary r-colouring of $[1, S]$.

Define an r-colouring $\bar{c}$ of $[1, T]$ by setting

$$\bar{c}(x) = c(S/x),\ 1 \leqslant x \leqslant T.$$

By the definition of T, there is a solution of $G(y_1, \cdots, y_n) = 0$ in distinct $y_1, \cdots, y_n$ such that

$$\bar{c}(y_1) = \bar{c}(y_2) = \cdots = \bar{c}(y_n).$$

By the definition of $\bar{c}$, this means that

$$c(S/y_1) = c(S/y_2) = \cdots = c(S/y_n).$$

Setting $z_i = S/y_i$, $1 \leqslant i \leqslant n$, we have that $z_1, \cdots, z_n$ are distinct, are monochromatic relative to the colouring c of $[1, S]$, and that

$$G\left(\frac{1}{z_1}, \cdots, \frac{1}{z_n}\right) = 0.$$

□

Omitting all references to distinctness, one gets the following.

THEOREM 2.1A. *Let* $G(x_1, \cdots, x_n) = 0$ *be a system of homogeneous equations such that for every finite colouring of the positive integers there is a monochromatic solution of* $G(x_1, \cdots, x_n) = 0$. *Then, for every finite colouring of the positive integers there is a monochromatic solution of* $G(1/z_1, \cdots, 1/z_n) = 0$.

COROLLARY 2.2. *Let* $a_1, \cdots, a_m, b_1, \cdots, b_n$ *be positive integers such that*

(1) *some non-empty subset of the* a_i*'s has the same sum as some non-empty subset of the* b_j*'s and*

(2) *there exist distinct integers* $u_1, \cdots, u_m, v_1, \cdots, v_n$ *such that* $a_1u_1 + \cdots + a_mu_m = b_1v_1 + \cdots + b_nv_n$.

Then, given any r-colouring of the positive integers, there is a monochromatic solution of

$$\frac{a_1}{x_1} + \cdots + \frac{a_m}{x_m} = \frac{b_1}{y_1} + \cdots + \frac{b_n}{y_n}$$

in distinct $x_1, \cdots, x_m, y_1, \cdots, y_n$.

PROOF: Let $a_1, \cdots, a_m, b_1, \cdots, b_n$ satisfy conditions (1) and (2). According to Rado's theorem [3] (also see [2, p.59]), the equation

$$a_1x_1 + \cdots + a_mx_m = b_1y_1 + \cdots + b_ny_n$$

will always have a monochromatic solution $x_1, \cdots, x_m, y_1, \cdots, y_n$, for every r-colouring of the positive integers, because of condition (1). The additional condition (2) is enough (see [2, p.62 Corollary $8\frac{1}{2}$]) to ensure that the equation

$$a_1x_1 + \cdots + a_mx_m = b_1y_1 + \cdots + b_ny_n$$

will always have a monochromatic solution $x_1, \cdots, x_m, y_1, \cdots, y_n$, in *distinct* $x_1, \cdots, x_m, y_1, \cdots, y_n$. Theorem 2.1 now applies. □

COROLLARY 2.3. *Let an arbitrary r-colouring of the positive integers be given. Let n, a be positive integers, with $n \geqslant 2$ and $1 \leqslant a \leqslant n$. Then the equation*

$$\frac{a}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}$$

has a monochromatic solution in distinct $x_0, x_1, \cdots, x_n$.

PROOF: This follows immediately from Corollary 2.2. □

COROLLARY 2.4. *Let an arbitrary r-colouring of the positive integers be given. Then for every n there exist $m \geqslant n$ and monochromatic $x_1, \cdots, x_m$ (not necessarily distinct) such that $|\{x_1, \cdots, x_m\}| \geqslant n$ and $\sum_{k=1}^{m} 1/x_k = 1$.*

PROOF: Apply Corollary 2.3 (with $a = 1$) and take x_0 copies of each of $x_1, \cdots, x_m$. □

THEOREM 2.5. *For each $n \geqslant 2$, let $f(n)$ be the smallest N such that if $[1, N]$ is partitioned into two classes, then some class contains $x_0, x_1, \cdots, x_n$ such that*

$$\frac{1}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.$$

(Here, $x_0, x_1, \cdots, x_n$ are not necessarily distinct.) Then

$$f(n) \leqslant n^6 - (n^2 - n)^2.$$

PROOF: The proof is by contradiction. Fix $n \geqslant 2$, let $N = n^6 - \left(n^2 - n\right)^2$, and suppose throughout the proof that $c\colon [1, N] \to \{1, 2\}$ is some fixed 2-colouring of $[1, N]$ for which there does *not* exist any monochromatic solution of

$$\frac{1}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.$$

∎

LEMMA 2.6. *(a) If $nx \leqslant N$ then $c(nx) \neq c(x)$.*
(b) If $n^2 x \leqslant N$ then $c\left(n^2 x\right) = c(x)$.

PROOF: Part (a) follows from $1/x = 1/(nx) + \cdots + 1/(nx)$. Part (b) follows from part (a). ∎

LEMMA 2.7. *If $n^2\left(n^2 + n - 1\right)x \leqslant N$, then $c\left(\left(n^2 + n - 1\right)x\right) \neq c(x)$.*

PROOF: This follows from

$$\frac{1}{n^2 x} = \frac{1}{(n^2 + n - 1)x} + (n - 1)\frac{1}{n^2(n^2 + n - 1)x}$$

and Lemma 2.6. ∎

LEMMA 2.8. *If $n^2\left(n^2 - n + 1\right)x \leqslant N$, then $c\left(\left(n^2 - n + 1\right)x\right) \neq c(x)$.*

PROOF: This follows from

$$\frac{1}{(n^2 - n + 1)x} = \frac{1}{n^2 x} + (n - 1)\frac{1}{n^2(n^2 - n + 1)x}$$

and Lemma 2.6. ∎

LEMMA 2.9. *If $n^2\left(n^2 + n - 1\right)x \leqslant N$, then $c((n + 1)x) = c(x)$.*

PROOF: This follows from

$$\frac{1}{n(n + 1)x} = \frac{1}{(n^2 + n - 1)(n + 1)x} + (n - 1)\frac{1}{(n^2 + n - 1)nx},$$

and Lemmas 2.6 and 2.7. ∎

LEMMA 2.10. *If $n^2\left(n^2 + n - 1\right)\left(n^2 - n + 1\right)x \leqslant N$, then $c(2x) = c(x)$.*

PROOF: This follows from

$$\frac{1}{(n^2 - n + 1)2x} = \frac{1}{(n^2 + n - 1)2x} + (n - 1)\frac{1}{(n^2 + n - 1)(n^2 - n + 1)x}$$

and Lemmas 2.7 and 2.8. ∎

Finally, Theorem 2.5 is proved by observing that

$$\frac{1}{2 \cdot 1} = \frac{1}{(n + 1) \cdot 1} + (n - 1)\frac{1}{2(n + 1) \cdot 1},$$

and by Lemmas 2.9 and 2.10, $c(2 \cdot 1) = c((n + 1) \cdot 1) = c(2(n + 1) \cdot 1) = c(1)$, a contradiction.

REMARK. The authors have learned that Hanno Lefmann (Bielefeld) has independently obtained results which include our Theorem 2.1a.

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Department of Mathematics
and Statistics
Simon Fraser University
Burnaby BC
Canada V5A 1S6

Department of Mathematics
and Computer Science
Emory University
Atlanta GA 30322-9998
United States of America