Finite groups in which a given property of two-generator subgroups is a transitive relation are investigated. We obtain a description of such groups and prove in particular that every finite soluble-transitive group is soluble. A classification of finite nilpotent-transitive groups is also obtained.

1. Introduction

Let $\mathcal{X}$ be a group theoretical class. A group $G$ is said to be $\mathcal{X}$-transitive (or an $\mathcal{XT}$-group) if for all $x, y, z \in G \setminus \{1\}$ the relations $(x, y) \in \mathcal{X}$ and $(y, z) \in \mathcal{X}$ imply $(x, z) \in \mathcal{X}$. In graph theoretical terms, let $\Gamma_{\mathcal{X}}(G)$ be the simple graph whose vertices are the nontrivial elements of $G$, and $a$ and $b$ are connected by an edge if and only if $(a, b) \in \mathcal{X}$. Then $G$ is an $\mathcal{XT}$-group precisely when all the connected components of $\Gamma_{\mathcal{X}}(G)$ are complete graphs. Several authors have studied $\mathcal{XT}$-groups for some special classes $\mathcal{X}$. When $\mathcal{X}$ is the class of all Abelian groups, these groups are also known as commutative-transitive groups or CT-groups. Weisner [10] has shown that finite CT-groups are either soluble or simple. Finite nonabelian simple CT-groups have been classified by Suzuki [6]. These are precisely $\text{PSL}(2, 2^f)$, where $f > 1$. A characterisation of finite soluble CT-groups has been given by Wu [11] who has also obtained information on locally finite CT-groups and polycyclic CT-groups. When $\mathcal{X} = \mathcal{N}_c$, the class of all groups which are nilpotent of class $\leq c$, similar results have been obtained in [1].

The purpose of this note is to obtain a description of finite $\mathcal{XT}$-groups for the group theoretical classes $\mathcal{X}$ having the following properties:

$(\ast) \quad$ $\mathcal{X}$ is subgroup closed, it contains all finite Abelian groups and is bigenetic in the class of all finite groups.

Here a class $\mathcal{X}$ is said to be bigenetic (a terminology due to Lennox [4]) in the class of all finite groups when a finite group $G$ is in $\mathcal{X}$ if and only if all its two-generator subgroups are. Examples of classes satisfying $(\ast)$ are the class of all Abelian groups, all nilpotent...
groups, all supersoluble groups and all soluble groups. First we show that if $\mathfrak{X}$ is a class satisfying ($\ast$), then every finite $\mathfrak{X}T$-group which does not belong to $\mathfrak{X}$ is either a Frobenius group with kernel and complement belonging to $\mathfrak{X}$, or it has no normal $\mathfrak{X}$-subgroups, that is, it is $\mathfrak{X}$-semisimple as defined in [5]. We also show that in several cases, for example, in the soluble or supersoluble case, the second possibility does not occur. As a consequence we obtain that a finite group is soluble if and only if it is soluble-transitive. In the case when $\mathfrak{X} = \mathfrak{N}$, the class of all nilpotent groups, there exist simple $\mathfrak{N}T$-groups. We obtain a complete classification of finite $\mathfrak{N}T$-groups which generalises some results of [11].

2. Results

Given a group theoretical class $\mathfrak{X}$, let $R_\mathfrak{X}(G)$ be the product of all normal $\mathfrak{X}$-subgroups of $G$ (the $\mathfrak{X}$-radical of $G$). In general $R_\mathfrak{X}(G)$ does not belong to $\mathfrak{X}$. Our first result shows that this is however true within the class of all finite $\mathfrak{X}$-transitive groups when $\mathfrak{X}$ satisfies the properties ($\ast$).

**Lemma 2.1.** Let $\mathfrak{X}$ be a class of groups satisfying ($\ast$) and let $G$ be a finite $\mathfrak{X}$-group. Then $R_\mathfrak{X}(G)$ is an $\mathfrak{X}$-group.

**Proof:** Let $M$ and $N$ be normal $\mathfrak{X}$-subgroups of $G$. It suffices to show that $MN$ also belongs to $\mathfrak{X}$. Suppose first that $M \cap N \neq 1$ and let $x \in M \cap N \setminus \{1\}$. First note that for any $m \in M \setminus \{1\}$ and $n \in N \setminus \{1\}$ we have that $\langle m, x \rangle$ and $\langle x, n \rangle$ belong to $\mathfrak{X}$. As $G$ is an $\mathfrak{X}$-group, we conclude that $\langle m, n \rangle$ is an $\mathfrak{X}$-group. Now let $m_1, m_2 \in M \setminus \{1\}$ and $n \in N \setminus \{1\}$. We may suppose that $m_1n \neq 1$. Then $\langle m_1n, m_1 \rangle = \langle m_1, n \rangle$ is in $\mathfrak{X}$ and $\langle m_1, m_2 \rangle$ is in $\mathfrak{X}$. Thus it follows that $\langle m_1n, m_2 \rangle$ also belongs to $\mathfrak{X}$. Similarly we can prove that $\langle mn_1, n_2 \rangle$ is in $\mathfrak{X}$ for every $m \in M \setminus \{1\}$ and $n_1, n_2 \in N \setminus \{1\}$. Now take $m_1, m_2 \in M \setminus \{1\}$ and $n_1, n_2 \in N \setminus \{1\}$ and suppose that $m_1n_1 \neq 1, m_2n_2 \neq 1$. Then $\langle m_1n_1, m_2 \rangle \in \mathfrak{X}$, $\langle m_2, m_2n_2 \rangle \in \mathfrak{X}$, hence $\langle m_1n_1, m_2n_2 \rangle$ belongs to $\mathfrak{X}$. This shows that every two-generator subgroup of $MN$ belongs to $\mathfrak{X}$. Since $\mathfrak{X}$ is bigenetic in the class of all finite groups, we get that $MN$ is an $\mathfrak{X}$-group, as required.

Suppose now that $M \cap N = 1$. Then $[M, N] = 1$. As above it suffices to prove that every two-generator subgroup of $MN$ is in $\mathfrak{X}$. At first let $m_1, m_2 \in M \setminus \{1\}$ and $n \in N \setminus \{1\}$. Then the groups $\langle m_1n, n \rangle = \langle m_1, n \rangle$ and $\langle n, m_2 \rangle$ are Abelian, hence they belong to $\mathfrak{X}$. By the transitivity we have that $\langle m_1n, m_2 \rangle$ belongs to $\mathfrak{X}$. Similar argument shows that $\langle mn_1, n_2 \rangle \in \mathfrak{X}$ for every $m \in M \setminus \{1\}$ and $n_1, n_2 \in N \setminus \{1\}$. From this it follows that if $m_1, m_2 \in M \setminus \{1\}$ and $n_1, n_2 \in N \setminus \{1\}$, then $\langle m_1n_1, m_2 \rangle$ and $\langle n_1, m_2n_2 \rangle$ are in $\mathfrak{X}$, hence $\langle m_1n_1, m_2n_2 \rangle$ is also in $\mathfrak{X}$. This concludes the proof.

**Theorem 2.2.** Let $\mathfrak{X}$ be a class of groups satisfying ($\ast$). Let $G$ be a finite $\mathfrak{X}$-group. Then one of the following holds.

(i) $G$ belongs to $\mathfrak{X}$.

(ii) $G$ is $\mathfrak{X}$-semisimple.
(iii) $G$ is a Frobenius group with kernel and complement both belonging to $\mathcal{X}$.

**Proof:** Let $R$ be the $\mathcal{X}$-radical of $G$. By Lemma 2.1, $R$ belongs to $\mathcal{X}$. If $R = G$, then $G$ belongs to $\mathcal{X}$. If $R = 1$, then $G$ is $\mathcal{X}$-semisimple. So from now on we assume that $1 \neq R \neq G$.

Let $y \in R \setminus \{1\}$ and suppose that there exists $a \in C_G(y) \setminus R$. Then $(a, y)$ is Abelian, hence it belongs to $\mathcal{X}$. As $R$ is an $\mathcal{X}$-group and $G$ is an $\mathcal{X}$-group, we have that $(a, h)$ is in $\mathcal{X}$ for every $h \in R$. By conjugation we get that $(a^x, h) \in \mathcal{X}$ for every $x \in G$ and $h \in R$. Since $G$ is an $\mathcal{X}$-group, we get that

$$
(a^x, a^z) \in \mathcal{X}
$$

for every $x, z \in G$. We claim that $(u, v) \in \mathcal{X}$ for every $u, v \in a^G$. To prove this, we first introduce some notation. For $u \in a^G$ let $r$ be the smallest integer such that $u$ can be written as $a^{g_1} \cdots a^{g_r}$ for some $g_1, \ldots, g_r \in G$. Then we say that $u$ is of weight $r$ and denote $\text{wt}(u) = r$. The proof of our claim goes by induction on $\text{wt}(u) + \text{wt}(v)$. If $\text{wt}(u) + \text{wt}(v) \leq 2$, then the claim follows from (1). Suppose that the claim holds true for all $u, v \in a^G$ with $\text{wt}(u) + \text{wt}(v) \leq l$. Let now $u, v \in a^G$ be such that $\text{wt}(u) + \text{wt}(v) = l + 1$. Without loss of generality we may assume that $\text{wt}(u) > 1$ and $v \neq 1$. Then we can write $u = u'a^{g}$ for some $g \in G$ and $u' \in a^G \setminus \{1\}$ with $\text{wt}(u') = \text{wt}(u) - 1$. We have that $(u, a^{g}) = (u', a^{g})$ belongs to $\mathcal{X}$ by the induction assumption. For the same reason we have that $(a^{g}, v) \in \mathcal{X}$. As $G$ is an $\mathcal{X}$-group, we conclude that $(u, v) \in \mathcal{X}$. This proves that every two-generator subgroup of $a^G$ belongs to $\mathcal{X}$. As $\mathcal{X}$ is bigenetic in the class of all finite groups, we get $a^G \in \mathcal{X}$, hence $a \in R$, a contradiction. By Satz 8.5 in [2] we have that $G$ is a Frobenius group and $R$ is its kernel. In particular, it follows from here that $R$ is nilpotent. Let $H$ be its complement. Then $H$ is an $\mathcal{X}$-group with nontrivial centre. It follows from here that every two-generator subgroup of $H$ belongs to $\mathcal{X}$, hence $H \in \mathcal{X}$. \qed

A characterisation of $\mathcal{X}$-semisimple $\mathcal{X}$-groups is usually not easy and depends heavily on a choice of the class $\mathcal{X}$; see [1, 6, 11]. In the case of Frobenius groups we provide a general characterisation of $\mathcal{X}$-groups. At first we prove the following technical result.

**Lemma 2.3.** Let $\mathcal{X}$ satisfy $(\ast)$. Let $G$ be a finite $\mathcal{X}$-group and $H$ an $\mathcal{X}$-subgroup of $G$. Then

$$
C_1^\mathcal{X}(H) = \{ x \in G : (x, h) \in \mathcal{X} \text{ for some } h \in H \setminus \{1\} \}
$$

is an $\mathcal{X}$-subgroup of $G$ containing $H$.

**Proof:** Clearly $C_1^\mathcal{X}(H)$ contains $H$. Let $x, y \in C_1^\mathcal{X}(H) \setminus \{1\}$. Then there exist $h, k \in H \setminus \{1\}$ such that $(x, h) \in \mathcal{X}$ and $(y, k) \in \mathcal{X}$. Since $(h, k) \in \mathcal{X}$, we get that $(x, y)$ also belongs to $\mathcal{X}$. If $xy \neq 1$, then $(xy, y) = (x, y)$ belongs to $\mathcal{X}$, hence also $(xy, k) \in \mathcal{X}$. Thus $xy \in C_1^\mathcal{X}(H)$. Note also that every two-generator subgroup of $C_1^\mathcal{X}(H)$ is in $\mathcal{X}$, hence $C_0^\mathcal{X}(H)$ also belongs to $\mathcal{X}$. \qed
PROPOSITION 2.4. Let $\mathcal{X}$ be a group theoretical class satisfying $(\ast)$. Let $G$ be a Frobenius group with kernel $F$ and complement $H$. Then $G$ is an $\mathcal{X}T$-group if and only if $C^G_F(F)$ and $C^G_F(H)$ are $\mathcal{X}$-groups.

PROOF: Let $\mathcal{X}$ and $G$ be as above. If $G$ is an $\mathcal{X}T$-group, then it follows from Theorem 2.2 that $F$ and $H$ belong to $\mathcal{X}$. Consequently $C^G_F(F)$ and $C^G_F(H)$ are also $\mathcal{X}$-groups by Lemma 2.3. Conversely, suppose that $C^G_F(F)$ and $C^G_F(H)$ are $\mathcal{X}$-groups. Let $x, y, z \in G\setminus\{1\}$ and suppose that $(x,y) \in \mathcal{X}$ and $(y,z) \in \mathcal{X}$. Assume first that $y \notin F$. Then $x, z \in C^G_F(F)$ and consequently $(x,z) \in \mathcal{X}$. If $y \notin F$, then $y \in H$ for some $g \in G$. But then $x, z \in C^G_H(H^g) = (C^G_G(H))^g$, thus $(x,z)$ belongs to $\mathcal{X}$. Thus $G$ is an $\mathcal{X}T$-group. 

When $\mathcal{X}$ is the class of all Abelian groups, then all three possibilities of Theorem 2.2 can occur [6, 11]. In some cases, however, we can exclude the existence of $\mathcal{X}$-semisimple $\mathcal{X}T$-groups.

THEOREM 2.5. Let $\mathcal{X}$ be a class of groups satisfying $(\ast)$, and suppose that $\mathcal{X}$ contains all finite dihedral groups and that every finite $\mathcal{X}$-group is soluble. If $G$ is a finite $\mathcal{X}T$-group which is not in $\mathcal{X}$, then $G$ is a Frobenius group with complement belonging to $\mathcal{X}$. In particular, $G$ is soluble.

Before proving this result we mention here the well known Thompson's classification of minimal simple groups; that is, finite nonabelian simple groups all whose proper subgroups are soluble. It turns out [9] that every such group is isomorphic to one of the following groups.

(i) $\text{PSL}(2,p)$, where $p$ is a prime, $p > 3$ and $p^2 - 1 \neq 0 \mod 5$.
(ii) $\text{PSL}(2,2^f)$, where $f$ is a prime.
(iii) $\text{PSL}(2,3^f)$, where $f$ is an odd prime.
(iv) $\text{PSL}(3,3)$.
(v) $\text{Sz}(q)$, where $q = 2^{2n+1}$ and $2n + 1$ is a prime.

If $G = \text{PSL}(2,F)$ where $F$ is a Galois field of odd characteristic and $|F| > 5$, then $G$ can be generated by an involution and an element of even order. This can be easily seen as follows. Let $q = |F|$. By Dickson's theorem [2], $G$ contains elements $a$ and $b$ with $|a| = (q - 1)/2$ and $|b| = (q + 1)/2$. Note that precisely one of $|a|, |b|$ is even, without loss of generality we may assume that this is true for $|a|$. Then $N_G(a) = D_{q-1}$ and this is the only maximal subgroup of $G$ containing $a$; this follows from the proof of Dickson's theorem [2]. So if we choose any involution $u$ from $G/N_G(a)$, we have $(a,u) = G$, as required. A similar result holds true for $\text{PSL}(3,3)$ and $\text{Sz}(q)$. In the first case note that $\text{PSL}(3,3)$ can be generated by the canonical projections of matrices

$$
\begin{pmatrix}
0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{pmatrix} \quad \text{and} \quad 
\begin{pmatrix}
0 & 0 & -1 \\
1 & 0 & 0 \\
1 & -1 & 1
\end{pmatrix},
$$
which are of orders 2 and 8 in \( \text{PSL}(3, 3) \), respectively. For the Suzuki groups \( \text{Sz}(q) \) it follows from [8] that they can always be generated by an involution and an element of order 4. We summarise this in the following lemma.

**Lemma 2.6.** Let \( G \) be one of the following groups: \( \text{PSL}(2, F) \) where \( F \) is a Galois field of odd characteristic and \( |F| > 5 \), \( \text{PSL}(3, 3) \) or \( \text{Sz}(q) \). Then \( G \) can be generated by an involution and an element of even order.

Note that for the groups \( \text{PSL}(2, 2^f) \) the conclusion of the above lemma does not hold. In this case we have the following result that can be proved by straightforward calculation.

**Lemma 2.7.** Let \( G = \text{PSL}(2, 2^f), f > 1 \). Denote by \( \zeta \) a generator of \( \text{GF}(2^f) \) and let \( a, b \) and \( c \) be the elements of \( G \) which are projections of

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & \zeta \end{pmatrix},
\]

respectively. Then \( \langle a, b \rangle \) and \( \langle b, c \rangle \) are dihedral groups and \( \langle a, c \rangle = G \).

**Proof of Theorem 2.5:** We may suppose that \( G \) does not belong to \( \mathcal{X} \), hence \( R_X(G) \neq G \). If we prove that \( G \) is soluble, then \( R_X(G) \neq 1 \) and our claim follows from Theorem 2.2. So suppose that there exist finite insoluble \( \mathcal{X} \)-groups, and let \( G \) be a counterexample of minimal order. Then every proper subgroup of \( G \) is soluble. By Theorem 2.2 we have that \( R_X(G) = 1 \). Let \( R \) be the soluble radical of \( G \). Since \( \mathcal{X} \) contains all finite Abelian groups, we have that \( R = 1 \). It is now easy to see that \( G \) has to be simple. By Thompson's classification of minimal simple groups [9], \( G \) is isomorphic to one of the groups in the above mentioned list. By Lemma 2.7, \( G \) is not isomorphic to any of \( \text{PSL}(2, 2^f) \), where \( f \) is a prime. If \( G \) is one of the groups of Lemma 2.6, then \( G = \langle a, b \rangle \), where \( |a| = 2 \) and \( |b| = 2k, k > 1 \). We have that \( \langle a, b^k \rangle \) is a dihedral group and \( \langle b^k, b \rangle \) is a cyclic group, hence \( G \) is in \( \mathcal{X} \) by the \( \mathcal{X} \)-property, a contradiction. This concludes the proof.

Using Theorem 2.5, we obtain a rather surprising characterisation of finite soluble groups.

**Corollary 2.8.** Every finite soluble-transitive group is soluble.

Note that the class of all supersoluble groups also satisfies all the assumptions of Theorem 2.5. Thus we have the following.

**Corollary 2.9.** Let \( G \) be a finite supersoluble-transitive group. If \( G \) is not supersoluble, then \( G \) is a Frobenius group with supersoluble complement. In particular, \( G \) is always soluble.

In view of Corollary 2.8 we may ask if every finite supersoluble-transitive group is supersoluble. This is not true however, as the group \( A_4 \) shows. It is also not difficult
to find an example of a Frobenius group with supersoluble complement which is not
supersoluble-transitive. This example also shows that Proposition 2.4 is in a certain
sense best possible. Indeed, it is not possible to replace \( C^F_G(F) \) and \( C^Q_H(H) \) by \( F \) and \( H \),
respectively.

**Example 2.10.** Let \( A = \langle x \rangle \oplus \langle y \rangle \) be an elementary group of order 9 and let \( \alpha \) be the
automorphism of \( A \) given by the matrix

\[
\begin{pmatrix}
2 & 2 \\
2 & 1
\end{pmatrix}.
\]

Then \( \langle \alpha \rangle \) acts fixed-point-freely on \( A \). Let \( G = A \rtimes \langle \alpha \rangle \). This is a group of order
36 which is not supersoluble-transitive. To see this, note that \( \langle \alpha^2, (\alpha y)^2 \rangle \) is a dihedral
group, \( \langle (\alpha y)^2, \alpha y \rangle \) is cyclic, whereas \( \langle \alpha^2, \alpha y \rangle = G \) is not supersoluble. Denoting by \( \mathcal{G} \)
the class of all supersoluble groups, note that \( C^\mathcal{G}_G(\langle \alpha \rangle) \) has 20 elements and it is thus not
a subgroup of \( G \). On the other hand, \( C^\mathcal{G}_G(A) \) is a subgroup of index 2 in \( G \).

Theorem 2.5 cannot be applied in the case of \( \mathfrak{N}T \)-groups, where \( \mathfrak{N} \) denotes the class
of all nilpotent groups. Thus it is to be expected that there exist finite insoluble \( \mathfrak{N}T \)-
groups. This is confirmed by the following characterisation of finite \( \mathfrak{N}T \)-groups which is
essentially contained in [1]. We include a proof for the sake of completeness.

**Theorem 2.11.** Let \( G \) be a finite \( \mathfrak{N}T \)-group. Then one of the following holds.

(i) \( G \) is nilpotent.

(ii) \( G \) is a Frobenius group with nilpotent complement.

(iii) \( G \cong \text{PSL}(2, 2^f) \) for some \( f > 1 \).

(iv) \( G \cong \text{Sz}(q) \) with \( q = 2^{2n+1} > 2 \).

Conversely, every finite group under (i)–(iv) is an \( \mathfrak{N}T \)-group.

**Proof:** If \( G \) is soluble and not nilpotent, then the Fitting subgroup \( F \) of \( G \) is a
proper nontrivial subgroup of \( G \). By Theorem 2.2, \( G \) is a Frobenius group with nilpotent
complement. So suppose that \( G \) is not soluble. It is easy to see that in every finite \( \mathfrak{N}T \)-
group \( G \) the centralisers of nontrivial elements are nilpotent, that is, \( G \) is an CN-group.
By a result of Suzuki [7, Part I, Theorem 4], the centraliser of any involution in \( G \) is a
2-group. Let \( P \) and \( Q \) be any Sylow \( p \)-subgroups of \( G \) and suppose that \( P \cap Q \neq 1 \). Since
\( P \) and \( Q \) are nilpotent and \( G \) is an \( \mathfrak{N}T \)-group, we conclude that \( \langle P, Q \rangle \) is nilpotent. This
shows that the Sylow subgroups of \( G \) are independent. Combining Theorem 1 in Part I
and Theorem 3 in Part II of [7], we conclude that \( G \) has to be simple. Additionally, it
follows from [8] that \( G \) is isomorphic either to \( \text{PSL}(2, 2^f) \), where \( f > 1 \), or to \( \text{Sz}(q) \) with
\( q = 2^{2n+1} > 2 \).

Let \( G \) be a finite Frobenius group with the kernel \( N \) and a complement \( H \) and
suppose that \( H \) nilpotent. Let \( x, y, z \in G \setminus \{1\} \) and let the groups \( \langle x, y \rangle \) and \( \langle y, z \rangle \) be
nilpotent. Let \( c \) be the nilpotency class of \( \langle x, y \rangle \). First suppose that \( x \in N \) and \( y \notin N \).
Then \( [x, cy] = 1 \), which implies \([x, c_iy] = 1\), since \( H \) acts fixed-point-freely on \( N \). By the same argument we get \( x = 1 \), which is not possible. This shows that if \( x \in N \) then \( y \in N \) and similarly also \( z \in N \). But in this case \( (x, z) \) is clearly nilpotent, since \( N \) is nilpotent. Thus we may assume that \( x, y, z \notin N \). Let \( x \in H^g \) and \( y \in H^k \) for some \( g, k \in G \) and suppose \( H^g \neq H^k \). We clearly have \( C_G(x) \leq H^g \) and \( C_G(y) \leq H^k \). Let \( \omega \) be any commutator of weight \( c \) with entries in \( \{x, y\} \). Then \( \omega \in C_G(x) \cap C_G(y) = 1 \) implies that \( (x, y) \) is nilpotent of class \( \leq c - 1 \), a contradiction. Hence we conclude that \( (x, y) \leq H^g \) and similarly also \( (y, z) \leq H^g \). Therefore we have \( (x, z) \leq H^g \). But \( H^g \) is nilpotent, hence the group \( (x, z) \) is also nilpotent. This shows that the groups under (ii) are \( \mathfrak{S} \)-groups.

It remains to prove that the groups under (iii) and (iv) are \( \mathfrak{N} \)-groups. If \( G = \text{PSL}(2, 2^f) \), \( f > 1 \), then every centraliser of a nontrivial element of \( G \) is Abelian by [6]. It follows from here that \( G \) is an \( \mathfrak{N} \)-group. Now let \( G = S_2(q) \) where \( q = 2^{2n+1} > 2 \). By Theorem 3.10 (c) in [3], \( G \) has a nontrivial partition \((G_i)_{i \in I}, \) where for every \( i \in I \) the group \( G_i \) is nilpotent and contains centralisers of each of its nontrivial elements. Let \( x, y, z \in G \setminus \{1\} \) and suppose that the groups \( (x, y) \) and \( (y, z) \) are nilpotent. Let \( a \) and \( b \) be nontrivial elements in \( Z((x, y)) \) and \( Z((y, z)) \), respectively, and suppose that \( a \in G_i \) and \( b \in G_j \) for some \( i, j \in I \). Then \( y \in C_G(a) \cap C_G(b) \leq G_i \cap G_j \), hence \( i = j \). But now we get \( x, z \in G_i \) and since \( G_i \) is nilpotent, the same is true for the group \( (x, z) \). Hence \( G \) is an \( \mathfrak{N} \)-group. 

\( \square \)

**References**


