TENSOR PRODUCTS OF POSITIVE DEFINITE QUADRATIC FORMS III

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In the previous papers [2], [3] we treated the following two questions. Let $L, M, N$ be positive definite quadratic lattices over $\mathbb{Z}$:

(i) If $L, M$ are indecomposable, then is $L \otimes M$ indecomposable?

(ii) Does $L \otimes M \cong L \otimes N$ imply $M \cong N$?

In this paper we discuss the uniqueness of decompositions with respect to tensor products. Our aim is to prove the following two theorems.

**Theorem 1.** Let $L_i, M_i$ be indecomposable positive definite binary quadratic lattices with $L_i = \tilde{L}_i$, $M_i = \tilde{M}_i$, $m(L_i) = m(M_i) = 1$. For any isometry $\sigma : \otimes_{i=1}^n L_i \cong \otimes_{i=1}^n M_i$, we have $\sigma = \otimes_{i=1}^n \sigma_i$ where $\sigma_i$ is an isometry from $L_i$ on $M_i$, changing the suffix if necessary.

**Theorem 2.** Let $L_i, M_i$ be positive definite quadratic lattices with $[L_i; \tilde{L}_i] < \infty$, $[M_i; \tilde{M}_i] < \infty$. Assume that

(i) $L_i$ (resp. $M_i$) is of $E$-type except at most one,

(ii) $sL_i = sM_i = \mathbb{Z}$, and $m(L_i), m(M_i)$ are prime numbers, and

(iii) $\tilde{L}_i, \tilde{M}_i$ are indecomposable.

Then for any isometry $\sigma : \otimes_{i=1}^n L_i \cong \otimes_{i=1}^n M_i$ we have $n = m$ and $\sigma = \otimes \sigma_i$, where $\sigma_i$ is an isometry from $L_i$ on $M_i$, changing the suffix if necessary.

We must explain the notations and terminologies in two theorems. By a positive definite quadratic lattice we mean a lattice in a positive definite quadratic space over the rational number field $\mathbb{Q}$. For any quadratic space we use the same letter $Q$, which are the corresponding quadratic form and bilinear form $(2B(x, y) = Q(x + y) - Q(x) - Q(y))$. Let $L$ be a positive definite quadratic lattice; then $sL$ denotes $\{ \sum B(x_i, y_i) ; x_i, y_i \in L \}$ and we put $m(L) = \min Q(x)$ where $x$ runs over non-zero elements of $L$. $\mathfrak{M}(L)$ stands for $\{ x \in L ; Q(x) = m(L) \}$, and $\tilde{L}$ is the sub-

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module of $L$ spanned by elements of $\mathbb{M}(L)$. $L$ is called $E$-type if every element of $\mathbb{M}(L \otimes M)$ is of the form $x \otimes y$ ($x \in L$, $y \in M$) for any positive definite quadratic lattice $M$. If either $sL \subseteq \mathbb{Z}$, $m(L) \leq 6$ or rank $L \leq 42$, then $L$ is of $E$-type [1].

§1. In this section we define a weighted graph and prove some properties.

DEFINITION. Let $A$ be a finite set, and $[\ , \ ]$ be a mapping from $A \times A$ into $\{0 < t < 1\}$ such that

(i) $[a, a'] = 1$ if and only if $a = a'$, and

(ii) $[a, a'] = [a', a]$ for $a, a'$ in $A$.

Then we call $(A, [\ , \ ])$ or simply $A$ a weighted graph. A weighted graph $A$ is called connected if for any $x, y$ in $A$ there are elements $z_t$ of $A$ such that $x = z_0$, $y = z_r$, and $[z_i, z_{i+1}] \neq 0$ ($i = 1, \cdots, r - 1$). For weighted graphs $A, B$ we define the direct product $A \times B$ by $[(a, b), (a', b')] = [a, a'][b, b']$ ($a, a' \in A$, $b, b' \in B$); then $A \times B$ is clearly a weighted graph. It is also clear that the direct product of connected weighted graphs is connected. A bijection $f$ from $A$ on $B$ is called an isometry if $f$ satisfies $[f(a), f(a')] = [a, a']$ for $a, a' \in A$.

LEMMA 1. Let $A, B, C$ be connected weighted graphs, and let $\sigma$ be an isometry from $A \times B$ on $A \times C$. If there are $b_0 \in B, c_0 \in C$ such that $\sigma(a, b_0) = (f(a), c_0)$ for every $x$ in $A$, then $f$ is an isometry from $A$ on $A$ and there is an isometry $g$ from $B$ on $C$ with $\sigma(a, y) = (f(a), g(y))$ ($x \in A$, $y \in B$).

Proof. Since $\sigma$ is a bijection and $A$ is a finite set, $f$ is a bijection of $A$. Moreover for $a, a'$ in $A$ we have $[a, a'] = [(a, b_0), (a', b_0)] = [(f(a), c_0), (f(a'), c_0)] = [f(a), f(a')]$. This means that $f$ is an isometry of $A$. Multiplying $f^{-1} \times \text{id}_C$ to $\sigma$, we have only to prove the lemma in case of $f = 1$. Put $S = \{\tilde{B} \subset B; \sigma(a, b) = (a, c) \text{ for every } a \in A \text{ and } b \in \tilde{B},$ where $c$ is only dependent of $b\}$. $S$ is not empty since $S \ni \{b_0\}$. Take an element $B'$ in $S$ such that $\#B' \geq \#\tilde{B}$ for $\tilde{B}$ in $S$. If $B' = B$, then we have $\sigma(a, b) = (a, g(b))$ for $a \in A$, $b \in B$. It is easy to see that $g$ is an isometry from $B$ on $C$, and this completes the proof. Now we assume $B' \neq B$. We have to show that this implies a contradiction. Define a subset $C'$ by $\sigma(A, B') = (A, C')$. Put $m = \max [b, b']$ where $b \in B'$, $b' \in B'$, and we may assume $m \geq \max [c, c']$ where $c \in C'$, $c' \in C'$, taking $\sigma^{-1}$ instead of $\sigma$ if necessary. Since $B$ is connected, $m$ is positive.
Put $m = [b, b']$ ($b \in B'$, $b' \in B'$) and take any element $x \in A$. Put $\sigma(x, b') = (x', c)$; then $c$ is not in $C'$ since $c \in C'$ implies $(x, b') \in \sigma^{-1}(A, C') = (A, B')$. Putting $\sigma(x, b) = (x, c)$, we have $m = [b, b'] = [(x, b), (b, b')] = [(x, c), (x', e_c)] = [x, x'][c, c]$. If $x \neq x'$, then $0 < [x, x'] < 1$ implies a contradiction $m < [c, c] < m$. Hence $x' = x$ follows. Thus we get $\sigma(x, b') = (x, c(x))$ ($c(x) \in C$) for every $x$ in $A$. For $x, y$ in $A$ with $[x, y] \neq 0$, $[x, y] = [(x, b'), (y, b')] = [(x, c(x)), (y, c(y))] = [x, y][c(x), c(y)]$ implies $[c(x), c(y)] = 1$, and so $c(x) = c(y)$. Since $A$ is connected, this yields that $c(x)$ in $C$ is independent of $x$ in $A$, and then it implies a contradiction $B' \cup \{b'\} \in S$ and $\#(B' \cup \{b'\}) > \#B'$.

**Lemma 2.** Let $L$ be a positive definite quadratic lattice. For $x, y$ in $L$ we put $[x, y] = \|B(x, y)\|/m(L)$. Then $(\mathfrak{M}(L)/\pm1, [\ , \ ])$ is a weighted graph and it is connected if and only if $L$ is indecomposable.

**Proof.** Take $x, y$ in $\mathfrak{M}(L)$; then $x = \pm y$ if and only if $|B(x, y)| = m(L)$. Moreover $B(x, y)^2 \leq Q(x)Q(y) = m(L)^2$ implies that $\mathfrak{M}(L)/\pm1$ is a weighted graph. The latter part is obvious.

We say that $(\mathfrak{M}(L)/\pm1, [\ , \ ])$ is a weighted graph associated to $L$.

§2. Let $L_i, M_j$ be positive definite quadratic lattices and let $\sigma$ be an isometry from $\bigotimes_{i=1}^n L_i$ on $\bigotimes_{j=1}^n M_j$. Suppose that

(i) $\mathfrak{M}(\bigotimes L_i) = \bigotimes \mathfrak{M}(L_i), \mathfrak{M}(\bigotimes M_j) = \bigotimes \mathfrak{M}(M_j),$

(ii) $[L_i : \tilde{L}_i], [M_j : \tilde{M}_j] < \infty$ for every $i, j$,

(iii) $\mathfrak{M}(L_i)/\pm1, \mathfrak{M}(M_j)/\pm1$ are connected weighted graphs for every $i, j$.

Let $A, B, A_i, B_i$ be weighted graphs associated to $\bigotimes L_i, \bigotimes M_i, L_i, M_i$ respectively. Then $\sigma$ induces an isometry from $A = \prod_{i=1}^n A_i$ on $B = \prod_{i=1}^n B_i$ which is denoted by the same letter $\sigma$.

**Theorem.** If it follows that $n = m$, $\sigma = \prod_{i=1}^n \sigma_i$ where $\sigma_i$ is an isometry from $A_i$ on $B_i$, changing the suffix if necessary, then we have $\sigma = \bigotimes_{i=1}^n \mu_i$ where $\mu_i$ is an isometry from $L_i$ on $M_i$, changing the suffix if necessary.

**Proof.** We may assume $\sigma = \prod \sigma_i$ where $\sigma_i$ is an isometry from $A_i$ on $B_i$. By the same letter $\sigma_i$, we denote a mapping from $\mathfrak{M}(L_i)$ on $\mathfrak{M}(M_i)$ which induces an isometry $\sigma_i$ from $A_i = \mathfrak{M}(L_i)/\pm1$ on $B_i = \mathfrak{M}(M_i)/\pm1$. Fix any element $e_i$ in $\mathfrak{M}(L_i)$ ($i \geq 2$). Then $\sigma(e \otimes e_2 \otimes \cdots \otimes e_n) = \pm \sigma_i(e)$
\( \otimes \sigma(e_1) \otimes \cdots \otimes \sigma_n(e_n) \) holds for every \( e \) in \( \mathcal{M}(L) \). Putting \( \sigma(1) = \mu_1 \), then \( \sigma(e \otimes e_2 \otimes \cdots \otimes e_n) = \mu_1(1) \otimes \sigma(e_2) \otimes \cdots \otimes \sigma_n(e_n) \) for any \( e \) in \( \mathcal{M}(L) \).

This means that \( \mu_1 \) is an isometry from \( \tilde{L}_1 \) onto \( \tilde{M}_1 \). Since \( \mathcal{M}_1 \otimes \sigma(e_2) \otimes \cdots \otimes \sigma_n(e_n) \) is a direct summand of \( \otimes \mathcal{M}_1 \) and \([L_1; \tilde{L}_1] < \infty \), \( \mu_1 \) is an isometry from \( L_1 \) into \( M_1 \). Similarly we get an isometry \( \mu_4 \) from \( L_4 \) into \( M_4 \) so that \( \sigma(e \otimes \cdots \otimes e_n) = \pm \mu_4(e) \otimes \cdots \otimes \mu_n(e_n) \) for \( e \) in \( \mathcal{M}(L) \), where \( \pm \) may depend on the choice of \( e \).

This means that \( \mu_1 \) is an isometry from \( \tilde{L}_1 \) onto \( \tilde{M}_1 \).

§3. First we discuss the case of Theorem 1. Let \( L \) be an indecomposable binary positive definite quadratic lattice with \( L = L, m(L) = 1 \). Then \( L \) has a basis \( \{e_1, e_2\} \) so that \( Q(e_1, e_2) = 1 \), \( 0 < B(e_1, e_2) \leq \frac{1}{2} \), and moreover we have \( \mathcal{M}(L) = \{\pm e_1, \pm e_2, \pm (e_1 - e_2)\} \) \((\pm (e_1 - e_2) \) happens only when \( B(e_1, e_2) = \frac{1}{2} \). Let \( A_L \) be a weighted graph associated to \( L \); then \( A_L \) is connected. \( \# A_L \) is two for \( B(e_1, e_2) < \frac{1}{2} \). If \( B(e_1, e_2) = \frac{1}{2} \), then \( \# A_L = 3 \) and \( \{a_i, a_j\} = \frac{1}{2} \) for \( i \neq j \) where we put \( A_L = \{a_i, a_j, a_k\} \).

Let \( L_i, M_i, \sigma \) be as in Theorem 1; then \( L_i, M_i \) are of \( E\)-type, and define \( A, A_i, B, B_i \) and \( \sigma \) as in §2; then we have

**Lemma 3.** \( \sigma = \prod \sigma_i \) where \( \sigma_i \) is an isometry from \( A_i \) on \( B_i \), changing the suffix if necessary.

**Proof.** We prove this by the induction with respect to \( \# A \). Put \( m = \max \{a, a'\} = \max \{b, b'\} \) where \( a, a' \in A, a \neq a' \) and \( b, b' \in B, b \neq b' \). Since \( A, B_i \) are indecomposable, we get \( 0 < m \leq \frac{1}{2} \). Take \( a \neq a' \) in \( A \) with \( [a, a'] = m \). Putting \( a = \prod a_i, \quad a' = \prod a'_i, \quad m = \prod [a_i, a'_i] \) follows.

Noting \( [a_i, a'_i] < 1 \) for \( a_i \neq a'_i \), the maximality of \( m \) implies that there is an index \( j \) such that \( [a_i, a'_i] = 1 \), i.e., \( a_i = a'_i \) for \( i \neq j \), and \( a_j \neq a'_j \). We may assume \( j = 1 \), and similarly \( \sigma(a) = \prod b_i, \quad \sigma(a') = \prod b'_i, \quad b_i = b'_i \) for \( i > 1 \) and \( b_1 \neq b'_1 \). Then \( m = \prod [a_i, a'_i] = \prod [b_i, b'_i] \) follows. If \( m < \frac{1}{2} \), then \( A_i = \{a_i, a'_i\}, \quad B_i = \{b_i, b'_i\} \) and \( \sigma(A_i) \times \prod A_i = B_i \times \prod B_i \). Hence Lemma 1 and the assumption of the induction completes the proof.

Suppose \( m = \frac{1}{2} \); then there is an element \( a''_1 \) in \( A_1 \) so that \( A_1 = \{a_1, a'_1, a''_1\} \) and \( [a_i, a''_1] = [a'_i, a''_1] = \frac{1}{2} \). Put \( \sigma(a''_1) \times \prod A_i = \prod b''_i \); then \( [a_i, a''_1] = \prod A_i = \prod b''_i \) and \( [a'_i, a''_1] = \prod B_i = \prod b''_i \). Hence Lemma 1 and the assumption of the induction completes the proof.
[a', a''] = \frac{1}{2} implies \[ b_i, b'_i \] \prod_{i=2}^{n} [b''_i, b_i] = [b'_i, b'_i] \prod_{i=2}^{n} [b'_i, b_i] = \frac{1}{2}. Suppose \[ b_i = b''_i \]; then \[ \prod_{i=2}^{n} [b''_i, b_i] = \frac{1}{2} \], and so \[ [b'_i, b'_i] = 1 \], that is, \[ b'_i = b'_i = b_i \]. This is a contradiction. Hence we have \[ b_i \neq b''_i \], and then \[ [b_i, b'_i] = \frac{1}{2} \]. Therefore \[ b''_i = b_i \] for \( i \geq 2 \) and \( \sigma(A_i \times \prod_{i=2}^{n} a_i) = B_i \times \prod_{i=2}^{n} b_i \). This completes the proof as above.

Now Theorem 1 follows from Theorem in §2.

Next we discuss the case of Theorem 2.

**Lemma 4.** Let \( a_i, b_i \in \mathbb{Z} \) and \( 0 < b_i < a_i \), and let \( a_i \) be prime. Put \( \prod_{i=1}^{n} (b_i/a_i) = b/a, \ (a, b) = 1 \). Then \( a > a_i \) for some \( i \) if \( n \geq 2 \).

*Proof.* We may suppose \( a_i \leq \ldots \leq a_n \), and assume \( a_i \leq a_i \) for any \( i \). Since \( a_i \) divides \( \prod a_i \), we have \( a_i = a_i \). \( b_i \prod_{i=2}^{n} (b_i/a_i) = b \) and \( a_i \mid b_i \) imply \( \prod_{i=2}^{n} a_i \mid \prod_{i=2}^{n} b_i \). This contradicts \( 0 < b_i < a_i \).

**Lemma 5.** Let \( A_i, B_i \) be connected weighted graphs with \( \#A_i > 1 \), \( \#B_i > 1 \), and let \( p_i, q_i \) be primes. Suppose

\[
\{[x, y]; x, y \in A_i\} \subset \{a/p_i; a = 0, 1, \ldots, p_i\}
\]

and

\[
\{[x, y]; x, y \in B_i\} \subset \{b/q_i; b = 0, 1, \ldots, q_i\}.
\]

If \( \sigma \) is an isometry from \( \prod_{i=1}^{n} A_i \) on \( \prod_{i=1}^{n} B_i \), then \( n = m \) and \( \sigma = \prod \sigma_i \) where \( \sigma_i \) is an isometry from \( A_i \) on \( B_i \), changing the suffix if necessary.

*Proof.* We prove by the induction with respect to \( \# \prod_{i=1}^{n} A_i \). Since \( A_i \) is connected and \( \#A_i > 1 \), for any element \( a \) in \( A_i \) there is an element \( a' \) in \( A_i \) such that \( 0 < [a, a'] < 1 \). If \( [a, a'] \neq 0, 1 \) for \( a, a' \) in \( A_i \), then the denominator of \( [a, a'] \) is a prime \( p_i \). Without loss of generality we may assume \( p_i = \ldots = p_k < p_{k+1} \leq \ldots \leq p_m, q_i = \ldots = q_k < q_{k+1} \leq \ldots \leq q_m \). Put \( A = \prod_{i=1}^{n} A_i, B = \prod_{i=1}^{n} B_i \), and fix any element \( a = \prod a_i \) of \( A \). Suppose that the minimal value of the denominator of \( [a, a'] \) with \( [a, a'] \neq 0, 1 \) \( (a' \in A) \) is taken by \( a' = \prod a'_i \in A \). Then the above remark and Lemma 4 imply \( a'_i = a_i \) for \( i \neq j \), and \( a'_j \neq a_j \) for some \( j \) and so the minimal value is obviously \( p_i \), and \( j \leq k \). On the other hand, by virtue of Lemma 4 and the connectedness of \( A_i \), it is easy to see that \( A_i \times \cdots A_k \times a_{k+1} \times \cdots \times a_n \) is a subset of \( A \) consisting of elements \( z \) such that there are elements \( z_i, z_{i+1} \) of \( A \) satisfying that the denominator of \( [z_i, z_{i+1}] \) is \( p_i \) for \( i = 1, \ldots, r - 1 \).

From the similar argument for \( \sigma(a) = \prod b_i \) in \( B \) follows that the
corresponding minimal denominator is \( q_1 \), and the corresponding subset of \( B \) for \( q_1 \), \( \sigma(a) \) instead of \( p_1, a \) is \( B_1 \times \cdots \times B_h \times b_{k+1} \times \cdots \times b_m \). Since \( \sigma \) is an isometry, we have \( p_1 = q_1 \), and so \( \sigma(A_1 \times \cdots \times A_k \times a_{k+1} \times \cdots \times a_n) = B_1 \times \cdots \times B_h \times b_{k+1} \times \cdots \times b_m \) by their definitions. This implies that \( A_1 \times \cdots \times A_k \) and \( B_1 \times \cdots \times B_h \) are isometric. Therefore Lemma 1 and the assumption of the induction completes the proof if \( n > k \). Thus we may suppose \( n = k \). Then \( \sigma(A) = \prod_{i=1}^{k} B_i \times b_{k+1} \times \cdots \times b_m \) implies \( h = m \). Moreover we have \( n = m \) since the maximal value of the denominators of \( [a, a'] \) \((a, a' \in A)\) (resp. \([b, b'] \) \((b, b' \in B))\) is \( p_i^* \) (resp. \( p_i^n \)), and they are equal. For simplicity we put \( p_i = p \) in the following.

(1) Assume that \( A_1 \) contains distinct three elements \( x_1, x_2, x_3 \) such \([x_1, x_2, x_3]_{[x_2, x_3, x_1]} \neq 0 \). Fix any element \( a_i \) in \( A_i \) \((i \geq 2)\), and put \( \sigma(a_k \prod_{i=1}^{k} a_i) = \prod_{j=1}^{m} b_{k,j} \) \((b_{k,j} \in B_j)\); then \([x_k, x_h] = \prod_{j=1}^{m} [b_{k,j}, b_{h,j}] \). Since \( 0 < [b_{i,j}, b_{k,j}] \leq 1 \) and the denominator of \([b_{i,j}, b_{k,j}] \) is \( p \) if \( b_{i,j} \neq b_{k,j} \), comparing the denominators of both sides, we have \( b_{i,j} = b_{k,j} \) for any \( j \) except one index if \( i \neq k \). Without loss of generality we may assume \( b_{1,1} \neq b_{2,1}, b_{1,t} = b_{2,t} \) \((i \geq 2)\). Similarly we may assume \( b_{z,k} = b_{z,k} \) for \( k \neq t \). If \( t \geq 2 \), then \( b_{1,t} = b_{2,t} = b_{s,j} \) for \( j \neq 1, t \). This implies \([x_1, x_3] = [b_{1,1}, b_{1,t}][b_{1,t}, b_{2,t}] = [b_{1,1}, b_{2,t}][b_{1,t}, b_{2,t}] \). The denominator of the left (resp. right) side is \( p \) (resp. \( p_i^* \)) since \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{s,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction. Hence we get \( t = 1 \), and so \( b_{1,1} \neq b_{2,1}, b_{2,t} \neq b_{2,t} \). This is a contradiction.

We note that the denominator of the left side is \( p \). If \( b_{i} \neq b_{j} \) for \( i \neq j, j \), then \( [y_{k}, b_{j}] = 1 \), and so \( y_{i} = y_{j} = y_{2} \). This implies a contradiction \( x_{1} = x_{2} = x_{3} \). Hence \( b_{i} \neq b_{j} \) for \( i \neq j, j \). \( b_{i}' = y_{i} \) implies \( b_{j} = z_{1} \) (\( \neq z_{2}, z_{3} \)), and so we get \( y_{i} = y_{j} = y_{2} \). This is a contradiction. Hence we have \( b_{i}' = y_{i} \), and similarly \( b_{i}' = y_{i} \). This contradicts \( x_{1} = x_{2} = x_{3} \). Hence \( j \) equals 1, and we may put \( \sigma(x_{k} \times \prod_{i=1}^{k} a_{i} \times a_{n} = [x_{k}, x_{h}]_{[x_{k}, z_{k}]} \prod_{i=1}^{m} [b_{i}, b_{j}] \). Putting \( k = h \), and comparing the denominators we have \( b_{i} = b_{j}' \) for any \( i \geq 2 \) except at most one \( i \). Putting \( k = h \), the denominator of the left hand equals \( p_{i}^* \). Hence the exceptional suffix exists. Then putting \( k = h \) again, we have \( y_{k} = z_{k} \).
for $k = 1, 2, 3$. Thus we have $\sigma(x_k \times \prod_{i=2}^{k-1} a_i \times a'_i) = y_k \times \prod_{i=2}^{k-2} b'_i$. Doing the similar operations for $a_i, a'_i$, we have $\sigma(x_k \times \prod_{i=2}^{k-1} A_i) \subset y_k \times \prod_{i=2}^{k-2} B_i$, since $A_i$ is connected. Similarly $\sigma^{-1}(y_k \prod_{i=2}^{k-2} b_i) = x_k \prod_{i=2}^{k-1} a_i$ and $[y_k, y_2] \times [y_1, y_1] \neq 0$ imply $\sigma^{-1}(y_k \prod_{i=2}^{k-2} B_i) \subset x_k \times \prod_{i=2}^{k-1} A_i$, and so $\sigma(x_k \times \prod_{i=2}^{k-1} A_i) = y_k \times \prod_{i=2}^{k-2} B_i$. This implies $\prod_{i=2}^{k-1} A_i \cong \prod_{i=2}^{k-2} B_i$, and then Lemma 1 and the assumption of the induction completes the proof.

(ii) Suppose that $A_i$ contains distinct four elements $x_i$ such that $[x_1, x_2], [x_2, x_3], [x_1, x_3] \neq 0, [x_1, x_3] = [x_2, x_1] = [x_2, x_3] = 0$. Fix any element $a_i$ in $A_i$ ($i \geq 2$). Put $\sigma(x_k \times \prod_{i=1}^{k-1} a_i) = \prod_{i=1}^{k-1} b_i, i$; then $[x_k, x_i] = [\prod_{i=1}^{k-1} b_i, b_i]$ $\neq 0$. Since the denominator of the left hand is $p$ for $k \neq 1$, there is a number $t_k$ such that $b_{k,t} = b_{i,t}$ for $i \neq t_k$, and $b_{k,t} \neq b_{i,t}$.

a) Suppose that $t_2, t_3, t_4$ are distinct.

$x_2, x_3 = 0$ implies $[b_{3,t}, b_{4,t}] = 0$ for some $i$. Since $b_{k,j} = b_{i,j}$ for $j \neq t_2, t_3, t_4$, $i$ equals $t_2, t_3$ or $t_4$. If $i = t_2$, then $b_{k,t_2} = b_{i,t} = b_{i,t}$ implies a contradiction $[b_{i,t}, b_{i,t}] = 1$. Similarly $i = t_3$ or $i = t_4$ implies a contradiction.

b) Suppose that $t_2 = t_3 \neq t_4$.

$x_2, x_3 = 0$ implies $[b_{3,t}, b_{4,t}] = 0$ for some $i$. $b_{k,j} = b_{i,j}$ for $j \neq t_k$ yields $i = t_2$ or $t_4$. $i = t_2$ implies $b_{t_2} = b_{i,t} = b_{i,t}$, and so $[b_{i,t}, b_{i,t}] = 0$. This contradicts $[x_2, x_3] = 0$. Similarly $i = t_4$ is a contradiction.

Similary $t_2 \neq t_3 = t_4$ or $t_2 = t_4 \neq t_3$ implies a contradiction. Hence we have $t_2 = t_3 = t_4 = 1$ (say). Thus we may assume $\sigma(x_k \times \prod_{i=1}^{k-1} a_i) = y_k \times \prod_{i=1}^{k-2} b_i, i$ ($y_k \in B_1, b_i \in B_i$). Take an element $a_i'$ in $A_i$ with $[a_n, a'_n] \neq 0, 1, \text{ and put } \sigma(x_k \times \prod_{i=2}^{k-1} a_i \times a'_i) = z_k \prod_{i=1}^{k-2} b'_i, i$. Assume $j \neq 1$; then $[x_k \times \prod_{i=2}^{k-1} a_i, x_i] \times \prod_{i=2}^{k-1} a_i, a'_i] = [x_k, x_i] \times [a_n, a'_n] = [y_k, b_i][b_{j, z_i} \prod_{i=1}^{k-2} b_i, b_i] \neq 0$ implies $[b_j, z_i] \neq 0$ ($t = 1, 2, 3, 4$), $[b_i, b'_j] \neq 0$ for $i \neq 1, j$. Similarly $[x_k, x_i] \neq 0$ implies $[y_k, b'_i] \neq 0$ ($k = 1, 2, 3, 4$). This means $[x_k, x_i][a_n, a'_n] \neq 0$ for any $k, t$ and contradicts $[x_2, x_3] = 0$. Thus we have $j = 1$, and $[x_k, x_i]$. $[a_n, a'_n] = [y_k, z_i] \times \prod_{i=2}^{k-2} b_i, b_i$. Since the denominator of the left hand for $k = 1, t = 2$ is $p^2$, there is at least one suffix $i$ such that $b_i \neq b'_i$. Moreover the denominator of the left side for $k = t$ is $p$. Hence there is no such suffix except $i$, and this yields $[y_k, x_k] = 1$, i.e., $y_k = z_k$. As the proof of the case (i) we have $\sigma(x_k \times \prod_{i=2}^{k-2} A_i) = y_k \times \prod_{i=2}^{k-2} B_i$ and complete the proof for the case (ii) by the induction and Lemma 1.

For a weighted graph $W$ we make a usual graph, joining two ele-

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ments $x, y$ with $[x, y] \neq 0$. Then, by virtue of (i), (ii), we may assume that $A_i, B_i$ do not contain subgraphs \( \begin{array}{c}
 extracting and drawing an similar triangle here \end{array} \). Hence $A_i, B_i$ are \( \begin{array}{c}
 extracting and drawing an similar triangle here \end{array} \) or \( \begin{array}{c}
 extracting and drawing an similar triangle here \end{array} \) as graphs.

(iii) Suppose that $A_i$ contains three distinct elements $x_1, x_2, x_3$ such that $[x_1, x_2] \neq 0$, $[x_2, x_3] \neq 0$, $[x_1, x_3] = 0$, i.e., \( \begin{array}{c}
 extracting and drawing an similar triangle here \end{array} \). Take any element $a_i$ in $A_i$, and put $\sigma(x_k \prod_{t=2}^n a_t) = \prod_{t=1}^n b_{k,t}$ $(b_{k,t} \in B_t)$. Comparing the denominators of $[a, a'] = f(a) \prod_{t=2}^n b_t$, we have numbers $q, s$ so that $b_{1,t} = b_{2,t}$ for $i \neq q$, $b_{1,t} = b_{s,t}$ for $i \neq s$. $q \neq s$ implies $b_{1,t} = b_{s,t}$, and then we have \( \begin{array}{c}
 extracting and drawing an similar triangle here \end{array} \). This contradicts \( \begin{array}{c}
 extracting and drawing an similar triangle here \end{array} \). Thus we may assume $q = s = 1$ (say), and $\sigma(x_k \prod_{t=2}^n a_t) = y_k \prod_{t=2}^n b_t$ $(y_k \in B_1, b_t \in B_t)$. Doing the similar thing for $\begin{array}{c}
 extracting and drawing an similar triangle here \end{array}$, we have $\sigma(x_k \prod_{t=2}^n a_t) = z_k \prod_{t=2}^n b_t \quad (z_k \in B_j, b_t \in B_t)$ for $k = 2, 3, 4$. Comparing the case $k = 2, 3$, we get $z_3 = b_3 = z_3$ if $j \neq 1$. This is a contradiction, and so $j = 1$. This means $b_i = b_i$ for $i \geq 2$ and $\sigma(x_k \prod_{t=2}^n a_t) = z_k \prod_{t=2}^n b_t$. Since $A_i$ is \( \begin{array}{c}
 extracting and drawing an similar triangle here \end{array} \), we have $\sigma(a \prod_{t=2}^n a_t) = f(a) \prod_{t=2}^n b_t$ for any $a$ in $A_i$, that is, $\sigma(A_i \prod_{t=2}^n a_t) \subset B_i \prod_{t=2}^n b_t$. Similarly we have $\sigma^{-1}(B_i \prod_{t=2}^n b_t) \subset A_i \prod_{t=2}^n a_t$ and so $\sigma(A_i \prod_{t=2}^n a_t) = B_i \prod_{t=2}^n b_t$. Lemma 1 and the induction complete the proof.

(iv) By virtue of (i), (ii), (iii) we have only to prove the case that $\# A_i \neq \# B_i = 2$. Put $m = \max [a, a'] (a, a' \in A, a \neq a')$ and assume $m = [a, a']$ for $a = \prod_{t=1}^n a_t, a' = \prod_{t=1}^n a'_t$. Since $[a_t, a'_t] < 1$ if $a_t = a'_t$, by the definition, there is a suffix $t$ so that $a_t = a'_t$ for $i \neq t$ and $a_t \neq a'_t$. Putting $\sigma(a) = \prod b_t, \sigma(a') = \prod b'_t$, there is a suffix $s$ so that $b_s = b'_s$ for $i \neq s$, and $b_s \neq b'_s$. Without loss of generality we may assume $t = s = 1$; then $A_i = \{a_t, a'_t\}$, $B_i = \{b_t, b'_t\}$ and $[a_t, a'_t] = [b_t, b'_t] = m$. Hence $A_i \cong B_i$ and $\sigma(A_i \prod_{t=2}^n a_t) = B_i \prod_{t=2}^n b_t$. Lemma 1 and the assumption of the induction complete the proof of Lemma 4.

To complete the proof of Theorem 2 we need only to prove that the cardinalities of weighted graphs associated to $L_t, M_t$ are not 1. It follows immediately from the assumption (ii).

Let $L$ be an indecomposable positive definite quadratic lattice, and...
put $A = \mathbb{M}(L)/\pm 1$ and we consider $A$ as a weighted graph by $[x, y] = |B(x, y)|/m(L)$ for $x, y \in \mathbb{M}(L)/\pm 1$ as above. We call such a weighted graph a quadratic weighted graph associated to $L$. Then the following questions arise.

(i) Let $A_i, B_i$ be connected quadratic weighted graphs and $f$ be an isometry from $\prod_{t=1}^n A_t$ on $\prod_{t=1}^n B_t$. What is a sufficient condition to the following assertion?

$n = m$ and $f = \prod f_t$ (changing the suffix if necessary), where $f_t$ is an isometry from $A_t$ on $B_t$.

(ii) Let $L$ be an indecomposable positive definite quadratic lattice with $L = \tilde{L}$, and let $A$ be an associated quadratic weighted graph. If $A \cong B \times C$ where $B, C$ are quadratic weighted graphs, then is there a decomposition $L \cong M \otimes N$ so that $B$ (resp. $C$) is a quadratic weighted graph associated to $M$ (resp. $N$)?

Remark 1. For $M \cong \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$, $N \cong \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}$, associated quadratic graphs are isometric but $M, N$ are not isometric.

Remark 2. Let $L$ be a positive definite quadratic lattice with $L = \tilde{L}$, $m(L) = 1$, and assume that $\mathbb{M}(L)/\pm 1 = A \times B$ where $A, B$ are weighted graphs with $\# A, \# B > 1$. Put $\mathbb{M}(L)/\pm 1 = \{e_t\}$ and $e_t = (a_t, b_t)$ ($a_t \in A$, $b_t \in B$). Suppose that there is a mapping $s_1$ (resp. $s_2$) from $A \times A$ (resp. $B \times B$) into $\{\pm 1\}$ so that $s_1(a, a) = s_2(b, b) = 1$ for every $a$ in $A$ and every $b$ in $B$, and $B(e_i, e_j) = s_1(a_i, a_j)s_2(b_i, b_j)[a_i, a_j][b_i, b_j]$ for any $i, j$. Then we can show that there are positive definite quadratic lattices $M, N$ such that $L \cong M \otimes N$, $M = \tilde{M}$, $N = \tilde{N}$, $m(M) = m(N) = 1$ and $A, B$ are quadratic graphs associated to $M, N$ respectively. The assumption on $s_1, s_2$ is not satisfied for a decomposable lattice $M \perp N$ in Remark 1.

References