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THE RELATIONSHIP BETWEEN FIXING SUBGRAPHS AND SMOOTHLY EMBEDDABLE SUBGRAPHS

KEITH CHIDZEY

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Abstract

Grant (1976) has attempted to establish a relationship between fixing subgraphs and smoothly embeddable subgraphs. Here we give counterexamples to his two main lemmas and two characterizing theorems. We then go on to give our own version of these lemmas and theorems.

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1. Introduction

We study finite, simple graphs G with vertex set $V(G)$, edge set $E(G)$ and automorphism group $\Gamma(G)$. $L(G)$ denotes the line graph of G, $\mathscr{S}(G)$ the set of spanning subgraphs of G and $\mathscr{S}_0(G)$ the set of induced subgraphs of G.

Fixing and smoothly embeddable subgraphs were introduced by Sheehan (1972a, 1972b). We now give definitions of these concepts.

DEFINITION. Let $H \in \mathscr{S}(G)$. If K is any spanning subgraph of G isomorphic to H and if for any permutation α such that $H^{\alpha} = K$, then $\alpha \in \Gamma(G)$, we say that H is a *fixing subgraph* of G. We denote the set of fixing subgraphs of G by $\mathscr{F}(G)$.

DEFINITION. Let $H \in \mathscr{S}_0(G)$. If K is any induced subgraph of G isomorphic to H and if for any isomorphism β such that $H^{\beta} = K$, then $\beta = \alpha | V(H)$, the restriction of α to $V(H)$, for some $\alpha \in \Gamma(G)$, we say that H is a *smoothly embeddable subgraph* of G. We denote the set of smoothly embeddable subgraphs of G by $\mathscr{F}_0(G)$.

As there is a unique correspondence between the spanning subgraphs of a graph G and the induced subgraphs of $L(G)$ it has been indicated by Sheehan

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(1972b) that "the relationship between fixing and smoothly embeddable subgraphs of G can be made explicit by a consideration of the line graph of G". This can be done by comparing when spanning subgraphs of G are fixing subgraphs, with when the corresponding induced subgraphs of $L(G)$ are smoothly embeddable subgraphs. In Section 2 we give some preliminary results which aid this comparison. We then list four claimed results of Grant (1976) in Section 3 and give counter-examples. The remainder of this note is devoted to reformulating these statements. In Section 4 we obtain two lemmas on fixing subgraphs and smoothly embeddable subgraphs of disconnected graphs which could be considered separately but are required for obtaining the relationship between fixing and smoothly embeddable subgraphs in Section 5. Section 4 also includes a lemma which is basically that of Grant (1976) on the same relationship when automorphism groups are somewhat restricted.

2. Preliminary results

For $H \in \mathscr{S}(G)$, let $\Gamma(H, G) \equiv \{\alpha \in \Gamma(G): \alpha | V(H) \in \Gamma(H)\}$ and for $H \in \mathscr{S}_0(G)$, let $c(H, G) \equiv |\{K \in \mathscr{S}_0(G): K \cong H\}|$. It follows immediately from the definitions that $H \in \mathscr{F}(G)$ implies $\Gamma(H) \leqslant \Gamma(G)$ and that $H \in \mathscr{F}_0(G)$ implies

$$\Gamma(H) = \{\beta: \beta = \alpha | V(H), \alpha \in \Gamma(H, G)\}.$$

We now list the characterizations of fixing subgraphs and smoothly embeddable subgraphs of a graph G in terms of the number of copies of them in G.

LEMMA 1 (Sheehan (1972a)). *Given $H \in \mathscr{S}(G)$ then $H \in \mathscr{F}(G)$ if and only if G contains exactly $|\Gamma(G)|/|\Gamma(H)|$ distinct copies of H in G.*

LEMMA 2 (Sheehan (1974)). *Given $H \in \mathscr{S}_0(G)$ then $H \in \mathscr{F}_0(G)$ if and only if $c(H, G) = |\Gamma(G)|/|\Gamma(H, G)|$ and $\Gamma(H) = \Gamma(H, G)| V(H)$.*

The following notation will be of use later on.

NOTATION.
1. Let $\mathbf{N}_n = \{1, 2, \ldots, n\}$.
2. Let $c(G)$ denote the number of components of graph G.
3. Let nG denote the union of n copies of graph G.
4. For graphs A, B let $\Gamma(A) \times \Gamma(B)$ be the permutation group acting on the disjoint union $V(A) \cup V(B)$ whose elements are the ordered pairs of permutations

α in $\Gamma(A)$ and β in $\Gamma(B)$, written $\alpha\beta$, and whose action is given by:

$$v^{\alpha\beta} = \begin{cases} v^{\alpha} & \text{if } v \in V(A), \\ v^{\beta} & \text{if } v \in V(B). \end{cases}$$

5. Let $\prod_{i \in \mathbf{N}_n} \Gamma(A_i) = \Gamma(A_1) \times \Gamma(A_2) \times \ldots \times \Gamma(A_n)$ for graphs A_i, $i \in \mathbf{N}_n$.

The following lemmas are relevant to studying the relationship between fixing and smoothly embeddable subgraphs. Note that $\Gamma_1(G)$ is the edge automorphism group of G, $\Gamma^*(G)$ is the subgroup of $\Gamma_1(G)$ whose elements are induced by elements of $\Gamma(G)$, and $\Gamma_1(G) = \Gamma(L(G))$.

LEMMA 3 (Whitney (1932)). *Let G and H be connected graphs such that $L(G) \cong L(H)$. Then $G \cong H$ unless one of G and H is K_3 and the other is $K_{1,3}$.*

LEMMA 4 (Behzad and Chartrand (1971)). *Let G be a non-trivial connected graph. Then $\Gamma^*(G) \cong \Gamma(G)$ unless G is K_2.*

COROLLARY. *For a non-trivial graph G, $\Gamma^*(G) \cong \Gamma(G)$ if and only if G has neither K_2 as a component nor two or more isolated vertices.*

LEMMA 5 (Whitney (1932)). *Let G be a non-empty graph. Then $\Gamma_1(G) = \Gamma^*(G)$ if and only if*

(1) *not both K_3 and $K_{1,3}$ are components of G, and*

(2) *none of the graphs G_1, G_2 (of Fig. 1) and K_4 is a component of G.*

FIG. 1.

COROLLARY. *Let G be a connected graph with $|V(G)| \geqslant 3$. Then $\Gamma_1(G) = \Gamma^*(G)$ if and only if G is none of G_1, G_2 and K_4.*

LEMMA 6 (Sheehan (1974)). *All induced subgraphs of $L(K_4)$ are smoothly embeddable.*

These results motivate the following definition.

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DEFINITION. Let $H \in \mathscr{S}(G)$. We define G^H to be

$$\{K \in \mathscr{S}(G) \colon K \cong H' \cup mK_{1,3} \cup (n_1+n_2-m)K_3 \cup (n_1+n_2-m)K_1, m \neq n_1\},$$

where $H \cong H' \cup n_1 K_{1,3} \cup n_2 K_3 \cup n_2 K_1$ and H' has neither $K_{1,3}$ nor both K_3 and K_1 as components.

REMARK. $K \in G^H$ implies $L(K) \cong L(H)$.

3. Counterexamples

First we note some definitions of Grant (1976). We then state four results he asserts and give counterexamples to each.

If M is a graph with a component isomorphic to $K_{1,3}$ let $M^\perp$ denote the graph obtained from M by replacing a component isomorphic to $K_{1,3}$ by one component isomorphic to K_1 and one component isomorphic to K_3. If M has no components isomorphic to $K_{1,3}$, let $M^\perp = M$. If M has components isomorphic to both K_1 and K_3, let $M^\top$ denote the graph obtained from M by replacing two components, one isomorphic to K_1 and the other to K_3 by one component isomorphic to $K_{1,3}$. If M does not have components isomorphic to K_1 and K_3, let $M^\top = M$. Note that by Lemma 3, $L(M) \cong L(M^\perp) \cong L(M^\top)$. If G is a graph with spanning subgraph isomorphic to H, we say that H conforms to G provided either $H = H^\perp = H^\top$ or if $H \neq H^\perp$ then G has no spanning subgraph isomorphic to $H^\top$ and if $H \neq H^\top$ then G has no spanning subgraph isomorphic to $H^\perp$.

STATEMENT A. *Let G be a graph with components $A_1, A_2, \ldots, A_k$. Let H be a spanning subgraph of G and for $i = 1, 2, \ldots, k$ let A'_i be the subgraph of H induced by $V(A_i)$. Then*

(a) *if $H \in \mathscr{F}(G)$, it follows that $A'_i \in \mathscr{F}(A_i)$ for $i = 1, 2, \ldots, k$, and*

(b) *if $H \notin \mathscr{F}(G)$, it follows that either* (i) *for some i, with $1 \leqslant i \leqslant k$, $A'_i \notin \mathscr{F}(A_i)$, or* (ii) *for some $j, l \neq j$, with $1 \leqslant j, l \leqslant k$, there exist components B'_j, B'_l of A'_j and A'_l respectively which are isomorphic and are such that there is no $\alpha \in \Gamma(G)$ which maps B'_j onto B'_l and vice versa, and fixes $V(G) - (V(B'_j) \cup V(B'_l))$.*

STATEMENT B. (a) *Let $U \in \mathscr{F}_0(G)$. Let K be a component of G such that $V(U) \cap V(K) \neq \varnothing$. Then if K' is the subgraph of K induced by $V(U) \cap V(K)$, we have $K' \in \mathscr{F}_0(K)$.*

(*b*) *Suppose $U \notin \mathscr{F}_0(G)$. Then either*

(i) *there exists a component M of G with $V(U) \cap V(M) \neq \varnothing$ such that if M' is the subgraph of M induced by $V(U) \cap V(M)$, then $M' \notin \mathscr{F}_0(M)$, or*

(ii) *there are components M and N of G, with corresponding subgraphs M' and N' of U respectively such that there are components M'_0 and N'_0 of M' and N' respectively which are isomorphic and, yet, are such that there is no automorphism of G which interchanges M'_0 and N'_0 but fixes all other vertices of $M' \cup N'$.*

STATEMENT C. *Suppose $H \in \mathscr{F}(G)$. Then $L(H) \in \mathscr{F}_0(L(G))$ if and only if none of the following hold.*

(i) *There is a component of H isomorphic to one of the graphs G_1, G_2, K_4 which is not a component of G.*

(ii) *H has at least one component isomorphic to K_3 and at least one component isomorphic to $K_{1,3}$, not both of which are components of G.*

(iii) *Neither* (i) *nor* (ii) *holds, and there is a component C of G, such that if C' is the subgraph of H induced by $V(C)$, then C' does not conform to C.*

STATEMENT D. *Suppose $L(H) \in \mathscr{F}_0(L(G))$. Then $H \in \mathscr{F}(G)$ if and only if none of the following hold.*

(i) *There is a component M of G, such that if M' is the subgraph of H induced by $V(M)$, then the ordered pair (M, M') is either (G_1, P_4), (G_2, C_4) or (G_2, P_4).*

(ii) *H has at least two isolated vertices which do not share the same open neighbourhood in G, or has at least one component isomorphic to K_2 whose vertices do not share the same closed neighbourhood in G.*

FIG. 2.

COUNTEREXAMPLE A. Let $G = A_1 \cup A_2$ and $H = A'_1 \cup A'_2$ of Fig. 2. Then $A'_i \in \mathscr{F}(A_i)$, $i = 1, 2$. Also A'_1 and A'_2 have no isomorphic components and so H satisfies the conditions of Statement A for $H \in \mathscr{F}(G)$. But $H \notin \mathscr{F}(G)$ since there exists B_1 and B_2 such that $A'_1 \cong B_2 \in \mathscr{S}(A_2)$ and $A'_2 \cong B_1 \in \mathscr{S}(A_1)$ and so $B_1 \cup B_2 \cong H$ and as $A_1 \not\cong A_2$ clearly $H^\alpha \neq B_1 \cup B_2$ for any $\alpha \in \Gamma(G)$.

Statement A is false as it is not necessary given $A'_i \in \mathscr{F}(A_i)$, $i = 1, 2$, and $H \notin \mathscr{F}(G)$ that there must be two isomorphic components of H satisfying the stated conditions. There are infinitely more counterexamples as the existence of further spanning subgraphs, isomorphic to, but not similar to H is not as rare as implied by Statement A.

COUNTEREXAMPLE B. Let $G = A_1 \cup A_2$ and $H = A_1' \cup A_2'$ of Fig. 3. Then $A_i' \in \mathscr{F}_0(A_i)$, $i = 1, 2$. Also A_1' and A_2' have no isomorphic components. Hence by Statement B, $H \in \mathscr{F}_0(G)$. However, $H \notin \mathscr{F}_0(G)$ since there exists B_1 and B_2 such that $A_1' \cong B_2 \in \mathscr{S}_0(A_2)$ and $A_2' \cong B_1 \in \mathscr{S}_0(A_1)$ so $B_1 \cup B_2 \cong H$ and as $A_1 \not\cong A_2$ clearly $H^\alpha \neq B_1 \cup B_2$ for any $\alpha \in \Gamma(G)$.

FIG. 3.

Statement B is false for similar reasons to Statement A but instead with reference to induced subgraphs.

COUNTEREXAMPLE C. Let $G \cong K_4$ and $H \cong K_{1,3}$. Then $H \in \mathscr{F}(G)$. Now conditions (i) and (ii) of Statement C do not hold for G and H but as $H^\perp \not\cong H$ and $H^\top = H$ it follows that H does not conform to G and thus condition (iii) holds so Statement C asserts that $L(H) \notin \mathscr{F}_0(L(G))$. This is false because $L(H) \cong K_3$ which is smoothly embeddable in $L(G) \cong K_{2,2,2}$, the complete tripartite graph with two vertices in each part.

Statement C is false since if condition (iii) holds it does not follow that $L(H) \notin \mathscr{F}_0(L(G))$ as claimed. Furthermore, the proof relies on Statements A and B. Also condition (i) holding does not imply $\Gamma(L(H)) \nleq \Gamma(L(G))$ as claimed in the proof, for example $G \cong K_4$, $H \cong G_2$.

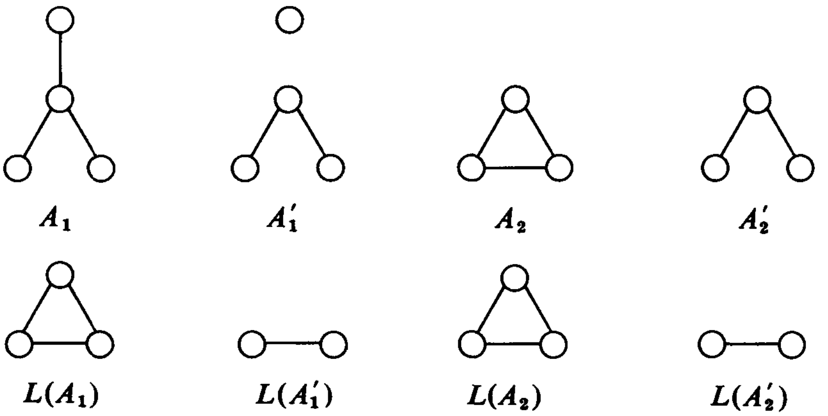

FIG. 4.

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Counterexample D. Let $G = A_1 \cup A_2$ and $H = A_1' \cup A_2'$ of Fig. 4. Then $L(H) \in \mathscr{F}_0(L(G))$. Conditions (i) and (ii) of Statement D do not hold for G and H so the statement asserts that $H \in \mathscr{F}(G)$ which is false. The proof breaks down because of use of Statement A and as an incorrect deduction was made from the preceding lemma.

In the following two sections we reformulate Statements A–D.

4. Fixing and smoothly embeddable subgraphs of disconnected graphs

In this section we assume G is a graph with components A_i, $i \in \mathbf{N}_n$. Also we assume H is a subgraph of G and for $i \in \mathbf{N}_n$, A_i' is the subgraph of H induced by $V(A_i) \cap V(H)$ with components A_{ij}', $j \in \mathbf{N}_{c(A_i')}$.

We now determine which spanning subgraphs of a disconnected graph are fixing subgraphs. (Compare with Statement A.)

Lemma 7. *Given $H \in \mathscr{S}(G)$ then $H \in \mathscr{F}(G)$ if and only if*

(1) *$A_i' \in \mathscr{F}(A_i)$, $i \in \mathbf{N}_n$, and*

(2) *if $K \in \mathscr{S}(G)$ is a copy of H in G and α is a permutation of $V(G)$ such that $H^\alpha = K$ then, given any $i \in \mathbf{N}_n$ and $j \in \mathbf{N}_{c(A_i')}$, $A_{ij}'^{\alpha} \subseteq A_l$ implies $A_i^\alpha = A_l$.*

Proof. Let G and H be as given. Let $V_i = V(A_i)$. As $A_i' \in \mathscr{S}(A_i)$,

$$V(A_i') = V(A_i) = V_i.$$

($\Rightarrow$) First we show by the contrapositive that $H \in \mathscr{F}(G)$ implies $A_i' \in \mathscr{F}(A_i)$, $i \in \mathbf{N}_n$. Suppose $A_i' \notin \mathscr{F}(A_i)$ for some $i \in \mathbf{N}_n$. Then there exists a copy B of A_i' in A_i and a permutation γ of V_i such that $A_i'^{\gamma} = B$ but $\gamma \notin \Gamma(A_i)$. Let

$$K = B \cup \bigcup_{j \in \mathbf{N}_n \setminus \{i\}} A_j'$$

and let β be a permutation on $V(G) = V(H)$ such that $H^\beta = K$ with $\beta | V_i = \gamma$ so that $A_i'^{\beta} = A_i'^{\gamma} = B$ and $A_j'^{\beta} = A_j$ for $j \in \mathbf{N}_n \setminus \{i\}$. As $V_j^\beta = V_j$ for each $j \in \mathbf{N}_n$ and $\beta | V_i = \gamma \notin \Gamma(A_i)$, then $\beta \notin \Gamma(G)$. Consequently $H \notin \mathscr{F}(G)$.

We now show that $H \in \mathscr{F}(G)$ implies condition (2) holds. Suppose $H \in \mathscr{F}(G)$. Let $K \in \mathscr{S}(G)$ such that $K \cong H$. Let α be a permutation of $V(G)$ such that $H^\alpha = K$. Since $H \in \mathscr{F}(G)$, $\alpha \in \Gamma(G)$. Now for $i \in \mathbf{N}_n$ and $j \in \mathbf{N}_{c(A_i')}$, $A_{ij}'^{\alpha} \subseteq A_l$ for some $l \in \mathbf{N}_n$. It follows that $A_i^\alpha = A_l$ as α preserves connectedness when acting on G.

($\Leftarrow$) Suppose (1) and (2) hold. Let K be a copy of H in G and α a permutation of $V(G)$ such that $H^\alpha = K$. Let $i \in \mathbf{N}_n$ and $j \in \mathbf{N}_{c(A_i')}$. Then $A_{ij}'^{\alpha} \subseteq A_l$ for some $l \in \mathbf{N}_n$. By (2), $A_i^\alpha = A_l$ so $A_i'^{\alpha} \in \mathscr{S}(A_l)$. Let Ψ be a permutation on $\mathbf{N}_n$ such that $i^\psi = l$.

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As $A_i^\alpha = A_{i\psi}$, $A_{i\psi} \cong A_i$ so there exists $\beta \in \Gamma(G)$ such that $A_i^\beta = A_{i\psi}$ for each $i \in \mathbf{N}_n$. Now $A_i'^{\alpha\beta^{-1}} \cong A_i'$ and $A_i'^{\alpha\beta^{-1}} \in \mathscr{S}(A_i)$. Since $A_i' \in \mathscr{F}(A_i)$ there exists $\gamma_i \in \Gamma(A_i)$ such that $A_i'^{\alpha\beta^{-1}\gamma_i} = A_i'$. Therefore $a\beta^{-1}\gamma_i | V_i \in \Gamma(A_i') \leqslant \Gamma(A_i)$ as $A_i' \in \mathscr{F}(A_i)$. Let $\delta_i = \alpha\beta^{-1}\gamma_i | V_i$ so $\alpha\beta^{-1} | V_i = \delta_i \gamma_i^{-1} \in \Gamma(A_i)$. Let $\xi = \prod_{i \in \mathbf{N}_n} \delta_i \gamma_i^{-1}$. Then

$$\xi \in \prod_{i \in \mathbf{N}_n} \Gamma(A_i) \leqslant \Gamma(G).$$

Now

$$\xi | V_i = \prod_{j \in \mathbf{N}_n} \delta_j \gamma_j^{-1} | V_i = \delta_i \gamma_i^{-1}$$

as $\delta_j \gamma_j$ acts only on V_j for each $j \in \mathbf{N}_n$ and $V_j \cap V_i = \varnothing$ for $j \neq i$. Hence $\alpha\beta^{-1} | V_i = \xi | V_i$ for each $i \in \mathbf{N}_n$. Therefore $\alpha\beta^{-1} = \xi$ and $\alpha = \xi\beta \in \Gamma(G)$. Thus $H \in \mathscr{F}(G)$.

The next result is analogous to the last indicating which induced subgraphs of a disconnected graph are smoothly embeddable. (Compare with Statement B.)

LEMMA 8. *Given* $H \in \mathscr{S}_0(G)$ *then* $H \in \mathscr{F}_0(G)$ *if and only if*

(1) $A_i' \in \mathscr{F}_0(A_i)$, $i \in \mathbf{N}_n$, *and*

(2) *if* $K \in \mathscr{S}_0(G)$ *is a copy of* H *in* G *and* β *is an isomorphism such that* $H^\beta = K$ *then given any* $i \in \mathbf{N}_n$ *and* $j \in \mathbf{N}_{c(A_i')}$, $A_i'^\beta \in \mathscr{S}_0(A_l)$ *implies* $A_i'^\beta \in \mathscr{S}_0(A_l)$ *and* $A_l \cong A_i$.

The proof is also analogous to that of Lemma 7.

The following lemma is based on one of Grant (1976) which shows that the relationship between fixing and smoothly embeddable subgraphs is much simpler when the automorphism groups are suitably restricted.

LEMMA 9. *Given* $H \in \mathscr{S}(G)$,

$$\Gamma(L(G)) = \Gamma^*(G) \cong \Gamma(G) \quad \textit{and} \quad \Gamma(L(H)) = \Gamma^*(H) \cong \Gamma(H)$$

then

(1) *if* $H \in \mathscr{F}(G)$, *it follows that* $L(H) \in \mathscr{F}_0(L(G))$ *if and only if* $G^H = \varnothing$; *and*

(2) *if* $L(H) \in \mathscr{F}_0(L(G))$, *it follows that* $G^H = \varnothing$ *and* $H \in \mathscr{F}(G)$.

The proof is by the arguments of Grant's Lemma 12.

5. The relationship between fixing and smoothly embeddable subgraphs

We are now in a position to prove Theorems 1 and 2 which tell us for any graph G, which fixing subgraphs of G have line graphs smoothly embeddable in $L(G)$ and vice versa. Compare with Statements C and D.

THEOREM 1. *Given* $H \in \mathscr{F}(G)$ *then* $L(H) \in \mathscr{F}_0(L(G))$ *if and only if*

(1) *whenever a component* T *of* H *is isomorphic to* G_1, G_2 *or* K_4 *then* T *is either a component of* G *or* T *is a spanning subgraph of a component of* G *isomorphic to* K_4, *and*

(2) *whenever copies of* H *and graphs of* G^H *have between them components isomorphic to both* K_3 *and* $K_{1,3}$ *then these components are all components of* G *or these are all subgraphs of components of* G *isomorphic to* K_4.

PROOF. ($\Leftarrow$) We assume $H \in \mathscr{F}(G)$ but $L(H) \notin \mathscr{F}_0(L(G))$ and it is required to show that (1) or (2) does not hold. If

$$\Gamma(L(G)) = \Gamma^*(G) \cong \Gamma(G) \quad \text{and} \quad \Gamma(L(H)) = \Gamma^*(H) \cong \Gamma(H),$$

then $G^H \neq \emptyset$ by Lemma 9. Thus H and any $K \in G^H$ have between them a copy of both K_3 and $K_{1,3}$. Now by hypothesis and Lemma 5 it follows that (2) does not hold. Henceforth we assume that these statements about automorphism groups do not both hold.

Case 1. G connected.

1.1. $\Gamma^*(G) \ncong \Gamma(G)$. By Lemma 4, G is isomorphic to K_2, whence trivially $L(H) \in \mathscr{F}_0(L(G))$, contrary to hypothesis.

1.2. $\Gamma(L(G)) \neq \Gamma^*(G)$. By Lemma 5, G is isomorphic to one of the graphs G_1, G_2 or K_4. However, $\mathscr{F}(G_1) = \{G_1\}$ and $\mathscr{F}(G_2) = \{G_2\}$, so that in these cases $H \in \mathscr{F}(G)$ implies $L(H) \in \mathscr{F}_0(L(G))$, contrary to hypothesis. Moreover, if $G \cong K_4$, then by Lemma 6, $L(H) \in \mathscr{F}_0(L(G))$ and we again contradict our hypothesis.

1.3. $\Gamma^*(H) \ncong \Gamma(H)$ but $\Gamma(L(H)) = \Gamma^*(H)$ and $\Gamma(L(G)) = \Gamma^*(G) \cong \Gamma(G)$. By Corollary to Lemma 4, H has at least one component isomorphic to K_2 or at least two components isomorphic to K_1. As $L(H) \notin \mathscr{F}_0(L(G))$ there exists $L(K) \cong L(H)$ where $L(K) \in \mathscr{S}_0(L(G))$ and an isomorphism η_1 such that $L(H)^{\eta_1} = L(K)$ but there is no $\mu_1 \in \Gamma(L(G))$ such that $\mu_1 | V(L(H)) = \eta_1$. Now suppose $K \cong H$. Then as $\Gamma(L(H)) = \Gamma^*(H)$, H does not have components isomorphic to G_1, G_2, K_4 or both K_3 and $K_{1,3}$ by Lemma 5. Therefore η_1 is induced by an isomorphism η such that $H^\eta = K$. As $H \in \mathscr{F}(G)$, $\eta \in \Gamma(G)$. Now $\Gamma(L(G)) \cong \Gamma(G)$ so η induces $\eta_2 \in \Gamma(L(G))$. Clearly $\eta_2 | V(L(H)) = \eta_1$, a contradiction. Thus $K \ncong H$. Now $L(K) \cong L(H)$ so H and K have between them components isomorphic to K_3 and $K_{1,3}$. As G is connected we deduce that (2) does not hold.

1.4. $\Gamma(L(H)) \neq \Gamma^*(H)$ and $\Gamma(L(G)) = \Gamma^*(G) \cong \Gamma(G)$. By Lemma 5 it follows that either (1) or (2) does not hold.

Case 2. G disconnected.

Let G have components A_i, $i \in \mathrm{N}_n$. Let A_i' be the subgraphs of H induced by $V(A_i)$ with components A_{i_j}, $j \in \mathrm{N}_{c(A_i')}$. As $L(H) \notin \mathscr{F}_0(L(G))$, we deduce from Lemma

8 that either (a) $L(A_i') \notin \mathscr{F}_0(L(A_i))$ for some $i \in \mathbf{N}_n$ or (b) $L(A_t') \in \mathscr{F}_0(L(A_t))$ for each $t \in \mathbf{N}_n$ but there exists $L(K) \in \mathscr{S}_0(L(G))$ and an isomorphism β_1 such that $L(H)^{\beta_1} = L(K)$ and for some $i \in \mathbf{N}_n$ and $j \in \mathbf{N}_{c(A_i')}$, $L(A_{i_j}') \in \mathscr{S}_0(L(A_l))$ but $L(A_i')^{\beta_1} \notin \mathscr{S}_0(L(A_l))$ or $L(A_l) \ncong L(A_i)$.

2.1. Suppose (a) holds. As $H \in \mathscr{F}(G)$, by Lemma 7 we deduce that $A_i' \in \mathscr{F}(A_i)$. Since $L(A_i') \notin \mathscr{F}_0(L(A_i))$ the previous argument for G connected shows that (1) or (2) does not hold for A_i and so not for G.

2.2. Suppose (b) holds, $K \cong H$ and (1) holds. Now β_1 is not induced by a permutation β of $V(H)$ such that $H^\beta = K$ for then $A_i'^\beta \notin \mathscr{S}(A_l)$ or $A_l \ncong A_i$ and so $H \notin \mathscr{F}(G)$ by Lemma 7, contrary to hypothesis. Thus we deduce from Lemma 5 that H has a component isomorphic to G_1, G_2, K_4 or has components isomorphic to both K_3 and $K_{1,3}$.

Let μ_1 be an isomorphism such that $L(H)^{\mu_1} = L(K)$ be the same as β_1 except that if a component A_{t_s}' of A_t' is isomorphic to G_1, G_2 or K_4 then $L(A_{t_s}')^{\mu_1} = L(A_{t_s}')^{\beta_1}$, but $\mu_1 \mid V(L(A_{t_s}'))$ is induced by an isomorphism between A_{t_s}' and B where $L(B) = L(A_{t_s})^{\mu_1}$.

If μ_1 is induced by a permutation μ of $V(H)$ such that $H^\mu = K$ then as under μ_1 and β_1 each component of $L(H)$ is mapped into the same component of $L(K)$ we have $A_i'^\mu \notin \mathscr{S}(A_l)$ or $A_l \ncong A_i$, so $H \notin \mathscr{F}(G)$ by Lemma 7, contrary to hypothesis. Thus μ_1 and so β_1 maps components of $L(H)$ corresponding to components of H isomorphic to K_3 and $K_{1,3}$ into components of $L(K)$ corresponding to components of K isomorphic to $K_{1,3}$ and K_3 respectively. As H and K have the same number of components isomorphic to K_3 and the same number of components isomorphic to $K_{1,3}$, there exists an isomorphism η_1 which is the same as μ_1 except that components of $L(H)$ corresponding to components of H isomorphic to K_3 and $K_{1,3}$, are mapped into components of $L(K)$ corresponding to components of K isomorphic to K_3 and $K_{1,3}$ respectively. Clearly η_1 is induced by a permutation η of $V(H)$ such that $H^\eta = K$. Thus as $H \in \mathscr{F}(G)$, by Lemma 7 we deduce that $A_i^\eta = A_m$ for some $m \in \mathbf{N}_n$.

2.2.1. Consider $m = l$. Then $L(A_l) \cong L(A_i)$ and since (b) holds, $L(A_i')^{\beta_1} \notin \mathscr{S}_0(L(A_l))$. It follows that β_1 maps a component of $L(A_i')$ isomorphic to K_3 onto an induced subgraph of $L(A_r)$ for some $r \neq l$. But $L(A_{i_j}')^{\beta_1} \in \mathscr{S}_0(L(A_l))$ and thus $c(A_i') \geqslant c(L(A_i')) \geqslant 2$. Consequently (2) does not hold.

2.2.2. Consider $m \neq l$. Then we deduce that $L(A_{i_j}') \cong K_3$. If $L(A_i')^{\beta_1} \notin \mathscr{S}_0(L(A_l))$, then $c(L(A_i')) \geqslant 2$ and so (2) does not hold. If $L(A_i) \ncong L(A_l)$ and $c(L(A_i')) \geqslant 2$ then again (2) does not hold. If $L(A_i) \ncong L(A_l)$ and $c(L(A_i')) = 1$ then it follows that $L(A_{i_j}') = L(A_i') \cong K_3$. Now $L(A_i) \cong K_3$ or $L(K_4)$ since otherwise (2) does not hold. Let B_r be the subgraph of K induced by $V(A_r)$, $r \in \mathbf{N}_n$, with components B_{r_j}, $j \in \mathbf{N}_{c(B_r)}$. As $L(A_i') \in \mathscr{S}_0(L(A_l))$ it follows that $L(A_i')^{\beta_1} = L(B_{l_k}) \cong K_3$ forso me $k \in \mathbf{N}_{c(B_l)}$. Assume $L(A_i) \cong K_3$. Then as $L(A_l) \ncong L(A_i)$ we have $L(A_l) \ncong K_3$ but $K_3 \cong L(B_{l_k}) \in \mathscr{S}_0(L(A_l))$ so B_{l_k} is a component of K for which (2) does not hold. Assume $L(A_i) \cong L(K_4)$. Then as $L(A_l) \ncong L(A_i)$ we have $L(A_l) \ncong L(K_4)$ but $K_3 \cong L(B_{l_k}) \in \mathscr{S}_0(L(A_l))$ and so B_{l_k} is a component of K for which (2) does not hold.

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2.3. Suppose (b) holds, $K \not\cong H$ and (1) holds. By considering Lemma 3, we deduce that $K \in G^H$. We assume further that (2) holds and show that this leads to a contradiction. Thus the components isomorphic to K_3 in copies of $L(H)$ in $\mathscr{S}_0(L(G))$ are either (i) all components of $L(G)$ or (ii) all subgraphs of components in $L(G)$ isomorphic to $L(K_4)$. Without loss of generality suppose (i) $\{r: L(A_r) \cong K_3\} = \mathbf{N}_m$ or (ii) $\{r: L(A_r) \cong L(K_4)\} = \mathbf{N}_m$.

2.3.1. Suppose given any $r \in \mathbf{N}_m$ and $j \in \mathbf{N}_{c(A_r')}$ with $A'_{r_j} \not\cong K_1$, that $L(A'_{r_j})^{\beta_1} \in \mathscr{S}_0(L(A_s))$ for some $s \in \mathbf{N}_m$. Then counting components we must have

$$\sum_{r \in \mathbf{N}_m} c(L(A'_r)) \leqslant \sum_{r \in \mathbf{N}_m} c(L(B_r)).$$

Now (i) $L(A'_r) \cong K_3, K_2, K_1$ or $\varnothing$, or (ii) $L(A'_r) \cong K_3, P_3, K_2, 2K_1, K_1$ or $\varnothing$, $r \in \mathbf{N}_m$. Now suppose for some $p \in \mathbf{N}_n \setminus \mathbf{N}_m$ and $w \in \mathbf{N}_{c(A_p')}$ that $L(A'_{p_w})^{\beta_1} \in \mathscr{S}_0(L(A_q))$ for some $q \in \mathbf{N}_m$. Then either $L(A'_t) \cong \varnothing$ for some $t \in \mathbf{N}_m$ or $L(A'_x)^{\beta_1} \in \mathscr{S}_0(L(A_z))$ and $L(A'_y)^{\beta_1} \in \mathscr{S}_0(L(A_z))$ for some $x, y, z \in \mathbf{N}_m$, $x \neq y$. If the former, then

$$L(A'_{p_w}) \cong D \in \mathscr{S}_0(L(A_t))$$

and as $A'_t \cong 3K_1$ or $4K_1$ we deduce that $H \notin \mathscr{F}(G)$ contrary to hypothesis. If the latter, then it follows that $B_z \cong 2K_1$ and $L(A'_x) \cong L(A'_y) \cong K_1$. Therefore

$$A'_x \cong A'_y \cong K_2 \cup 2K_1.$$

Consequently $H \notin \mathscr{F}(G)$ contrary to hypothesis. Thus for each $r \in \mathbf{N}_n \setminus \mathbf{N}_m$, $j \in \mathbf{N}_{c(A_r')}$ such that $L(A'_{r_j}) \not\cong \varnothing$, $L(A'_{r_j})^{\beta_1} \in \mathscr{S}_0(L(A_s))$ for some $s \in \mathbf{N}_n \setminus \mathbf{N}_m$.

Let $\gamma_1 = \beta_1 | V(L(H \setminus \bigcup_{r \in \mathbf{N}_m} A'_r))$. Let δ_1 be the same as γ_1 except that if $A'_r \cong G_1, G_2$ or K_4 and $L(A'_r)^{\gamma_1} = L(B_s)$ then $\delta_1 | V(L(A'_r))$ is induced by an isomorphism between A'_r and B_s. As $L(H \setminus \bigcup_{r \in \mathbf{N}_m} A'_r)$ has no components isomorphic to K_3 it follows that δ_1 is induced by an isomorphism δ such that

$$[H \setminus \bigcup_{r \in \mathbf{N}_m} A'_r]^\delta = K \setminus \bigcup_{r \in \mathbf{N}_m} B_r$$

but $A_i'^{\delta} \notin \mathscr{S}(A_l)$ or $A_i \not\cong A_l$ and thus by Lemma 7, $H \setminus \bigcup_{r \in \mathbf{N}_m} A'_r \notin \mathscr{F}(G \setminus \bigcup_{r \in \mathbf{N}_m} A_r)$ which implies $H \notin \mathscr{F}(G)$ contrary to hypothesis.

2.3.2. Suppose for some $p \in \mathbf{N}_m$ and $j \in \mathbf{N}_{c(A_p')}$ that $L(A'_{p_j})^{\beta_1} \in \mathscr{S}_0(L(A_s))$ for some $s \in \mathbf{N}_n \setminus \mathbf{N}_m$. Let

$$P = \{A'_r: r \in \mathbf{N}_m \text{ and } L(A'_{r_s})^{\beta_1} \in \mathscr{S}_0(L(A_t)) \text{ where } s \in \mathbf{N}_{c(A_r')} \text{ and } t \in \mathbf{N}_n \setminus \mathbf{N}_m\}$$

and let $\mathbf{N}^P = \{r: A'_r \subseteq P\}$. Let

$$Q = \{B_t: t \in \mathbf{N}_m \text{ and } L(A'_{r_s})^{\beta_1} \in \mathscr{S}_0(L(A_t)) \text{ where } r \in \mathbf{N}_n \setminus \mathbf{N}_m \text{ and } s \in \mathbf{N}_{c(A_r')}\}$$

and let $\mathbf{N}^Q = \{t: A'_t \subseteq Q\}$. Let $P_1 \subseteq K$ be such that $L(P_1) = L(P)^{\beta_1}$ and let $Q_1 \subseteq H$ be such that $L(Q_1)^{\beta_1} = L(Q)$ choosing P_1 and Q_1 without isolated vertices. Let

$R = H \setminus [\bigcup_{r \in \mathbf{N}_m} A'_r \cup Q_1]$ and $S = K \setminus [\bigcup_{r \in \mathbf{N}_m} B_r \cup P_1]$. As $L(K) \cong L(H)$ and

$$L(\bigcup_{r \in \mathbf{N}_m} A'_r \cup Q_1)^{\beta_1} = L(\bigcup_{r \in \mathbf{N}_m} B_r \cup P_1)$$

it follows that $L(R)^{\beta_1} = L(S)$.

Suppose $|\mathbf{N}^P| \neq |\mathbf{N}^Q|$. Then if $|\mathbf{N}^P| < |\mathbf{N}^Q|$ we deduce from 2.3.1 that $H \notin \mathscr{F}(G)$ contrary to hypothesis. Therefore $|\mathbf{N}^P| > |\mathbf{N}^Q|$ and either (A) $L(B_t) \cong \varnothing$ for some $t \in \mathbf{N}_m$ or (B) $L(A'_{z_1}) \cong L(A'_{z_2}) \cong K_1$ for some $z \in \mathbf{N}_m$ and $L(A_{z_1})^{\beta_1} \in \mathscr{S}_0(L(A_x))$ and $L(A_{z_2})^{\beta_1} \in \mathscr{S}_0(L(A_y))$ for some $x \neq y$ in $\mathbf{N}_m$. (A) Assume $B'_t \cong 3K_1$ then as $H \ncong K$ we deduce that $A'_q \cong P_3 \cup K_1$ for some $q \in \mathbf{N}_m$ and as there must be more isolated vertices in H it follows that $H \notin \mathscr{F}(G)$ contrary to hypothesis. Assume $B'_t \cong 4K_1$ or (B) holds. Let $M \in \mathscr{S}(G)$ be such that

$$L(M) = \bigcup_{r \in \mathbf{N}_m \setminus \mathbf{N}^P} L(A'_r) \cup L(P_1) \cup L(S) \cup L(E),$$

where $L(E) \cong L(Q_1)$ and $E \in \mathscr{S}(\bigcup_{r \in \mathbf{N}^P} A_r)$ which is possible as $|\mathbf{N}^P| > |\mathbf{N}^Q|$ and as $A_r \cong K_4$, $r \in \mathbf{N}_m$. Let ω be an isomorphism such that $H^\omega = M$ with the action of ω on P and R inducing that of β_1 on $L(P)$ and $L(R)$. Then $A'^{\omega}_{p_j} \subseteq A_s$ where $A^p \nsubseteq A_s$ so $H \notin \mathscr{F}(G)$ contrary to hypothesis.

Hence we may assume $|\mathbf{N}^P| = |\mathbf{N}^Q|$. As Q_1 consists of components isomorphic to (i) P_3 or K_2, (ii) P_4, P_3 or K_2 and as A_r, $r \in \mathbf{N}^P$, is isomorphic to (i) K_3 or $K_{1,3}$, or (ii) K_4 and $|\mathbf{N}^P| = |\mathbf{N}^Q|$ there exists $Q_2 \subseteq \bigcup_{r \in \mathbf{N}^P} A_r$ such that $Q_2 \cong Q_1$. Now let μ_1 be an isomorphism the same as β_1, except that if $A'_r \cong G_1$, G_2 or K_4 and $L(A'_r)^{\beta_1} = L(B_s)$ then $\mu_1 | V(L(A'_r))$ is induced by an isomorphism between A'_r and B_s.

Let ξ_1 be an isomorphism such that $L(A_r)$, $r \in \mathbf{N}_m \setminus \mathbf{N}^P$ is fixed,

$$\xi_1 | V(L(P)) = \mu_1 | V(L(P)), \quad L(Q_1)^{\xi_1} = L(Q_2) \quad \text{and} \quad \xi_1 | V(L(R)) = \mu_1 | V(L(R)).$$

Now

$$L(H)^{\xi_1} = \bigcup_{r \in \mathbf{N}_n \setminus \mathbf{N}^P} L(A'_r) \cup L(P) \cup L(Q_1) \cup L(R)]^{\xi_1}$$

$$= \bigcup_{r \in \mathbf{N}_n \setminus \mathbf{N}^P} L(A'_r) \cup L(P_1) \cup L(Q_2) \cup L(S) = L(M)$$

for some $M \in \mathscr{S}(G)$. As components of $L(H)$ isomorphic to K_3 are fixed by ξ_1 and as components $L(A'_r) \cong L(G_1)$, $L(G_2)$ or $L(K_4)$ of $L(H)$ are mapped by ξ_1 such that $\xi_1 | V(L(A'_r))$ is induced by an isomorphism between A'_r and the appropriate component of M, we thus have by construction that ξ_1 is induced by an isomorphism ξ such that $H^\xi = M$. Now for $A'_{r_s} \subseteq P$ we have $A'^{\xi}_{r_s} \subseteq A_t$ where $t \notin \mathbf{N}_m$ and thus $A_r \nsubseteq A_t$ contrary to $H \in \mathscr{F}(G)$.

($\Rightarrow$) We now show the necessity of (1) and (2). Assume $H \in \mathscr{F}(G)$ and $L(H) \in \mathscr{F}_0(L(G))$. Then $\Gamma(L(H)) = \Gamma(L(H), L(G)) | V(L(H))$ and by applying Lemma 5 we deduce that (1) holds.

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Suppose H has components A'_{i_k} and A'_{j_m} isomorphic to K_3 and $K_{1,3}$ respectively. As $\Gamma(L(H)) = \Gamma(L(H), L(G)) \mid V(L(H))$ we have

$$\Gamma(L(A'_{i_k})) = \Gamma(L(A'_{i_k}), L(A_i)) \mid V(L(A'_{i_k})) \text{ and } \Gamma(L(A'_{j_m})) = \Gamma(L(A'_{j_m}), L(A_j)) \mid V(L(A'_{j_m})).$$

If $i = j$ then $L(A'_i)$ has components $L(A'_{i_k}) \cong L(A'_{i_m}) \cong K_3$ and each vertex of $L(A'_{i_k})$ must have the same neighbourhood in $L(A_i)$ since $\Gamma(L(A'_{i_k})) \cong \Gamma(K_3)$. As $L(A'_{i_k})$ has a non-empty neighbourhood in $L(A_i)$ the subgraph $L(M)$, $M \subseteq A_i$, of $L(A_i)$ induced by $L(A'_{i_k})$ and one of its neighbourhood vertices is isomorphic to K_4. Now $L(M) \cong K_4$ implies $M \cong K_{1,4}$ but $A'_{i_k} \cong K_3$ implies $A'_{i_k} \nsubseteq M$, a contradiction. Whence $i \neq j$. Thus $L(A'_i) \cong L(A'_j) \cong K_3$. As $\Gamma(L(H)) = \Gamma(L(H), L(G)) \mid V(L(H))$ we have $L(A_i) \cong L(A_j)$. Also it follows that $A'_i \cong K_3 \cup xK_1$ and $A'_j \cong K_{1,3} \cup yK_1$, $x, y \geqslant 0$. As $H \in \mathscr{F}(G)$ and so $\Gamma(H) \leqslant \Gamma(G)$ we have $x = 0$ or $y = 0$. Since $L(A_i) \cong L(A_j)$ are connected as are A_i and A_j we deduce that $A_i \cong K_3$ and $A_j \cong K_{1,3}$ or $A_i \cong A_j \cong K_4$. Clearly the same results holds for any copy of H in G.

Suppose $A'_{i_k} \cong K_3$ and $K_{1,3} \cong B_{l_m} \subseteq K \in G^H$ where B_{l_m} is a component of B_l, the subgraph of K induced by $V(A_l)$. Let γ_1 be an isomorphism such that $L(H)^{\gamma_1} = L(K)$ with $L(A'_{i_k})^{\gamma_1} = L(B_{l_m})$. Thus as $L(H) \in \mathscr{F}_0(L(G))$ we have

$$L(A'_i) = L(A_i) \cong L(A_l) = L(B_l).$$

Hence $A_i \cong A_l$ or $A'_{i_k} = A_i \cong K_3$ and $B_{l_m} = A_l \cong K_{1,3}$. Assume $A_i \cong A_l$. Then $A'_i \cong K_3 \cup xK_1$ and $B_l \cong K_{1,3} \cup (x-1)K_1$, $x \geqslant 1$. As $A'_i \in \mathscr{F}(A_i)$ we deduce $A_i \cong K_{x+3}$. Now $L(A'_i) \cong K_3 \in \mathscr{F}_0(L(K_{x+3}))$ implies $x = 1$. Thus $A_i \cong A_l \cong K_4$.

Suppose $A'_{i_k} \cong K_{1,3}$ and $K_3 \cong B_{l_m} \subseteq K \in G^H$. Let δ_1 be an isomorphism such that $L(H)^{\delta_1} = L(K)$ with $L(A'_{i_k})^{\delta_1} = L(B_{l_m})$. Thus as $L(H) \in \mathscr{F}_0(L(G))$ we have $L(A'_i) = L(A_i) \cong L(A_l) = L(B_l)$. Hence $A_i \cong A_l$ or $A'_{i_k} = A_i \cong K_{1,3}$ and $B_{l_m} = A_l \cong K_3$. Assume $A_i \cong A_l$. Then $A'_i \cong K_{1,3} \cup (x-1)K_1$ and $B_l \cong K_3 \cup xK_1$, $x \geqslant 1$. As $A'_i \in \mathscr{F}(A_i)$ we deduce that $A_i \cong K_{x+3}$ or $K_{1,x+2}$. Now $B_{l_m} \cong K_3 \nsubseteq K_{1,x+2}$ so $A_l \cong A_i \ncong K_{1,x+2}$. Thus $A_i \cong K_{x+3}$. Now $L(A'_i) \cong K_3 \in \mathscr{S}_0(L(K_{x+3}))$ implies $x = 1$. Hence $A_i \cong A_l \cong K_4$. We conclude that (2) holds.

THEOREM 2. *Given $L(H) \in \mathscr{F}_0(L(G))$ then $H \in \mathscr{F}(G)$ if and only if*

(1) *if M is a component of G and M' the subgraph of H induced by $V(M)$, then (M, M') is not isomorphic to (G_1, P_4), (G_2, C_4) or (G_2, P_4); and*

(2) *all isolated vertices of H share the same open neighbourhood in G and the vertices of any component of H isomorphic to K_2 share the same closed neighbourhood in G; and*

(3) *G does not have components isomorphic to K_3 and $K_{1,3}$ with subgraphs induced in H isomorphic to P_3 and $P_3 \cup K_1$ respectively.*

PROOF. Let H be as given.

($\Rightarrow$) Then if (1) does not hold, by inspection $H \notin \mathscr{F}(G)$. If (2) or (3) does not hold then $\Gamma(H) \nleq \Gamma(G)$ and so again $H \notin \mathscr{F}(G)$.

($\Leftarrow$) Now suppose that $L(H) \in \mathscr{F}_0(L(G))$ but that $H \notin \mathscr{F}(G)$. By Lemma 9 we can assume that statements $\Gamma(L(G)) = \Gamma^*(G) \cong \Gamma(G)$ and $\Gamma(L(H)) = \Gamma^*(H) \cong \Gamma(H)$ do not both hold.

Case 1. Assume to begin with that G is connected. Arguments of Grant's Theorem 2 give the required result.

Case 2. Now suppose that G is not connected. Let G have components A_i, $i \in \mathbf{N}_n$. Let A'_i be the subgraphs of H induced by $V(A_i)$ with components A'_{i_j}, $j \in \mathbf{N}_{c(A_i')}$. As $H \notin \mathscr{F}(G)$, we deduce from Lemma 7, that either (a) $A'_i \notin \mathscr{F}(A_i)$ for some $i \in \mathbf{N}_n$ or (b) $A'_r \in \mathscr{F}(A_r)$ for each $r \in \mathbf{N}_n$ but there exists $K \in \mathscr{S}(G)$ and a permutation β of $V(H)$ such that $H^\beta = K$ and for some $i \in \mathbf{N}_n$, $A'^{\beta}_{i_j} \subseteq A_l$ but $A^\beta_i \neq A_l$.

Suppose (a) holds. Then $A'_i \notin \mathscr{F}(A_i)$ for some $i \in \mathbf{N}_n$. As $L(H) \in \mathscr{F}_0(L(G))$, by Lemma 8 $L(A'_i) \in \mathscr{F}_0(L(A_i))$ and the previous argument for G connected shows that (1) or (2) does not hold for A_i, and so not for G.

Suppose (b) holds. Then β induces an isomorphism β_1 such that $L(H)^{\beta_1} = L(K)$. As $L(H) \in \mathscr{F}_0(L(G))$, by Lemma 8 if $A'_{r_s} \ncong K_1$ and $A'_{r_s} \subseteq A_l$ then $L(A'_r)^{\beta_1} \in \mathscr{S}_0(L(A_l))$ and $L(A_r) \cong L(A_l)$. Assume $L(A'_i) \cong \varnothing$. Then we deduce that (2) does not hold. Now we can assume without loss of generality that $A'_{i_j} \ncong K_1$, so $L(A'_i) \in \mathscr{S}_0(L(A_l))$ and $L(A_i) \cong L(A_l)$. Therefore $A^\beta_{i_s} \subseteq A_l$ for each $A'_{i_s} \ncong K_1$, $s \in \mathbf{N}_{c(A_i')}$. Assume $A_i \cong A_l$. Then there exists $A'_{i_k} \cong K_1$ such that $A'^{\beta}_{i_k} \subseteq A_m$, $m \neq l$, and we deduce that there are isolated vertices in H not sharing the same open neighbourhood in G so (2) does not hold. Assume $A_i \ncong A_l$. Then either (i) $A_i \cong K_3$ and $A_l \cong K_{1,3}$ or (ii) $A_i \cong K_{1,3}$ and $A_l \cong K_3$. If (i) holds then $A'_i \cong P_3$ for otherwise $A'_l \notin \mathscr{F}(A_l)$. Thus $A'_l \cong P_3 \cup K_1$ and so (3) does not hold. If (ii) holds then $A'_i \cong P_3 \cup K_1$ and consequently $A'_l \cong P_3$ so (3) does not hold.

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Department of Mathematics
University of Melbourne
Parkville, Victoria 3052
Australia