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Finiteness conditions in soluble groups and Lie algebras

Ian N. Stewart

We prove some new theorems and reprove some old ones about finitely generated soluble groups and Lie algebras by a uniform method. Among the applications are Gruenberg's Theorem on Engel groups, for which we obtain a very short proof; and the Milnor and Wolf polynomial growth theorem. It is shown that a finitely generated soluble group with all 2-generator subgroups polycyclic is itself polycyclic, and that a finitely generated soluble Lie algebra, all of whose inner derivations are algebraic, is finite-dimensional. This last result enables us to give a partial answer to a question of Jacobson.

The aim of this note is to illustrate a procedure for proving a number of theorems about finiteness conditions in soluble groups, based on quite simple considerations centred around the module-theoretic methods of Hall [5, 6, 7]. By virtue of the results of Amayo and Stewart [1] the procedure also applies to Lie algebras. Some of these theorems are well known, although others seem to be new. Among the former are:

(a) the theorem of Gruenberg [3] that finitely generated soluble Engel groups are nilpotent,

(b) the Lie algebra analogue of (a), also due to Gruenberg [3],

(c) the theorem of Milnor [9] and Wolf [10] that finitely generated soluble groups with polynomial growth are nilpotent-by-finite.

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Our proofs of (a) and (b) are very short, and use very little machinery: they are more conceptual than those of Gruenberg, which rely on properties of basic commutators. The proof of (c) is just a tactical variant of that given by Bass [2] and in consequence will be given in outline only. Our procedure is presumably 'folklore': in particular it seems that Hall has given in lectures a proof of (a) roughly along the same lines. However it is interesting that the same method proves such a variety of results.

1. Notation

We let $\underline{\underline{A}}$, $\underline{\underline{F}}$, $\underline{\underline{G}}$, $\underline{\underline{N}}$, $\underline{\underline{P}}$ denote the classes of abelian, finite, finitely generated, nilpotent, and polycyclic groups. We use Hall's closure operations Q and E : for any class $\underline{\underline{X}}$ of groups $\mathrm{Q}\underline{\underline{X}}$ consists of all quotient groups of $\underline{\underline{X}}$-groups; whilst $\mathrm{E}\underline{\underline{X}}$ comprises those groups having a finite series

$$1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G$$

with each factor $G_{i+1}/G_i \in \underline{\underline{X}}$. Thus $\mathrm{E}\underline{\underline{A}}$ is the class of soluble groups. If $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are group classes then $\underline{\underline{XY}}$ denotes the class of *$\underline{\underline{X}}$-by-$\underline{\underline{Y}}$-groups* G , having a normal subgroup $H \in \underline{\underline{X}}$ such that $G/H \in \underline{\underline{Y}}$.

Let $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ be classes of groups satisfying

(i) all finitely generated $\underline{\underline{X}}$-groups are polycyclic-by-finite,

(ii) $\underline{\underline{Y}}$ is Q-closed.

Suppose we wish to prove that *all finitely generated soluble $\underline{\underline{Y}}$-groups are $\underline{\underline{X}}$-groups*. Then we may try the following

Procedure. Let G be a finitely generated soluble $\underline{\underline{Y}}$-group. Argue by induction on the derived length of G . If A is the last nontrivial term of the derived series of G , then A is abelian and normal in G , whilst G/A is polycyclic-by-finite by induction. Therefore G lies in the class $\underline{\underline{G}} \cap \underline{\underline{APF}}$, which is studied by Hall [5, 6, 7]. In particular A is a module for the integral group ring of the $\underline{\underline{PF}}$-group G/A , and G satisfies the maximal condition for normal subgroups. This is a very strong condition, and often suffices to carry out the induction step.

2. The theorems of Gruenberg, Milnor and Wolf

If G is a group and $g, h \in G$ we write

$$(g, h) = g^{-1}h^{-1}gh$$

for the group commutator, and define recursively

$$(g, {}_{n+1}h) = ((g, {}_n h), h) .$$

Then G is an Engel group if for each $g, h \in G$ there exists $n = n(g, h)$ such that $(g, {}_n h) = 1$.

To prove Gruenberg's Theorem we argue as above, with $\underline{\underline{X}} = \underline{\underline{N}}$, and $\underline{\underline{Y}}$ = the class of Engel groups. By induction we may assume that our group G lies in $\underline{\underline{G}} \cap \underline{\underline{AN}}$. To prove G nilpotent it is sufficient to show that the abelian normal subgroup A lies in the hypercentre of G , for then G is hypercentral (since G/A is nilpotent) and finitely generated, and hence G is nilpotent. If this is not the case we may quotient out the intersection of the hypercentre with A . We may then assume, for a contradiction, that A contains no nontrivial element centralised by G/A .

Let $N = G/A$ and argue by induction on the length (necessarily finite) of a cyclic series for N that A contains a nontrivial N-invariant element. This is clear if $N = 1$. Otherwise we can find $K \triangleleft N$ such that N/K is cyclic, K has smaller cyclic length than N , and $N = \langle K, x \rangle$ for some $x \in N$. By induction there is an element a of A , $a \neq 1$, which is invariant under K . Consider the subgroup T of A generated by all conjugates a^{x^i} of a by a power of x . This is clearly invariant under x . It is centralised by K , since if $k \in K$ then

$$a^{x^i k} = a^{x^i k x^{-i} . x^i} = a^{x^i} .$$

By the Engel condition $(a, {}_t x) = 1$. Let t be smallest with this property. Then $(a, {}_{t-1} x) \neq 1$, lies in T , and is centralised by x . Hence it is centralised by N . This is a contradiction, and Gruenberg's Theorem is proved.

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A very similar argument gives the Lie algebra version of the theorem.

The proof of the Milnor and Wolf Theorem follows the same lines, but is harder. We take $\underline{\underline{Y}}$ to be the class of groups with polynomial growth, $\underline{\underline{X}} = \underline{\underline{NF}}$. By Milnor [9], Lemma 1, we have A finitely generated as an abelian group (which at once makes G polycyclic). Since subgroups of finite index in finitely generated groups are also finitely generated we may assume G/A nilpotent. Induction on the cyclic length of G/A , arguing as in Bass [2] p. 605 and using his Lemma 2 (which, as he remarks, is the essential point of the proof) completes the induction step.

3. Other results

The theorems of this section may all be proved by the same method.

THEOREM 1. *Let G be a finitely generated soluble group, all of whose 2-generator subgroups are polycyclic. Then G is polycyclic.*

Proof. We use the standard procedure, with $\underline{\underline{Y}}$ the class of groups all of whose 2-generator subgroups are polycyclic, and $\underline{\underline{X}} = \underline{\underline{P}}$. With the usual notation, we may assume that $P = G/A$ is polycyclic, and that A is nontrivial, having no nontrivial P-invariant subgroup which is finitely generated as an abelian group.

We show by induction on a cyclic series for P that on the contrary such a subgroup exists. Take $K \triangleleft P$ with P/K cyclic, $P = \langle K, x \rangle$. There is a nontrivial subgroup $B = \langle b_1, \ldots, b_t \rangle$ of A which is K-invariant. Now each $\langle b_i, x \rangle$ is polycyclic, so that each b_i lies inside an x-invariant subgroup T_i of A which is finitely generated as an abelian group. Thus B is contained in $T_1 \ldots T_t$, a finitely generated x-invariant group. Let C be the product of the conjugates of B under powers of x . Then C is finitely generated and x-invariant. But if y is any power of x then

$$B^{yK} = B^{Ky} = B^y ,$$

so that C is also K-invariant, hence P-invariant. This completes the induction, and the resulting contradiction proves the theorem.

A similar argument yields:

THEOREM 2. *Let L be a finitely generated soluble Lie algebra, all of whose 2-generator subalgebras are finite dimensional. Then L is finite dimensional.*

Examples mentioned by Golod [4] p. 103 (footnote) show that the hypothesis of solubility cannot be omitted from Theorems 1 and 2.

Define the Lie algebra L to be *algebraic* if every inner derivation satisfies some polynomial equation (which is allowed to vary from element to element). Then our procedure easily yields:

THEOREM 3. *Every finitely generated soluble algebraic Lie algebra is finite dimensional.*

Again an example of Golod [4] shows that we cannot omit the hypothesis of solubility.

COROLLARY 4. *A locally soluble algebraic Lie algebra is locally finite.*

We can apply Theorem 3 to a question of Jacobson [8] p. 196. In Exercise 17 he states:

> "Conjecture (probably false and probably true under additional hypotheses): If the restricted Lie algebra L of characteristic p is finitely generated, and every element of L is algebraic in the sense that there exists a non-zero p-polynomial $\mu_a(\lambda)$ such that $\mu_a(a) = 0$, then L is finite dimensional."

For the relevant definitions see Jacobson [8] pp. 185-194.

We show that the conjecture is true if, in addition, L is required to be soluble. For in any restricted Lie algebra we have the equation (Jacobson [8] p. 188)

$$[b, a^p] = [b, \underbrace{a, \ldots, a}_{p}]$$

and in consequence the inner derivation induced by a is algebraic (in our sense) if it is algebraic (in Jacobson's sense). Theorem 3 is now applicable, and for completeness we state:

THEOREM 5. *If L is a finitely generated soluble restricted Lie algebra of characteristic p, and if every element of L is algebraic in* *the sense of* Jacobson, *then* L *is finite-dimensional.*

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Mathematics Institute,
University of Warwick,
Coventry, England.