FIXED POINT THEOREMS BY ALTERING
DISTANCES BETWEEN THE POINTS

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In this paper we have established some fixed point theorems
in complete and compact metric spaces.

1. Introduction

Let \( \mathbb{R}^+ \) be the set of nonnegative real numbers and \( \mathbb{N} \) the set of
positive integers.

Delbosco [1] and Skof [8] have established fixed point theorems for
selfmaps of complete metric spaces by altering the distances between the
points with the use of a function \( \varphi: \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying the following
properties:

1. \( \varphi \) is continuous and strictly increasing in \( \mathbb{R}^+ \);
2. \( \varphi(t) = 0 \) if and only if \( t = 0 \);
3. \( \varphi(t) \geq M \cdot t^\mu \) for every \( t > 0 \), where \( M > 0 \), \( \mu > 0 \) are constant.

We denote the set of above functions \( \varphi \) with \( \Phi \).

Precisely in [8, corol. 2] the following theorem was proved:

**THEOREM 1.** Let \( T \) be a selfmap of a complete metric space \( (X,d) \)
and \( \varphi \in \Phi \) such that for every \( x,y \) in \( X \),

\[
\varphi(d(Tx,Ty)) \leq a \cdot \varphi(d(x,y)) + b \cdot \varphi(d(x,Tx)) + c \cdot \varphi(d(y,Ty))
\]

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where $0 < a + b + c < 1$. Then $T$ has a unique fixed point.

In [1], the author has considered functions $\phi \in \Phi$ such that $\phi(t) = t^n$, $n \in \mathbb{N}$, for every $t > 0$.

REMARK 1. Note that $\phi$ is not necessarily a metric: for example, $\phi(t) = t^2$.

REMARK 2. By symmetry of metric $d$, we may assume $b = a$ in (A). The purpose of this paper is to study a stronger condition than (A) and to remove the hypothesis (3) which seems superfluous. Furthermore, our main theorem is an improvement upon some fixed point theorems of Rakotch [5], Reich [6], and a result of Fisher [3] in compact metric spaces. Other related results can be found in Sessa [7].

2. Main theorem

We shall prove a fixed point theorem offering a condition closely related to that used by Massa [4] in Banach spaces. Strictly speaking, the following theorem holds:

THEOREM 2. Let $(X,d)$ be a complete metric space, $T$ a selfmap of $X$, and $\phi : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ an increasing, continuous function satisfying property (2). Furthermore, let $a, b, c$ be three decreasing functions from $\mathbb{R}^+ \setminus \{0\}$ into $[0,1[\)$ such that $a(t) + 2b(t) + c(t) < 1$ for every $t > 0$. Suppose that $T$ satisfies the following condition:

\[
\phi(d(Tx,Ty)) \leq a(d(x,y)) \cdot \phi(d(x,y)) + b(d(x,y)) \cdot \min\{\phi(d(x,Tx)), \phi(d(y,Ty))\} + c(d(x,y)) \cdot \min\{\phi(d(x,Ty)), \phi(d(y,Tx))\},
\]

where $x, y \in X$ and $x \neq y$. Then $T$ has a unique fixed point.

Proof. Let $x_0$ be a point of $X$. We define

\[
(*) \quad x_{n+1} = Tx_n, \quad \tau_n = d(x_n, x_{n+1}), \quad \text{for all } n \in \mathbb{N} \cup \{0\}.
\]

We first prove that $T$ has a fixed point. We may assume $\tau_n > 0$ for each $n$. From (B), we obtain:

\[
\phi(\tau_{n+1}) \leq a(\tau_n) \cdot \phi(\tau_n) + b(\tau_n) \cdot \min\{\phi(\tau_n), \phi(\tau_{n+1})\} + c(\tau_n) \cdot \min\{\phi(d(x_n, x_{n+2})), \phi(d(x_{n+1}, x_{n+1}'))\}.
\]
Hence we obtain:

\[
\varphi(\tau_{n+1}) \leq \frac{a(\tau_n) + b(\tau_n)}{1 - b(\tau_n)} \cdot \varphi(\tau_n) < \varphi(\tau).
\]

Since \( \varphi \) is increasing, \( \{\tau_n\} \) is a decreasing sequence.

We put \( \lim_{n \to \infty} \tau_n = \tau \) and suppose that \( \tau > 0 \). By (2.1), then

\[
\tau_n > \tau \implies \varphi(\tau_{n+1}) < \frac{a(\tau) + b(\tau)}{1 - b(\tau)} \cdot \varphi(\tau).
\]

By letting \( n \to \infty \), since \( \varphi \) is continuous, we have:

\[
\varphi(\tau) \leq \frac{a(\tau) + b(\tau)}{1 - b(\tau)} \cdot \varphi(\tau) < \varphi(\tau),
\]

which is inadmissible. So \( \tau = 0 \). Now we prove that \( \{x_n\} \) is a Cauchy sequence. Suppose it is not. Then there exist \( \varepsilon > 0 \) and two sequences \( \{p(n)\}, \{q(n)\} \) such that for every \( n \in \mathbb{N} \cup \{0\} \), we find that

\[
p(n) > q(n) \geq n, \quad d(x_{p(n)}, x_{q(n)}) > \varepsilon \quad \text{and} \quad d(x_{p(n)-1}, x_{q(n)}) < \varepsilon.
\]

For each \( n \geq 0 \), we put \( s_n = d(x_{p(n)}, x_{q(n)}) \). Then we have

\[
\varepsilon \leq s_n \leq d(x_{p(n)-1}, x_{p(n)}) + d(x_{p(n)-1}, x_{q(n)}) < \tau_{p(n)-1} \varepsilon.
\]

Since \( \{\tau_n\} \) converges to \( 0 \), \( \{s_n\} \) converges to \( \varepsilon \).

Furthermore, the triangular inequality implies, for each \( n \geq 0 \),

\[
-\tau_{p(n)} - \tau_{q(n)} + s_n \leq d(x_{p(n)+1}, x_{q(n)+1}) \leq \tau_{p(n)} + \tau_{q(n)} + s_n,
\]

and therefore also the sequence \( \{d(x_{p(n)+1}, x_{q(n)+1})\} \) converges to \( \varepsilon \).

From (B), we also deduce:

\[
\varphi(d(x_{p(n)+1}, x_{q(n)+1})) \leq a(s_n) \cdot \varphi(s_n) + b(s_n) \cdot (\varphi(\tau_{p(n)}) + \varphi(\tau_{q(n)})) + c(s_n) \cdot \min\{\varphi(d(x_{p(n)}, x_{q(n)+1})), \varphi(d(x_{q(n)}, x_{p(n)+1}))\} \leq a(\varepsilon) \cdot \varphi(s_n) + b(\varepsilon) \cdot (\varphi(\tau_{p(n)}) + \varphi(\tau_{q(n)})) + c(\varepsilon) \cdot \varphi(s_n + \tau_{q(n)} + \tau_{p(n)}).
\]
For \( n \to \infty \) we are left with
\[
\varphi(\epsilon) \leq \{a(\epsilon) + c(\epsilon) \}.\varphi(\epsilon) < \varphi(\epsilon),
\]
which is absurd. Therefore \( \{x_n\} \) is a Cauchy sequence. By completeness of \( X \), \( \{x_n\} \) converges to some point \( z \). Now we show that \( z \) is a fixed point of \( T \). Since each \( \tau_n > 0 \), there is a subsequence \( \{x_{\tau(n)}\} \) of \( \{x_n\} \) such that \( x_{\tau(n)} \neq z \) for each \( n \geq 0 \) and we put \( \rho_n = d(z, x_n) \).

Since \( b < 1/2 \), we obtain from (A):
\[
\varphi(d(x_{\tau(n)} + 1, Tz)) \leq a(\rho_{\tau(n)}).\varphi(\rho_{\tau(n)}) + b(\rho_{\tau(n)}).\{\varphi(\tau_{\tau(n)}) + \varphi(d(z, Tz))\}
+ c(\rho_{\tau(n)}).\min\{\varphi(\rho_{\tau(n)} + 1), \varphi(d(x_{\tau(n)}, Tz))\}
\leq \varphi(\rho_{\tau(n)}) + 1/2(\varphi(\tau_{\tau(n)}) + \varphi(d(z, Tz)) + \varphi(\rho_{\tau(n)} + \tau_{\tau(n)})).
\]

Since \( \{\rho_n\} \) converges to \( 0 \), for \( n \to \infty \) the last inequality yields
\[
(2.2) \quad \limsup_{n \to \infty} \varphi(d(x_{\tau(n)} + 1, Tz)) \leq 1/2 \varphi(d(z, Tz)).
\]

On the other hand, the triangular inequality implies that
\[
d(z, Tz) \leq \rho_{\tau(n)} + \tau_{\tau(n)} + d(x_{\tau(n)} + 1, Tz),
\]
which in turn implies that
\[
(2.3) \quad \varphi(d(z, Tz)) \leq \limsup_{n \to \infty} \varphi(d(x_{\tau(n)} + 1, Tz)).
\]

From (2.2) and (2.3), then we deduce
\[
\varphi(d(z, Tz)) \leq 1/2 \varphi(d(z, Tz));
\]
that is, \( \varphi(d(z, Tz)) = 0 \) and therefore \( d(z, Tz) = 0 \).

If \( T \) has two distinct fixed points \( x, y \) in \( X \), then
\[
\varphi(d(x, y)) = \varphi(d(Tx, Ty)) \leq \{a(d(x, y)) + c(d(x, y)) \}.\varphi(d(x, y)) < \varphi(d(x, y)),
\]
a contradiction. This completes the proof.

REMARK 3. Note that we have not supposed the continuity of \( T \).
3. Some consequences and examples

If we assume $c = 0$ in Theorem 2 and take $a, b$ as constants, we obtain Theorem 1. The following examples show that condition (B) is more general than condition (A):

**EXAMPLE 1.** Let $X$ be the subset of $\mathbb{R}^2$ defined by

$$X = \{A, B, C, D, E\},$$

where $A = (-1, 0), B = (0, 0), C = (0, 1/2), D = (0, 1), E = (-1, 1)$.

Let $T : X \to X$ be given by

$$T(A) = B, \quad T(B) = T(C) = T(D) = C, \quad T(E) = D.$$ 

Then $T$ satisfies condition (B) by letting:

$$a(t) = 3/4, \quad b(t) = 0, \quad c(t) = 1/5 \quad \text{and} \quad \varphi(t) = t^2 \quad \text{for any} \quad t \in \mathbb{R}^+.$$ 

However, $T$ does not satisfy condition (A). For otherwise, choosing $x = A$ and $y = E$, we would have

$$\varphi(d(TA, TE)) = \varphi(1) \leq a \cdot \varphi(1) + b \cdot \varphi(1) + c \cdot \varphi(1) < \varphi(1),$$

which is a contradiction.
In Theorem 2 if we assume $c = 0$ and $\varphi(t) = t$ for every $t \geq 0$, we obtain the following condition indebted to Reich [6],

\[ d(Tx, Ty) \leq a(d(x, y)) \cdot d(x, y) + b(d(x, y)) \cdot (d(x, Tx) + d(y, Ty)). \]

The example given below proves that condition (B) is more general than condition (C):

**EXAMPLE 2.** Consider the set $X = \{1, 2, 3, 4\}$ equipped with the metric $d$ which is defined by

\[
\begin{align*}
d(1, 2) &= 2/5, \\
d(1, 3) &= 1/5, \\
d(1, 4) &= 3/5, \\
d(2, 3) &= 2/5, \\
d(2, 4) &= 1, \\
d(3, 4) &= \sqrt{2}/2.
\end{align*}
\]

Let $T$ be a selfmap of $X$ such that

\[ T(1) = T(3) = T(4) = 3, \quad T(2) = 4. \]

Here all the assumptions of Theorem 2 are satisfied with

\[ a(t) = 1/16, \quad b(t) = 1/3, \quad c(t) = 1/16 \quad \text{and} \quad \varphi(t) = t^4 \quad \text{for any} \quad t \in \mathbb{R}^+. \]

But the condition (C) is not fulfilled, otherwise for $x = 1$ and $y = 2$, and all functions $a, b$ from $\mathbb{R}^+ \setminus \{0\}$ into $[0, 1]$ with $a + 2b < 1$, we would have

\[ d(T1, T2) = \sqrt{2}/2 \leq a(2/5) \cdot 2/5 + b(2/5) \cdot 6/5 \leq a(2/5) \cdot 3/5 + 2b(2/5) \cdot 3/5 < 3/5, \]

which is a contradiction as $\sqrt{2}/2 > 3/5$.

If we assume $b = c = 0$ in Theorem 2, we get the following:

**THEOREM 3.** Let $(X, d)$ be a complete metric space, $T$ a selfmap of $X$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing, continuous function for which property (2) holds. Let $a$ be a decreasing function from $\mathbb{R}^+ \setminus \{0\}$ into $[0, 1]$ such that

\[ \varphi(d(Tx, Ty)) \leq a(d(x, y)) \cdot \varphi(d(x, y)), \]

where $x, y \in X$ and $x \neq y$. Then $T$ has a unique fixed point.

**REMARK 4.** For $\varphi(t) = t$ Theorem 3 yields Rakočević's fixed point theorem [5].
4. A result in compact metric spaces

In a paper of Fisher [3], the following theorem has been given:

**THEOREM 4.** Let $T$ be a continuous selfmap of a compact metric space $(X,d)$ such that

$$(E) \quad d(Tx,Ty) < \frac{1}{2}(d(x,Tx) + d(y,Ty))$$

for all distinct $x,y$ in $X$. Then $T$ has a unique fixed point.

Following the fundamental idea of our work presented in section 2 we now generalize Theorem 4 as follows:

**THEOREM 5.** Let $T$ be a continuous selfmap of a metric space $(X,d)$ such that for some $x_0 \in X$ the sequence $\{T^n x_0\}$ has a cluster point $z \in X$. Let there exist a continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying property (2). Furthermore, for all distinct $x,y$ in $X$ the inequality

$$(F) \quad \psi(d(Tx,Ty)) < c\psi(d(x,y)) + \left(\frac{1-c}{2}\right)\left[\psi(d(x,Tx)) + \psi(d(y,Ty))\right]$$

holds, where $0 < c < 1$. Then $z$ is the unique fixed point of $T$.

**Proof.** If $T^n x_0 = T^{n+1} x_0$ for some $n \in \mathbb{N}$, then $z = T^k x_0$ for all $k \geq n$ and therefore the thesis. So we may assume that $T^n x_0 \neq T^{n+1} x_0$ for every $n \in \mathbb{N}$. Let $\{k(n)\}$ be a sequence of positive integers such that $\{T^{k(n)} x_0\}$ converges to $z$. By maintaining the notations (*) of Theorem 2, and using the continuity of $T$, we have

$$\lim_{n \to +\infty} x_{k(n)+1} = T(z) \quad \text{and} \quad \lim_{n \to +\infty} x_{k(n)+2} = T^2(z).$$

As $\psi$ is continuous, it also follows that

$$\phi(d(z,Tz)) = \lim_{n \to +\infty} \phi(T_{k(n)}) = \lim_{n \to +\infty} \phi(T_{k(n)+1}) = \phi(d(Tz,T^2 z)).$$

Now we claim that $z = Tz$, otherwise, by condition (F) when $x = z$ and $y = Tz$, we have

$$\phi(d(Tz,T^2 z)) < c \phi(d(z,Tz)) + \frac{1-c}{2} \{\phi(d(z,Tz)) + \phi(d(Tz,T^2 z))\}.$$
This last inequality implies that
\[ \varphi(d(Tz, T^2z)) < \varphi(d(z, Tz)), \]
which contradicts (4.1).

Property (2) assures the uniqueness of the fixed point.

**REMARK 5.** If \( \varphi(t) = t \) for any \( t \geq 0 \) and \( c = 1 \), Theorem 5 becomes a well-known result of Edelstein [2].

**REMARK 6.** If \( \varphi(t) = t \) for any \( t \geq 0 \) and \( c = 0 \), Theorem 5 reduces to Theorem 4 as every sequence in a compact metric space necessarily has a cluster point.

Using the following example, we show that condition (F) is more general than condition (E):

**EXAMPLE 3.** Consider the set \( X = \{1, 2, 3, 4\} \) with the metric \( d \) defined as
\[
\begin{align*}
d(1, 2) &= \sqrt{3}, \\
d(1, 3) &= 3\sqrt{3}, \\
d(1, 4) &= 12\sqrt{3}, \\
d(2, 3) &= \sqrt{3}, \\
d(2, 4) &= 21\sqrt{3}, \\
d(3, 4) &= 21.
\end{align*}
\]

Let \( T : X \rightarrow X \) be defined by
\[ T(1) = T(3) = T(4) = 3, \quad T(2) = 4. \]

Then condition (F) is clearly verified for \( \varphi(t) = t^2 \) and \( c = 1/3 \).

But condition (E) does not hold because for \( x = 1 \) and \( y = 2 \), we have:
\[
d(T1, T2) = 21 > 12\sqrt{3} = \frac{1}{2} (3\sqrt{3} + 21\sqrt{3}) = \frac{1}{2}(d(1, T1) + d(2, T2)).
\]

The idea of this example appears in [1].

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