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GLOBAL ATTRACTOR FOR THE CAHN-HILLIARD SYSTEM

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The Cahn-Hilliard system, a natural extension of the single Cahn-Hilliard equation in the case of multicomponent alloys, will be shown to generate a dissipative semigroup on the phase space $\mathcal{H} = [H^2(\Omega)]^m$. Following Hale's ideas and based on the existence and form of the Lyapunov functional, our main result will be the existence of a global attractor on a subset of $\mathcal{H}$. New difficulties specific to the system case make our problem interesting.

1. INTRODUCTION AND NOTATION

This paper justifies the existence of a global attractor for the system of Cahn-Hilliard equations

$$w_t = \Delta[-\Gamma\Delta w + \nabla_w \Phi(w)] \quad (t, x) \in \mathbb{R}^+ \times \Omega, \tag{1}$$

where $w : \mathbb{R}^+ \times \Omega \to \mathbb{R}^m$, $\Gamma = [\Gamma_{ij}] \in \mathbb{R}^{m\times m}$ is a symmetric and positive definite matrix, Ω is a bounded domain in $\mathbb{R}^n$ $(n \leqslant 3)$ having C^4 regular boundary $\partial\Omega$ and $\Phi: \mathbb{R}^m \to \mathbb{R}$ has locally Lipschitz continuous partial derivatives up to the third order. The system (1) is considered with homogeneous boundary conditions

$$\nabla_x w N \,|_{x\in\partial\Omega} = \nabla_x(\Delta w) N \,|_{x\in\partial\Omega} = 0, \tag{2}$$

(here $\nabla_x w = [(\partial w_i)/(\partial x_j)]$ is a gradient $m \times n$ matrix, while N denotes an outward normal vector to $\partial\Omega$) and with an initial condition

$$w(0, x) = w_0(x), \tag{3}$$

for a vector function w_0 from the product space $[H^2(\Omega)]^m$, satisfying the compatibility condition $\nabla_x w_0 N = 0$ on $\partial\Omega$.

For $m = 1$, the problem (1)–(3) reduces to the original Cahn-Hilliard model, studied by many authors (for example [1, 10, 2]), describing decomposition of a binary alloy. In general the system (1)–(3) is proposed as a phase separation model in the

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case where the alloy consists of $m+1$ components (decomposition of a multicomponent alloy). The derivation of (1)–(3), as well as many theoretical results concerning this problem, can be found for example in [3] or [4]. However, based on the single Cahn-Hilliard equation considerations of [10] and [2] it is possible to investigate the dynamics of (1)–(3) from the point of view of *dissipative systems* presented in [5] and general theory given in [6].

Since we are dealing with a system we use mostly matrix (vector) notation and calculus, writing the components directly only when it is needed for the clarity of local calculations. The components of a vector $s \in \mathbb{R}^d$ are written as s_i, whereas partial derivatives are denoted in the standard way, with explicitly given variable with respect to which differentiation is taken; for example the components of the nonlinear term $\nabla_w \Phi(w)$ are written simply as $\frac{\partial \Phi}{\partial w_i}(w)$. The differential operators Δ, ∇ are understood to be taken with respect to spatial variable x; otherwise we always indicate the variable explicitly in the index, as in the case of $\nabla_w \Phi(w)$. For a matrix B, $tr(B)$ is the sum of the diagonal elements of B and B^T is the transposed matrix of B; also by the integral (or derivative) of a matrix we understand the matrix of the integrals (derivatives) of its elements. Notation of Sobolev spaces is standard [10, 11]. Product spaces $[H^k(\Omega)]^m$ and $[L^p(\Omega)]^m$ are written as $\mathcal{H}^k$ and $\mathcal{L}^p$ respectively and we generally omit dependence on Ω in notation of these spaces. $|\Omega|$ denotes the Lebesgue measure of Ω. Because of the semigroup approach only the time argument of the solution w is explicitly distinguished. On the other hand, the integration is usually taken with respect to $x \in \Omega$, hence throughout the paper all integrals left unspecified should be understood to be taken over Ω, also all constants mentioned are positive numbers.

Let us specify the additional conditions that are required of the function Φ staying under the gradient on the right side of (1). We claim (besides the $C^{3+\text{Lipschitz}}$ regularity mentioned at the beginning of this paper) that Φ fulfills the following growth conditions:

$$\exists_{M \in \mathbb{R}} \forall_{s \in \mathbb{R}^m} \, \Phi(s) \geqslant M, \tag{4}$$

$$-s^T(\nabla_s \Phi(s)) \leqslant -k_0 |s|^{2p} + k_1, \quad s \in \mathbb{R}^m, \tag{5}$$

$$\left| \frac{\partial^2 \Phi}{\partial s_i \partial s_j}(s) \right| \leqslant k_2 \left(1 + |s|^{2p-2} \right), \quad 1 \leqslant i,\, j \leqslant m, \quad s \in \mathbb{R}^m, \tag{6}$$

$$\left| \frac{\partial^3 \Phi}{\partial s_i \partial s_j \partial s_l}(s) \right| \leqslant k_3 \left(1 + |s|^{2p-3} \right), \quad 1 \leqslant i,\, j,\, l \leqslant m, \quad s \in \mathbb{R}^m, \tag{7}$$

$$|\nabla_s \Phi(s)| \leqslant k_4 |s|^{2p-1} + k_4', \quad s \in \mathbb{R}^m, \tag{8}$$

and additionally, everywhere in (4)–(8) we require $p = 2$ if $n = 3$.

REMARK 1. As a simple application of the Main Value Theorem it may be seen that the assumption (6) is a consequence of (7) and, in order, (8) is a consequence of (6). Therefore (7) is the only necessary assumption among (6)–(8), and we decided to formulate (6) and (8) explicitly only for simplicity of further references.

REMARK 2. Although conditions (4)–(8) seem to be complicated, there are many functions satisfying all of them. As a "model function" which satisfies (4)–(8), we can take the real polynomial of order $2p$, having all leading coefficients positive; that is a function of the variable $s = (s_1, \dots, s_m)$ of the form:

$$\Phi(s) = \sum_{i=1}^{m} a_i^2 s_i^{2p} + \ (\text{a polynomial of order} \leqslant 2p-1).$$

However, our Φ need not to be a polynomial function as are those considered in [**10**] or [**2**]. Moreover, the lower bound (5) will be used in our paper only in the proof of Lemma 5 which offers the possibility of further generalisations. For example, condition (5) can be omitted in the case when Φ is convex, that is, the Hessian Φ'' is nonnegative definite (compare Remark 4 below).

2. SYSTEM (1)–(3) AS A CAUCHY PROBLEM IN A BANACH SPACE. EXISTENCE OF A GLOBAL SOLUTION

Let us write (1)–(3) as an evolution problem

$$(9) \qquad \begin{cases} \dot{w} + Aw = F(w), & t > 0, \\ w(0) = w_0, \end{cases}$$

where $F(w) = \Delta \nabla_w \Phi(w)$ and the operator $A = \Gamma \Delta^2$ is considered on the Hilbert space $\mathcal{L}^2$ with dense domain

$$(10) \qquad D(A) = \{\phi \in \mathcal{H}^4;\ \nabla_x \phi N = \nabla_x(\Delta\phi) N = 0 \text{ on } \partial\Omega\}.$$

In order to use the existence theory developed in [**6**] with the improvements mentioned in [**5**, p.73], we shall show that A is sectorial, whereas the nonlinear term F is locally Lipschitz continuous as the operation between the spaces $D(A^{1/2}) = \{\phi \in \mathcal{H}^2;\ \nabla_x \phi N = 0 \text{ on } \partial\Omega\}$ and $\mathcal{L}^2$.

Denoting by $\langle\cdot,\cdot\rangle$ the scalar product in $\mathcal{L}^2$ and defining

$$\mathcal{A} := A + \delta_0 \Gamma,$$

where $\delta_0 > 0$ is taken so large that $\left(\Delta^2 + \delta_0\right)$ is an isomorphism between $D(A)$ and $\mathcal{L}^2$ (see [**11**, Theorem 5.5.1]), we find for any $\phi, \psi \in D(A)$, that

$$\begin{aligned} \text{(11)} \qquad \langle \mathcal{A}\phi, \psi\rangle &= \int \psi^T \Gamma \Delta^2 \phi dx + \delta_0 \int \psi^T \Gamma \phi dx \\ &= -\int tr\left(\nabla_x \psi^T \Gamma \nabla_x(\Delta\phi)\right) dx + \delta_0 \int \psi^T \Gamma \phi dx \\ &= \int \Delta\psi^T \Gamma \Delta\phi dx + \delta_0 \int \psi^T \Gamma \phi dx. \end{aligned}$$

Since Γ is symmetric we have analogously from (11) that

$$\begin{aligned} \text{(12)} \qquad \int \Delta\psi^T \Gamma \Delta\phi dx + \delta_0 \int \phi^T \Gamma \psi dx &= \int \Delta^2 \psi^T \Gamma \phi dx + \delta_0 \int \psi^T \Gamma \phi dx \\ &= \langle \phi, \mathcal{A}\psi\rangle, \end{aligned}$$

which proves the symmetry of $\mathcal{A}$. Next, since Γ is positive definite, the operator $\mathcal{A}$ is bounded below, that is, for each $\phi \in D(A)$

$$\begin{aligned} \text{(13)} \qquad \langle \mathcal{A}\phi, \phi\rangle &= \int \phi^T \Gamma \Delta^2 \phi dx + \delta_0 \int \phi^T \Gamma \phi dx \\ &= \int \Delta\phi^T \Gamma \Delta\phi dx + \delta_0 \int \phi^T \Gamma \phi dx \geqslant \delta_0 \langle \phi, \phi\rangle. \end{aligned}$$

Moreover, thanks to the choice of δ_0, we have

$$D(A) \underset{\text{isomorphism}}{\overset{\Delta^2+\delta_0}{\longrightarrow}} \mathcal{L}^2 \underset{\text{isomorphism}}{\overset{\Gamma}{\longrightarrow}} \mathcal{L}^2,$$

so that the range of $\mathcal{A}$ is the whole space $\mathcal{L}^2$. From these observations it is clear that $\mathcal{A}$ is self-adjoint and bounded below and hence, in particular, sectorial. Then using [**6**, Exercise 6 or Theorem 1.3.2] we obtain finally that A is itself sectorial.

The second property we should investigate is the local Lipschitz continuity of the nonlinear function F standing on the right side in (9), that is, we need to check the following condition

$$\text{(14)} \qquad \forall_{\substack{\phi,\psi \in U \subset D\left(A^{1/2}\right) \\ U \text{ bounded}}} \left\| \Delta\nabla_\phi \Phi(\phi) - \Delta\nabla_\psi \Phi(\psi)\right\|_{\mathcal{L}^2} \leqslant k_U \left\|\phi - \psi\right\|_{D\left(A^{1/2}\right)}.$$

We find by differentiation that

$$\begin{aligned} \text{(15)} \qquad \Delta \frac{\partial \Phi}{\partial \phi_i}(\phi) - \Delta \frac{\partial \Phi}{\partial \psi_i}(\psi) &= \sum_{l=1}^{m} \left(\frac{\partial^2 \Phi}{\partial \psi_l \partial \phi_i}(\phi)\Delta\phi_l - \frac{\partial^2 \Phi}{\partial \psi_l \partial \psi_i}(\psi)\Delta\psi_l \right) \\ &+ \sum_{k,l=1}^{m} \left(\frac{\partial^3 \Phi}{\partial \phi_k \partial \phi_l \partial \phi_i}(\phi)(\nabla\phi_k \nabla\phi_l) - \frac{\partial^3 \Phi}{\partial \psi_k \partial \psi_l \partial \psi_i}(\psi)(\nabla\psi_k \nabla\psi_l) \right). \end{aligned}$$

Transforming the right side of (15) and taking L^2 norms of both sides we obtain further

(16)
$$
\begin{aligned}
\left\|\Delta \frac{\partial \Phi}{\partial \phi_i}(\phi) - \Delta \frac{\partial \Phi}{\partial \psi_i}(\psi)\right\|_{L^2} \leqslant & \max_{1 \leqslant i,l \leqslant m} \left\| \frac{\partial^2 \Phi}{\partial \psi_l \partial \psi_i}(\psi)\right\|_{L^\infty} \sum_{l=1}^{m} \|\Delta \phi_l - \Delta \psi_l\|_{L^2} \\
&+ \max_{1 \leqslant i,l \leqslant m} \left\| \frac{\partial^2 \Phi}{\partial \phi_l \partial \phi_i}(\phi) - \frac{\partial^2 \Phi}{\partial \psi_l \partial \psi_i}(\psi)\right\|_{L^\infty} \sum_{l=1}^{m} \|\Delta \phi_l\|_{L^2} \\
&+ \max_{1 \leqslant i,k,l \leqslant m} \left\| \frac{\partial^3 \Phi}{\partial \psi_k \partial \psi_l \partial \psi_i}(\psi)\right\|_{L^\infty} \sum_{k,l=1}^{m} \|(\nabla \phi_k - \nabla \psi_k)\nabla \phi_l\|_{L^2} \\
&+ \max_{1 \leqslant i,k.l \leqslant m} \left\| \frac{\partial^3 \Phi}{\partial \psi_k \partial \psi_l \partial \psi_i}(\psi)\right\|_{L^\infty} \sum_{k,l=1}^{m} \|(\nabla \phi_l - \nabla \psi_l)\nabla \psi_k\|_{L^2} \\
&+ \max_{1 \leqslant i,k,l \leqslant m} \left\| \frac{\partial^3 \Phi}{\partial \phi_k \partial \phi_l \partial \phi_i}(\phi) - \frac{\partial^3 \Phi}{\partial \psi_k \partial \psi_l \partial \psi_i}(\psi)\right\|_{L^\infty} \sum_{k,l=1}^{m} \|\nabla \phi_k \nabla \phi_l\|_{L^2}.
\end{aligned}
$$

Remember that ϕ, ψ are taken from a bounded set $U \subset \mathcal{H}^2$ and also that Sobolev embeddings $H^1 \subset L^4$, $H^2 \subset L^\infty$ hold for space dimensions $n \leqslant 3$. Hence since the partial derivatives of Φ are locally Lipschitz continuous and locally bounded (Lipschitz constants and upper bounds are denoted by common symbols L_U and M_U, respectively), inequality (16) gives:

$$
\begin{aligned}
\left\|\Delta \frac{\partial \Phi}{\partial \phi_i}(\phi) - \Delta \frac{\partial \Phi}{\partial \psi_i}(\psi)\right\|_{L^2} &\leqslant M_U \sum_{l=1}^{m} \|\Delta \phi_l - \Delta \psi_l\|_{L^2} \\
+ L_U \|\phi - \psi\|_{L^\infty} \sum_{l=1}^{m} \|\Delta \phi_l\|_{L^2} &+ 2M_U \sum_{k,l=1}^{m} \|\nabla \phi_k - \nabla \psi_k\|_{L^4} \|\nabla \phi_l\|_{L^4} \\
+ L_U \|\phi - \psi\|_{L^\infty} \sum_{k,l=1}^{m} \|\nabla \phi_k\|_{L^4} \|\nabla \phi_l\|_{L^4} &\leqslant k_U \|\phi - \psi\|_{\mathcal{H}^2},
\end{aligned}
$$

where the constant k_U depends only on the set U. Condition (14) is thus justified.

The general theory of [**6**, p.54] now guarantees the existence of a local solution for the problem (9), that is, there are a positive time $\tau = \tau(w_0)$ and a function $w\colon [0, \tau) \to \mathcal{L}^2$ which satisfies (9) for $t < \tau$. Moreover, $w(t)$ belongs to $D(A)$ for $t \in (0, \tau)$ and also

(17) $$w \in C\Big([0, \tau); D\Big(A^{1/2}\Big)\Big),\ \Delta \nabla_w \Phi(w) \in C\big([0, \tau); \mathcal{L}^2\big).$$

In fact, as will be shown in Section 3.D, the $\mathcal{H}^2$ norm of the solution is *a priori* bounded uniformly for $t \in [0, \infty)$. This property of the solution implies, in particular, global boundedness in time of the quotient

$$\frac{\|\Delta\nabla_w(\Phi(w))\|_{\mathcal{L}^2}}{1+\|w(t)\|_{D(A^{1/2})}},$$

so that in consequence, according to [**6**, Exercise 1, p.58], the solution w exists on the whole half-line $[0,+\infty)$. Moreover, the family of operators $\{T(t)\}_{t\geqslant 0}$ defined by $T(t)w_0 := w(t,\, w_0)$ forms a strongly continuous semigroup on the space $D(A^{1/2})$ (compare [**5**, p.73]).

3. Some properties of the system (1)–(3)

A. Preservation of spatial average of the solution.

In the physical model described by (1)–(3), preservation of spatial average of the solution corresponds to mass conservation. This property follows immediately by integration of (1) over $x \in \Omega$ and by parts. Because of boundary conditions (2) the integrals over $\partial\Omega$ disappear and we obtain

$$\frac{d}{dt}\frac{1}{|\Omega|}\int w(t)dx = \frac{d}{dt}\overline{w}(t) = 0,$$

or equivalently

$$\overline{w}(t) = \overline{w}_0, \qquad t \geqslant 0. \tag{18}$$

B. Existence of the Lyapunov functional.

Multiplying (1) by $[-\Gamma\Delta w + \nabla_w\Phi(w)]^T$ we find by integration that

$$\int [-\Gamma\Delta w + \nabla_w\Phi(w)]^T w_t dx = -\sum_{i=1}^{m}\int\left|\nabla\left(-\sum_{j=1}^{m}\Gamma_{ij}\Delta w_j + \frac{\partial\Phi}{\partial w_i}(w)\right)\right|^2 dx, \tag{19}$$

that is, for all $t > 0$ the left side of (19) is non-positive, moreover

$$0 \geqslant \int [-\Gamma\Delta w + \nabla_w\Phi(w)]^T w_t dx = \frac{d}{dt}\left(\int \frac{1}{2} tr\big(\nabla_x w^T \Gamma \nabla_x w\big)dx + \int \Phi(w)dx\right). \tag{20}$$

Thus we may define the Lyapunov functional $\mathcal{L}$ for (1)–(3) as

$$\mathcal{L}(w(t)) = \int \frac{1}{2} tr\big(\nabla_x w^T \Gamma \nabla_x w\big)dx + \int \Phi(w)dx. \tag{21}$$

It is clear from (19), (20) that $\mathcal{L}$ decreases along each trajectory. Furthermore:

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LEMMA 1. *If for a solution w of the system (1)–(3)*

$$\mathcal{L}(w(t)) = constant, \qquad t > 0, \tag{22}$$

then w is a stationary (time independent) solution of (1)–(3).

PROOF: Because for each positive t, $w(t)$ belongs to $D(A)$, conditions (22) and (19), (20) ensure that

(23)

$$\forall_{t>0}\forall_{i=1,2\ldots m}\left|\nabla\left(-\sum_{j=1}^{m}\Gamma_{ij}\Delta w_j(t)+\frac{\partial\Phi}{\partial w_i}(w(t))\right)\right|^2 = 0 \quad \text{almost everywhere in } \Omega.$$

Next, from the well known property of distributional derivatives (see [**9**, p.92]), the function under the gradient in (23) is independent of x. Thus, since $C^2(\overline{\Omega}) \subset D(A)$ for $n \leqslant 3$, condition (23) gives:

$$\forall_{t>0}\exists_{c(t)\in\mathbb{R}^m} \; -\Gamma\Delta w(t) + \nabla_w\Phi(w(t)) = c(t) \quad \text{for every } x \in \Omega. \tag{24}$$

Further, taking the Laplacian of both sides in (24) we obtain that

$$\Delta\left[-\Gamma\Delta w(t) + \nabla_w\Phi(w(t))\right] = 0, \qquad t \in (0, +\infty). \tag{25}$$

As $w(t)$ is a solution of (1)–(3), equality (25) implies that

$$\frac{dw}{dt}(t) = 0 \quad \text{for } t > 0,$$

therefore $w(t)$ is time independent for $t > 0$. Finally, because of (17), $w(t) = w_0$ for all $t \geqslant 0$. Lemma 1 is proved. ∎

We use the above considerations in the proof of the following:

LEMMA 2. *An element $w \in D(A)$ is a stationary solution of (1)–(3) if and only if w is a $\mathcal{H}^2$ solution of the Neumann type elliptic boundary value problem:*

$$\begin{cases} -\Gamma\Delta v + \nabla_v\Phi(v) = a & \text{in } \Omega, \\ \nabla_x v N = 0 & \text{on } \partial\Omega. \end{cases} \tag{26}$$

PROOF: Based on the considerations leading to (24), it is clear that if w is a stationary solution of (1)–(3) then, in particular, w is a solution of the elliptic problem (26) with some $a \in \mathbb{R}^m$. Moreover, by integrating (26) over $x \in \Omega$, it is possible to determine explicitly the constant vector a as:

$$a = \overline{\nabla_v\Phi(v)}. \tag{27}$$

For the converse, if $v \in D(A^{1/2})$ solves

$$-\Gamma\Delta v + \nabla_v \Phi(v) = a, \tag{28}$$

then from elliptic regularity theory [8], $v \in \mathcal{H}^4$ and also:

$$\nabla_x(\Delta v)N = \nabla_x\big(\Gamma^{-1}(-\nabla_v\Phi(v) + a)\big) = 0 \quad \text{on } \partial\Omega.$$

Thus $v \in D(A)$, whereas taking Laplician of both sides in (28)

$$\Delta(-\Gamma\Delta v + \nabla_v\Phi(v)) = 0, \tag{29}$$

so that $w(t) = v$ for all $t \geqslant 0$ is a time independent solution of (1)–(3) starting from $w(0) = v$. The proof is completed. □

C. Global $\mathcal{H}^1$ estimate of the solution.

Existence of the Lyapunov functional $\mathcal{L}$ guarantees a global in time estimate of the norm $\|w_i\|_{H^1}$ for each component w_i of the solution w. Since Φ is bounded below (condition (4)) and $\mathcal{L}$ decreases along each trajectory, we have from (21):

$$\int \frac{1}{2} tr\big(\nabla_x w^T \Gamma \nabla_x w\big)dx \leqslant \mathcal{L}(w_0) - \int \Phi(w)dx \leqslant \mathcal{L}(w_0) - M\,|\Omega|\,. \tag{30}$$

Next, since Γ is positive definite, then

$$\int \frac{1}{2} tr\big(\nabla_x w^T \Gamma \nabla_x w\big)dx \geqslant \gamma_0 \sum_{i=1}^{m} \|\nabla w_i\|^2_{L^2}\,. \tag{31}$$

Finally, as the expression

$$\Big(\|\nabla\phi\|^2_{\mathcal{L}^2} + \big|\overline{\phi}\big|^2\Big)^{1/2} \tag{32}$$

defines on $D(A^{1/2})$ the norm equivalent to the natural $\mathcal{H}^1$ norm, then collecting estimates (18), (30), (31) we reach the required property

$$\|w(t)\|_{\mathcal{H}^1} \leqslant constant, \tag{33}$$

with *constant* independent of $t \in [0, +\infty)$.

D. Global $\mathcal{H}^2$ estimate of the solution.

To obtain global in time boundedness of $\|w\|_{\mathcal{H}^2}$ it is necessary to find first a suitable estimate for the nonlinear term $\Delta\nabla_w\Phi(w)$. Hence, we start with the following auxiliary inequality, which is valid (with some $k > 0$) only on a solution $w(t)$ of the system (1)–(3);

LEMMA 3. *There exists a positive constant $k = k(\|w_0\|_{\mathcal{H}^1}, \Omega)$ such that for a solution $w(t)$ of the system (1)–(3):*

$$\|\Delta\nabla_w\Phi(w(t))\|^2_{L^2} \leqslant k\left(1 + \|\Delta^2 w(t)\|^{4/3}_{L^2}\right), \quad t \geqslant 0. \tag{34}$$

PROOF: Condition (34) stated above is similar to inequality (4.105) developed in [**10**, p.156], thus to derive (34) we shall follow the concept of [**10**].

Estimating the i-th component $\Delta\frac{\partial\Phi}{\partial w_i}(w)$ of the nonlinear term $\Delta\nabla_w\Phi(w)$ we have

(35)
$$\begin{aligned}\left|\Delta\frac{\partial\Phi}{\partial w_i}(w)\right| &= \left|\sum_{l,j=1}^{m}\sum_{k=1}^{n}\frac{\partial^3\Phi}{\partial w_j\partial w_l\partial w_i}(w)\frac{\partial w_l\partial w_j}{\partial x_k\partial x_k} + \sum_{j=1}^{m}\frac{\partial^2\Phi}{\partial w_j\partial w_i}(w)\Delta w_j\right| \\ &\leqslant \max_{1\leqslant j,l,i\leqslant m}\left|\frac{\partial^3\Phi}{\partial w_j\partial w_l\partial w_i}(w)\right|\sum_{l,j=1}^{m}|\nabla(w_l)\nabla(w_j)| \\ &\quad + \max_{1\leqslant i,j\leqslant m}\left|\frac{\partial^2\Phi}{\partial w_j\partial w_i}(w)\right|\sum_{j=1}^{m}|\Delta w_j|.\end{aligned}$$

Taking now L^2 norms of both sides of inequality (35) and using assumptions (6), (7) we obtain

$$\begin{aligned}\left\|\Delta\frac{\partial\Phi}{\partial w_i}(w)\right\|_{L^2} &\leqslant k_3\left(1 + \|w\|^{2p-3}_{L^\infty}\right)\sum_{l,j=1}^{m}\|\nabla(w_l)\nabla(w_j)\|_{L^2} \\ &\quad + k_2\left(1 + \|w\|^{2p-2}_{L^\infty}\right)\sum_{j=1}^{m}\|\Delta w_j\|_{L^2} \\ &\leqslant k'\left[\left(1 + \sum_{j=1}^{m}\|w_j\|^{2p-3}_{L^\infty}\right)\sum_{j=1}^{m}\|\nabla(w_j)\|^2_{L^4}\right. \\ &\quad \left. + \left(1 + \sum_{j=1}^{m}\|w_j\|^{2p-2}_{L^\infty}\right)\sum_{j=1}^{m}\|\Delta w_j\|_{L^2}\right].\end{aligned} \tag{36}$$

We are now in the position fully analogous to [**10**, condition (4.109), p.157]. If for each particular component w_j we follow the considerations of [**10**, between (4.110)–(4.116)], we find similarly as in [**10**] that:

$$\begin{cases}\|\Delta w_j\|_{L^2} \leqslant k_5\left\|\Delta^2 w_j\right\|^{1/3}_{L^2} \\ \|\nabla w_j\|_{L^4} \leqslant k_6\|\Delta w_j\|^{n/12}_{L^2},\end{cases} \tag{37}$$

and further, depending on the space dimension n,

$$(38)\qquad \begin{cases} \|w_j - \overline{w}_{0j}\|_{L^\infty} \leqslant k_7 \|\nabla w_j\|_{L^2} & \text{for } n = 1, \\ \|w_j - \overline{w}_{0j}\|_{L^\infty} \leqslant k_8 \|\nabla w_j\|_{L^2}^{(1-\varepsilon)} \|\Delta^2 w_j\|_{L^2}^{\varepsilon} & \text{for } n = 2 \text{ and any } \varepsilon > 0, \\ \|w_j - \overline{w}_{0j}\|_{L^\infty} \leqslant k_9 \|\nabla w_j\|_{L^2}^{5/6} \|\Delta^2 w_j\|_{L^2}^{1/6} & \text{for } n = 3. \end{cases}$$

Inserting (37) and (38) into (36), using the global $\mathcal{H}^1$ boundedness of $w(t)$ (shown in Section 3.C) and applying an obvious inequality

$$\forall_{a>a'>0}\forall_{b\geqslant 0} b^{1/a} \leqslant 1 + b^{1/a'},$$

we come finally to (34) with the constant k dependent on the $\mathcal{H}^1$ norm of the initial function w_0. Condition (34) is justified. □

We are now able to prove the following:

Lemma 4. *If w is a solution of (1)–(3), then for all $t \geqslant 0$:*

$$(39)\qquad \|\Delta w(t)\|_{\mathcal{L}^2}^2 \leqslant \max\{k_8 z_1, \|\Delta w_0\|_{\mathcal{L}^2}^2\},$$

where constants k_8, z_1 are given in (45), (46) respectively. Moreover, since the norm

$$(40)\qquad \left(\|\Delta \phi\|_{\mathcal{L}^2}^2 + |\overline{\phi}|^2\right)^{1/2}$$

on $D(A^{1/2})$ is equivalent to the natural $\mathcal{H}^2$ norm, then because of the mass conservation property shown in Section 3.A:

$$(41)\qquad \|w(t)\|_{\mathcal{H}^2} \leqslant constant(\|w_0\|_{\mathcal{H}^2}, \Omega), \quad t \geqslant 0.$$

Proof: By multiplying (1) by $\Delta^2 w^T$ and integrating over Ω we obtain:

$$(42)\qquad \int \Delta^2 w^T w_t dx = -\int \Delta^2 w^T \Gamma \Delta^2 w dx + \int \Delta^2 w^T \Delta(\nabla_w \Phi(w)) dx.$$

Next, since Γ is positive definite, the Hölder and Cauchy inequalities give:

$$(43)\qquad \begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Delta w\|_{\mathcal{L}^2}^2 &\leqslant -\gamma_0 \|\Delta^2 w\|_{\mathcal{L}^2}^2 + \|\Delta^2 w\|_{\mathcal{L}^2} \|\Delta(\nabla_w \Phi(w))\|_{\mathcal{L}^2} \\ &\leqslant -\frac{\gamma_0}{2}\|\Delta^2 w\|_{\mathcal{L}^2}^2 + \frac{1}{2\gamma_0}\|\Delta(\nabla_w \Phi(w))\|_{\mathcal{L}^2}^2. \end{aligned}$$

Inserting (34) into (43) we have

$$(44)\qquad \frac{d}{dt}\|\Delta w(t)\|_{\mathcal{L}^2}^2 \leqslant -\gamma_0 \|\Delta^2 w(t)\|_{\mathcal{L}^2}^2 + \frac{k}{\gamma_0}\left(1 + \left(\|\Delta^2 w(t)\|_{\mathcal{L}^2}^2\right)^{2/3}\right).$$

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Let us note further, that as a consequence of the boundary conditions (2), $\int \Delta w(t)dx = 0$ and hence, according to the Smoller inequality ([**7**, p.112]), we have

$$(45) \qquad \|\Delta w\|^2_{\mathcal{L}^2} \leqslant k_8 \left\|\Delta^2 w\right\|^2_{\mathcal{L}^2}.$$

We define the constant z_1 as the unique positive solution of the algebraic equation:

$$(46) \qquad -\gamma_0 z + \frac{k}{\gamma_0}\left(1 + z^{2/3}\right) = 0.$$

As a consequence of (45), whenever $\|\Delta w(t)\|^2_{\mathcal{L}^2} > k_8 z_1$ then, by the definition of z_1, the right side of the differential inequality (44) is negative, hence $\|\Delta w(t)\|^2_{\mathcal{L}^2}$ needs to decrease. This shows (compare [**2**, Lemma 5] for detailed proof), that (39) is satisfied. The proof of Lemma 4 is completed. $\square$

REMARK 3. To justify the correctness of the calculations in Lemma 4 we need additional smoothness of the solution w. As a result of [**6**, Theorem 3.5.2], the function $(0,\infty) \ni t \longrightarrow \dot{w}(t) \in D(A^{1/2})$ is continuous. Then, as a consequence of (17) and (9), also the function $(0, \infty) \ni t \longrightarrow Aw(t) \in \mathcal{L}^2$ is continuous and one can check that for $t > 0$:

$$\left|h^{-1}\left(\int |\Delta w(t+h)|^2\, dx - \int |\Delta w(t)|^2\, dx\right) - 2\int \Delta^2 w^T w_t dx\right| \longrightarrow 0 \text{ as } h \to 0^+$$

or

$$\frac{1}{2}\frac{d}{dt}\int |\Delta w(t)|^2\, dx = \int \Delta^2 w^T w_t dx \quad \text{for } t > 0.$$

E. BOUNDEDNESS OF THE SET OF STATIONARY SOLUTIONS.

Let us denote by $\mathcal{D}$ the subset of the space $D(A^{1/2})$ consisting of all fixed points of the semigroup $\{T(t)\}_{t\geqslant 0}$, that is, of all stationary solutions of the system (1)–(3) (the symbols $D(A^{1/2})$ and $\{T(t)\}_{t\geqslant 0}$ were introduced in Section 2). According to the considerations of Section 3.B (Lemma 2), the following characterisation holds (remembering that $\overline{\phi}$ is the spatial average of ϕ over Ω):

$$(47) \qquad \mathcal{D} = \{v \in D\left(A^{1/2}\right); -\Gamma\Delta v + \nabla_v\Phi(v) = a, \quad a \in \mathbb{R}^m, \quad a = \overline{\nabla_v\Phi(v)}\}.$$

Then for fixed $\alpha > 0$, let $\mathcal{D}_\alpha$ be the subset of $\mathcal{D}$ consisting of all elements with $|\overline{v}|$ not exceeding α. In the next section boundedness of the set of stationary solutions will play a crucial role in the construction of the global attractor. Thus we shall prove:

LEMMA 5. *For each $\alpha > 0$, $\mathcal{D}_\alpha$ is a bounded subset of $D(A^{1/2})$.*

PROOF: Since an element v of $\mathcal{D}$ is characterised in (47) as a solution of a Neumann type elliptic boundary value problem, then we have

$$(48) \qquad \int v^T[-\Gamma\Delta v + \nabla_v\Phi(v)]dx = \int v^T a\, dx.$$

Further, in the presence of the definition of a (stated in (47)), since Γ is positive definite (48) gives

$$\gamma_0 \sum_{i=1}^{m} \int |\nabla v_i|^2 \, dx \leqslant \int \overline{v}^T \nabla_v \Phi(v) dx - \int v^T \nabla_v \Phi(v) dx. \tag{49}$$

Next, it follows from (8) and the Young inequality that

$$\forall_{\nu>0} \exists_{C(\nu)>0} \forall_{s\in\mathbb{R}^m} \; |\nabla_s \Phi(s)| \leqslant \nu |s|^{2p} + C(\nu). \tag{50}$$

Hence, applying to the right side of (49) the Schwarz inequality and growth conditions (5), (50), we obtain:

$$\begin{aligned} \gamma_0 \sum_{i=1}^{m} \int |\nabla v_i|^2 \, dx &\leqslant \int |\nabla_v \Phi(v)| \, |\overline{v}| \, dx - k_0 \int |v|^{2p} \, dx + k_1 |\Omega| \\ &\leqslant (\nu |\overline{v}| - k_0) \int |v|^{2p} \, dx + (C(\nu) |\overline{v}| + k_1) |\Omega| . \end{aligned} \tag{51}$$

By substituting in (51) $\nu = \nu_0 = (k_0)/(|\overline{v}| + 1)$ in order to obtain a negative coefficient before $\int |v|^{2p} \, dx$, we reach the estimate

$$\gamma_0 \sum_{i=1}^{m} \int |\nabla v_i|^2 \, dx \leqslant (C(\nu_0) |\overline{v}| + k_1) |\Omega| , \tag{52}$$

where the right side depends only on given quantities. Moreover, for the equivalent norm (32) on $\mathcal{H}^1$, condition (52) ensures that

$$\|v\|_{\mathcal{H}^1} \leqslant k_9(|\overline{v}| , |\Omega|). \tag{53}$$

The inequality (53) is crucial for the rest of the proof. Using once again the characterisation of the elements of $\mathcal{D}$ (see (47)) we find:

$$\int \Delta v^T [-\Gamma \Delta v + \nabla_v \Phi(v)] dx = \int \Delta v^T a \, dx = 0. \tag{54}$$

Since the matrix Γ is positive definite, equality (54) gives

$$-\gamma_0 \int |\Delta v|^2 \, dx + \left(\int |\nabla_v \Phi(v)|^2 \, dx \right)^{1/2} \left(\int |\Delta v|^2 \, dx \right)^{1/2} \geqslant 0, \tag{55}$$

and further (from the Cauchy inequality),

$$\int |\Delta v|^2 \, dx \leqslant \frac{1}{\gamma_0^2} \int |\nabla_v \Phi(v)|^2 \, dx. \tag{56}$$

Now, based on the growth condition (8), we can increase the right side of (56) coming to

$$\int |\Delta v|^2\,dx \leqslant \frac{1}{\gamma_0^2}\int \left(k_4\,|v|^{2p-1}+k_4'\right)^2 dx \leqslant \frac{2k_4^2}{\gamma_0^2}\int |v|^{4p-2}\,dx + \frac{2k_4'^2\,|\Omega|}{\gamma_0^2}. \tag{57}$$

In the presence of the Sobolev embeddings (remembering that $p=2$ if $n=3$) the right side of (57) is estimated by $\left(constant_1\,\|v\|_{\mathcal{H}^1}^{4p-2}+constant_2\right)$. Moreover, since inequality (53) assures boundedness of $\|v\|_{\mathcal{H}^1}$ only in terms of $|\overline{v}|$ and $|\Omega|$, formula (57) gives immediately:

$$\|\Delta v\|_{\mathcal{L}^2} \leqslant k_{10}(|\overline{v}|\,,\,\Omega). \tag{58}$$

Then considering the equivalent norm (40) on $D\left(A^{1/2}\right)$, we obtain finally that every element v from $\mathcal{D}_\alpha$ satisfies:

$$\|v\|_{\mathcal{H}^2} \leqslant k_{11}(\alpha,\,\Omega). \tag{59}$$

Lemma 5 is thus proved. ∎

4. Dissipativeness and global attractor

Since the set $\mathcal{D}$ introduced in (47) contains all constant vector functions, that is $\{v(x)\equiv\overline{v};\ \overline{v}\in\mathbb{R}^m\}\subset\mathcal{D}$, it is impossible to construct the global attractor on the whole space $D\left(A^{1/2}\right)$. To overcome this difficulty, for fixed $\alpha>0$ we introduce the complete metric space $\mathcal{H}_\alpha$:

$$\mathcal{H}_\alpha := \left\{u\in D\left(A^{1/2}\right);\ |\overline{u}|\leqslant\alpha\right\}. \tag{60}$$

According to the mass conservation property (18) the set $\mathcal{H}_\alpha$ is positively invariant, hence we shall consider further $\{T(t)\}_{t\geqslant 0}$ restricted to the semigroup on $\mathcal{H}_\alpha$. With the use of our previous results we shall justify that:

Lemma 6. *There exists a bounded subset $\mathcal{B}$ of $\mathcal{H}_\alpha$ attracting each point of $\mathcal{H}_\alpha$ (that is, the semigroup $\{T(t)\}_{t\geqslant 0}$ on $\mathcal{H}_\alpha$ is point dissipative* [5, p.38]*).*

Proof: Let us define the set $\mathcal{B}$ as

$$\mathcal{B} := \bigcup_{u\in\mathcal{H}_\alpha}\omega(u), \tag{61}$$

where $\omega(u)$ denotes the *ω-limit set* of u. Because of the estimates (18), (41) $\{T(t)\}_{t\geqslant 0}$ takes bounded sets into bounded sets. Moreover, since the operator A (defined below

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formula (9)) is sectorial and, according to [**11**, Theorem 5.5.1.b] its resolvent is compact, then [**5**, Theorem 4.2.2] ensures that $\{T(t)\}_{t\geqslant 0}$ is compact. Hence $\mathcal{B}$ attracts each point of $\mathcal{H}_\alpha$, and to be able to use [**5**, Theorem 4.2.4] we only need to show that $\mathcal{B}$ is bounded in $\mathcal{H}_\alpha$.

For every $u_0 \in \mathcal{B}$ there is $w_0 \in \mathcal{H}_\alpha$ such that u_0 belongs to $\omega(w_0)$. Then by substituting $w := T(t)w_0$ as an argument of the Lyapunov functional $\mathcal{L}$ (21), it is obvious that:

$$\exists_{\beta\in\mathbb{R}} \lim_{t\to+\infty} \mathcal{L}(T(t)w_0) = \beta, \tag{62}$$

since $\mathcal{L}(T(t)w_0)$ is decreasing and bounded below. Moreover, from a characterisation of ω-limit sets the Lyapunov functional $\mathcal{L}$ is identically equal to β along the trajectory of every element from $\omega(w_0)$. Thus we have in particular:

$$\mathcal{L}(T(t)u_0) = \beta \quad \text{for all } t \geqslant 0, \tag{63}$$

so that, according to Lemma 1 (Section 3.B), u_0 must be a stationary solution of the system (1)–(3) belonging to $\mathcal{D}_\alpha$. Hence, due to Lemma 5, $\mathcal{B}$ is bounded in $D(A^{1/2})$. The proof of Lemma 6 is finished. □

The results we have obtained so far justify the validity of all the required assumptions of [**5**, Theorem 4.2.4]. As a direct consequence of this theorem we obtain the existence of the global attractor for the system (1)–(3); so we are able to formulate:

Theorem 1. *Semigroup $\{T(t)\}_{t\geqslant 0}$ generated by the Cahn-Hilliard system (1)–(3) on a metric space $\mathcal{H}_\alpha$ possesses a connected global attractor.*

5. Asymptotic behaviour of the trajectories

We have already mentioned in Lemma 6 that each element w_0 of the phase space $D(A^{1/2})$ is attracted by its ω-limit set $\omega(w_0)$. It has also been shown explicitly that ω-limit sets contain only stationary solutions of the system (1)–(3). However, if the gradient $\nabla\Phi$ is a monotone operator, then any ω-limit set consists of a single element, and in consequence $T(t)w_0$ must tend to a stationary solution. Thus let us introduce the set $\mathcal{W} \subset \mathbb{R}^m$:

$$\mathcal{W} := \{s' \in \mathbb{R}^m;\ \forall_{s\in\mathbb{R}^m} (s'-s)^T(\nabla_{s'}\Phi(s') - \nabla_s\Phi(s)) \geqslant 0\}. \tag{64}$$

We shall prove the following:

Theorem 2. *For each element w_0 from $D(A^{1/2})$ with the average $\overline{w}_0$ belonging to $\mathcal{W}$, $T(t)w_0 \to \overline{w}_0$ in $\mathcal{H}^2$ as t goes to infinity.*

Proof: It suffices to show that for arbitrary $w_0 \in D(A^{1/2})$ with $\overline{w}_0 \in \mathcal{W}$, the set $\omega(w_0)$ consists of a single element $\overline{w}_0$. If $v \in \omega(w_0)$, then v has the same spatial average as w_0. Moreover, since v and $\overline{w}_0$ are time independent solutions of (1)–(3), then with the use of Lemma 2 we find that

$$-\Gamma\Delta(\overline{w}_0 - v) + \nabla_{\overline{w}_0}\Phi(\overline{w}_0) - \nabla_v\Phi(v) = \overline{\nabla_{\overline{w}_0}\Phi(\overline{w}_0)} - \overline{\nabla_v\Phi(v)}. \tag{65}$$

Multiplying (65) in $\mathcal{L}^2$ by $\overline{w}_0 - v$, we obtain (Γ is positive definite):

$$\begin{aligned}\gamma_0 \sum_{i=1}^{m} \int |\nabla(\overline{w}_0 - v)|^2\, dx &\leqslant -\int (\overline{w}_0 - v)^T (\nabla_{\overline{w}_0}\Phi(\overline{w}_0) - \nabla_v\Phi(v))dx \\ &\quad + \int (\overline{w}_0 - v)^T \left(\overline{\nabla_{\overline{w}_0}\Phi(\overline{w}_0)} - \overline{\nabla_v\Phi(v)}\right)dx \\ &= -\int (\overline{w}_0 - v)^T (\nabla_{\overline{w}_0}\Phi(\overline{w}_0) - \nabla_v\Phi(v))dx.\end{aligned} \tag{66}$$

Since $\overline{w}_0 \in \mathcal{W}$, condition (66) gives:

$$\sum_{i=1}^{m} \int |\nabla(\overline{w}_0 - v)|^2\, dx = 0. \tag{67}$$

Then because the spatial average of v is equal to $\overline{w}_0$ and $v \in D(A^{1/2}) \subset C^0(\overline{\Omega})$, the function v must be identically equal to $\overline{w}_0$. The proof is completed. $\square$

REMARK 4. In order to ensure that $\mathcal{W}$ is a nontrivial set, further assumptions on the term Φ are needed, for example it is clear that $\mathcal{W} = \mathbb{R}^m$ if we assume that the function Φ is convex (that is, its Hessian matrix Φ'' is nonnegative definite). However, the set $\mathcal{W}$ can be "large" in $\mathbb{R}^m$ also in the case when some special forms of nonlinearities in (1) are considered. For example, let us take

$$\Phi(s_1, \ldots, s_m) = \sum_{i=1}^{m} W_i(s_i),$$

where W_i; $i = 1, \ldots, m$ are real polynomials of order $2p$ with positive leading coefficients (compare the conditions (4)–(8)). For such W_i the derivatives $W_i'(r)$ are monotone for $|r|$ sufficiently large, $i = 1, \ldots, m$, so we shall denote by a_i (b_i) the value of largest local maximum (smallest local minimum) of W_i'. Thus the set $\mathcal{W}$ consists of all $s' \in \mathbb{R}^m$ those components s_i' are outside the intervals (z_1^i, z_2^i) for each $i = 1, \ldots, m$, where constants z_1^i, z_2^i are defined as

$$\begin{cases} z_1^i = \sup\{z;\ W_i'(r) \leqslant b_i \text{ for all } r \leqslant z\}, \\ z_2^i = \inf\{z;\ W_i'(r) \geqslant a_i \text{ for all } r \geqslant z\}, \quad i = 1, \ldots, m. \end{cases}$$

6. Backward uniqueness result

At the end of the paper let us summarise some of the results we have obtained so far and consider the semiflow $\{T(t)\}_{t\geqslant 0}$ restricted to the global attractor in the phase space $\mathcal{H}_\alpha$. Since for two solutions w_1, w_2 of the problem (1)–(3) the Lipschitz condition (14) holds, then based on the abstract backward uniqueness result stated in [**10**, Lemma 6.2, p.170] it follows immediately that the restricted $T(t)$ is a one-to-one map for each $t \geqslant 0$. Then using the general semigroup property given in [**5**, Theorem 3.10, p.56] we obtain finally:

Lemma 7. *Semigroup $\{T(t)\}_{t\geqslant 0}$ restricted to the global attractor in the phase space $\mathcal{H}_\alpha$ can be extended to a group.*

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