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ISOMORPHISMS OF SOME SEGAL ALGEBRAS AND THEIR MULTIPLIER ALGEBRAS

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Let G_1, G_2 be locally compact groups and let S_1, S_2 be Segal algebras on G_1, G_2 respectively. Under certain conditions on G_1, G_2 and S_1, S_2, we prove that if there is a bipositive or isometric isomorphism between S_1, S_2 or between their multiplier algebras then G_1 and G_2 are topologically isomorphic.

1. Introduction

Let G be a locally compact group with a fixed left Haar measure dx. The topology of any locally compact group will be assumed to be Hausdorff. For a function f on G, the left translate ${}_xf$ and the right translate f_x of f are defined by

$$_xf(y) = f\left(x^{-1}y\right)$$

and

$$f_x(y) = f\left(yx^{-1}\right)\Delta\left(x^{-1}\right) , \quad x, y \in G ,$$

where Δ is the modular function of G.

A dense subalgebra $S(G)$ of $L_1(G)$ is said to be a *Segal algebra*, if it satisfies the following conditions:

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(1) $S(G)$ is a Banach space under a norm $\|\cdot\|_S$ such that $\|f\|_S \geq \|f\|_1$ for all $f \in S(G)$;

(2) $S(G)$ is left invariant $\left(f \in S(G) \Rightarrow {}_y f \in S(G)\right.$ for all $\left. y \in G\right)$ and for each $f \in S(G)$ the mapping $y \to {}_y f$ of G into $S(G)$ is continuous;

(3) $\|{}_y f\|_S = \|f\|_S$ for all $f \in S(G)$ and $y \in G$.

If G is discrete then every Segal algebra $S(G)$ coincides with $L_1(G)$.

A Segal algebra $S(G)$ is said to be *symmetric* if $S(G)$ satisfies the following conditions:

(1') $S(G)$ is right invariant and for each $f \in S(G)$ the mapping $y \to f_y$ of G into $S(G)$ is continuous;

(2') $\|f_y\|_S = \|f\|_S$ for all $f \in S(G)$ and $y \in G$.

If G is abelian then every Segal algebra $S(G)$ is symmetric.

A Segal algebra $S(G)$ is said to be *pseudosymmetric* if it satisfies (1') and $S(G)$ contains functions $u \geq 0$ with $\int u dx = 1$ and arbitrarily small support.

For various properties of Segal algebras, we refer to Reiter [13] and [12].

A *multiplier* T of a Segal algebra S on G is a linear map from S to S such that $T(f * g) = T(f) * g$ for $f, g \in S$. It is well known that a multiplier T of S is a continuous linear operator on S . The set $M(S)$ of all multipliers of S is a Banach algebra with multiplication as composition and the norm as operator norm. $M(S)$ is called the *multiplier algebra* of S .

The basic references for multipliers are Larsen [8] and Sections 35 and 36 of Hewitt and Ross [5].

Let S be a Segal algebra on G and let $M(G)$ denote the algebra of bounded regular Borel measures on G . $M(G)$ can be canonically imbedded in $M(S)$ by considering $\mu \in M(G)$ as a multiplier defined by $\mu(f) = \mu * f$, for all $f \in S$. This correspondence is norm decreasing; that is, $\|\mu\|_{M(S)} \leq \|\mu\|_{M(G)}$. For $S = L_1(G)$, this imbedding is an isometric isomorphism of $M(G)$ onto $M(S)$.

For $a \in G$, let $\delta_a \in M(G)$ denote the unit point mass at 'a' . As an element of $M(S)$, δ_a is nothing but left translation by 'a' ; that is, $\delta_a(f) = {}_af$. By the definition of Segal algebras $\|{}_af\|_S = \|f\|_S$, for every $f \in S$. Hence $\lambda\delta_a$ for $|\lambda| = 1$ and $a \in G$ is an isometric multiplier of S . The authors have proved in [11] that for a large class of Segal algebras, these are the only isometric multipliers. This result assumes special significance in relation to Theorem 2 of this paper.

A multiplier T of S is said to be *positive* if $Tf \geq 0$ almost everywhere whenever $f \geq 0$ almost everywhere and $f \in S$. If $\mu \in M(G)$ is a positive measure on G then μ is a positive multiplier of S . Conversely, we shall prove in Lemma 1 that for a large class of Segal algebras these are the only positive multipliers. This result will be the key to the proof of Theorem 1 of this paper.

Let G_1, G_2 be locally compact groups and let A_1, A_2 be spaces of measurable functions on G_1, G_2 respectively. A mapping $\Psi : A_1 \to A_2$ is said to be *bipositive* whenever $\Psi f \geq 0$ almost everywhere if and only if $f \geq 0$ almost everywhere. The bipositive mappings between the spaces of multipliers are defined analogously.

Gaudry [3] proved that if there is a bipositive or isometric isomorphism between the multiplier spaces of $L_p(G_1)$ and $L_p(G_2)$, $1 \leq p < \infty$ and $p \neq 2$, then G_1 and G_2 are topologically isomorphic. There has been considerable interest in results where the existence of some kind of isomorphism between spaces of functions on G_1 and G_2 or between their multiplier spaces implies that G_1 and G_2 are topologically isomorphic. See, Edwards [2], Johnson [6], Nagrajan [9], Parrot [10], Rigelhof [14], Strichartz [15], [16] and Tewari [17]. In this paper we consider isomorphisms between Segal algebras S_1, S_2 on G_1, G_2 or between their multiplier algebras and under some conditions on S_1, S_2, G_1, G_2 we prove that G_1 and G_2 are topologically isomorphic. Tewari [17] and Nagrajan [9] have proved special cases of our results. We also note that our proofs are simpler than those of Tewari and Nagrajan. Following are the main results of this paper.

THEOREM 1. *Let* G_1, G_2 *be locally compact groups and let* S_1, S_2 *be Segal algebras on* G_1, G_2 *respectively. Suppose that any one of the following holds:*

(i) G_1 *and* G_2 *are abelian;*

(ii) G_1 *and* G_2 *are compact and* S_1 *and* S_2 *are symmetric;*

(iii) S_1 *and* S_2 *are pseudosymmetric.*

If there is a bipositive isomorphism Ψ *of* $M(S_1)$ *onto* $M(S_2)$ *then* G_1 *and* G_2 *are topologically isomorphic.*

THEOREM 2. *Let* G_1, G_2 *be locally compact groups and let* S_1, S_2 *be Segal algebras on* G_1, G_2 *respectively. Suppose that the following conditions hold:*

(i) either G_1, G_2 *are abelian or* G_1, G_2 *are compact and* S_1, S_2 *are symmetric;*

(ii) the isometric multipliers of S_1 *and* S_2 *are just unimodular multiples of left translation operators.*

If there is an isometric isomorphism Ψ *of* $M(S_1)$ *onto* $M(S_2)$ *then* G_1 *and* G_2 *are topologically isomorphic.*

2. Bipositive isomorphisms

In this section our main aim is to give a proof of Theorem 1. We begin by proving a result about positive multipliers of a Segal algebra.

LEMMA 1. *Let* T *be a positive multiplier of a Segal algebra* S *on a locally compact group* G . *Suppose that any one of the following* *conditions holds:*

(i) G *is abelian;*

(ii) G *is compact and* S *is symmetric;*

(iii) S *is pseudosymmetric.*

Then there exists a positive measure $\mu \in M(G)$ *such that* $Tf = \mu * f$ *for every* $f \in S$.

Proof. Each of the conditions (i), (ii) and (iii) implies that there exists a two-sided approximate identity $\{f_\alpha\}$ of S such that $f_\alpha \in S$, $f_\alpha \geq 0$ and $\|f_\alpha\|_1 = 1$; see Theorems 33.12 and 28.53 of [5] and Proposition 1 on page 24 of [13]. Fixing $g \in S$ with $g \geq 0$ and $\|g\|_1 = 1$, we get

$$\begin{aligned} \|Tf_\alpha\|_1 = \|Tf_\alpha * g\|_1 \leq \|Tf_\alpha * g\|_S &= \|T(f_\alpha * g)\|_S \\ &\leq \|T\|\|f_\alpha * g\|_S \\ &\leq \|T\|\|g\|_S . \end{aligned}$$

The first equality occurs because Tf_α and g are non-negative and the last inequality follows because $\|f_\alpha\|_1 = 1$. $\{Tf_\alpha\}$ is now a bounded net in $M(G)$. Hence there is a subnet $\{Tf_\beta\}$ and a $\mu \in M(G)$ such that Tf_β converges to μ in the weak star topology of $M(G)$. Since each Tf_β is non-negative, μ is a positive measure. We shall show that $Tf = \mu * f$ for every $f \in S$.

Since Tf_β converges to μ in the weak star topology of $M(G)$, $Tf_\beta * h(x)$ converges to $\mu * h(x)$ for every $h \in C_0(G)$ and $x \in G$. In particular, $Tf_\beta * f * h(x)$ converges to $\mu * f * h(x)$ for every $f \in S$, $h \in C_0(G)$ and $x \in G$.

On the other hand, for any $f \in S$, $f_\beta * f$ converges to f in S . Therefore $T(f_\beta * f) = Tf_\beta * f$ converges in S , and hence in L_1 , to Tf . In particular, $Tf_\beta * f$ converges to Tf in the weak star topology of $M(G)$ and hence $Tf_\beta * f * h(x)$ converges to $Tf * h(x)$ for every

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$h \in C_0(G)$ and $x \in G$. Thus $Tf * h = \mu * f * h$ for every $f \in S$ and $h \in C_0(G)$. This implies that $Tf = \mu * f$ for every $f \in S$ and the proof of the lemma is complete.

Proof of Theorem 1. Let Ψ be a bipositive isomorphism of $M(S_1)$ onto $M(S_2)$ and let μ be a positive measure in $M(G_1)$. It follows that $\Psi\mu$ is a positive multiplier of S_2 and, by Lemma 1, we conclude that $\Psi\mu$ is a positive measure in $M(G_2)$. Conversely, every positive measure in $M(G_2)$ is the image under Ψ of a positive multiplier of S_1, that is, of a positive measure in $M(G_1)$.

Thus Ψ maps the positive cone of $M(G_1)$ onto that of $M(G_2)$. But any measure is a linear combination of positive measures. Hence Ψ restricted to $M(G_1)$ is a bipositive isomorphism of $M(G_1)$ onto $M(G_2)$. It follows from the L_1-case of Theorem 2 of Gaudry [3] that G_1 and G_2 are topologically isomorphic.

REMARKS. 1. In the abelian case Theorem 1 was proved by Tewari for multipliers of A_p-algebras. Nagrajan [9] modified the arguments of Tewari to prove the abelian case of Theorem 1. Our proof is simpler even in the general case.

2. It is obvious that our proof of Theorem 1 is valid for all pairs of Segal algebras S_1, S_2 whose positive multipliers are given by positive measures. The conditions in Theorem 1 are used only to ensure this; see Lemma 1.

3. The proof of Theorem 1, given here, has the further merit that it applies to more general situations than those considered in Theorem 1. For example, it holds good for any space of functions on the group whose multiplier algebra contains the measure algebra and all of whose positive multipliers are given by positive *bounded* measures. This is the case, for instance, for the L_p-spaces on a large class of groups, which include all abelian groups and compact groups; see Brainerd and Edwards [1]. However, it does not hold universally; see [3]. For those groups, for which it is

true, our arguments will yield a simpler proof of Theorem 2 of [3] for $p > 1$.

Rigelhof [14] proved a result about bipositive homomorphisms of measure algebras which contains Gaudry's L_1-case of Theorem 2 [3] as a special case. If, in the proof of Theorem 1, we use Rigelhof's result, we get the following more general version of Theorem 1. In view of Remark 3 above, it follows that a more general version of Gaudry's Theorem 2 [3] is true for certain groups which include all abelian groups and compact groups.

THEOREM 1'. *Let* G_1, G_2 *and* S_1, S_2 *be as in Theorem* 1. *Suppose that there is a bipositive homomorphism* Ψ *of* $M(S_1)$ *onto* $M(S_2)$ *such that if* $\Psi(\mu * M(G_1)) = 0$ *for some* $\mu \in M(G_1)$ *then* $\Psi(\mu) = 0$. *Then there is an open continuous homomorphism* α *of* G_1 *onto* G_2. *If* Ψ *is an isomorphism then so is* α.

COROLLARY 1. *Let* G_1, G_2 *and* S_1, S_2 *be as in Theorem* 1. *If there is a bipositive isomorphism* Ψ *of* S_1 *onto* S_2, *then* G_1 *and* G_2 *are topologically isomorphic.*

Proof. The map $T \to \Psi T \Psi^{-1}$ is a bipositive isomorphism of $M(S_1)$ onto $M(S_2)$ and the result follows from Theorem 1.

REMARK. Corollary 1, in the case of group algebras, was proved by Kawada [7]. His result was, perhaps, the first of the isomorphism theorems of the sort discussed in this paper. When G_1, G_2 are compact, Corollary 1 was proved for the algebras $L_p(G_i)$ and $C(G_i)$, $i = 1, 2$, by Edwards [2].

3. Isometric isomorphisms

Proof of Theorem 2. In the proof to follow, the multiplier corresponding to a measure μ will be denoted by T_μ except when μ is a point mass at a point, in which case δ_a will denote both the point measure concentrated at a as well as the corresponding multiplier. This

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is done in order to avoid the confusion which might arise by using the same symbol for the multiplier as well as the measure.

Condition *(ii)* in Theorem 2 implies that any isometric multiplier of S_1 or S_2 is of the form $\lambda\delta_a$ or $\lambda\delta_b$ where $|\lambda| = 1$ and $a \in G_1$ and $b \in G_2$.

For any $a \in G_1$, $\Psi\delta_a$ is an isometric multiplier of S_2 and hence there exists an element $b = \phi(a)$ of G_2 and a complex number $\lambda(a)$ with $|\lambda(a)| = 1$ such that $\Psi\delta_a = \lambda(a)\delta_b$. This follows from condition *(ii)* of the theorem. Thus we get a map $\phi : G_1 \to G_2$ defined by $\phi(a) = b$. It can be easily seen that ϕ is an isomorphism of G_1 onto G_2. If the continuity of ϕ is proved, then similar considerations using Ψ^{-1} would complete the proof.

So suppose that ϕ is not continuous. Then there is a net $\{a_i\}$ converging to the identity element e_1 of G_1 and a subnet $\{\phi(a_j)\}$ of $\{\phi(a_i)\}$ lying entirely outside some neighbourhood W of the identity element e_2 of G_2.

Now $\{\Psi\delta_{a_j}\}$ is a bounded net in $M(G_2)$ and hence some subnet $\{\Psi\delta_{a_k}\}$ of $\{\Psi\delta_{a_j}\}$ converges in the weak star topology to a measure $\mu \in M(G_2)$. Let A denote the set of all $h \in L_1(G_2)$ having compactly supported Fourier transform if G_2 is abelian and the set of all trigonometric polynomials if G_2 is compact. Then $\Psi T_f(h) \in A$ for every $h \in A$ and $f \in L_1(G_1)$. Hence, for any $h \in A$ and $f \in L_1(G_1)$,

(1) $$\Psi\delta_{a_k}\big(\Psi T_f(h)\big)(x) \to T_\mu\big(\Psi T_f(h)\big)(x) \quad \text{for every } x \in G_2 .$$

But $a_k \to e_1$, so that $\delta_{a_k} * f \to f$ in L_1 and hence $\delta_{a_k}T_f \to T_f$ in $M(S_1)$. This implies that $\Psi\delta_{a_k}\Psi T_f \to \Psi T_f$ in $M(S_2)$ and $\Psi\delta_{a_k}\Psi T_f(g) \to \Psi T_f(g)$ in S_2, and hence in the weak star topology of

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$M(G_2)$, for every $g \in S_2$. Therefore, we get

(2) $\Psi\delta_{a_k}\Psi T_f(g) * h(x) \to \Psi T_f(g) * h(x)$ for every $x \in G_2$ and $h \in A$.

(1) and (2) imply that

(3) $$T_\mu \Psi T_f(g) * h = \Psi T_f(g) * h \quad \text{for} \quad g \in S_2$$

and $h \in A$. Since A is dense in S_2 , (3) implies that

$$T_\mu \Psi T_f = \Psi T_f \text{ , for any } f \in L_1(G_1) \text{ ,}$$

so that $\Psi^{-1}T_\mu T_f = T_f$, which in turn implies that $\Psi^{-1}T_\mu$ equals the identity on $L_1 * S_1$ $(= S_1)$. This gives $T_\mu = \delta_{e_2}$ and so $\mu = \delta_{e_2}$.

Let V be a neighbourhood of e_2 in G_2 such that V^- is compact and $V^- \subset W$. Here V^- denotes the closure of V . Choose $f \in C_0(G_2)$ such that $f(e_2) = 1$ and support of f is contained in W . Since $\phi(a_k) \notin W$ for any k , we have

(4) $$f(\phi(a_k)) = 0 \text{ , for each } k \text{ .}$$

However $\Psi(\delta_{a_k}) = \lambda(a_k)\delta_{\phi(a_k)}$ and $\Psi\delta_{a_k} \to \delta_{e_2}$ in the weak star topology of $M(G_2)$ so that $\lambda(a_k)f(\phi(a_k)) \to f(e_2) = 1$. This contradicts (4). Therefore ϕ is continuous and the proof of the theorem is complete.

COROLLARY 2. *Let* G_1, G_2 *and* S_1, S_2 *be as in Theorem* 2. *If there is an isometric isomorphism* Ψ *of* S_1 *onto* S_2 , *then* G_1 *and* G_2 *are topologically isomorphic.*

Proof. The map $T \to \Psi T \Psi^{-1}$ is an isometric isomorphism of $M(S_1)$ onto $M(S_2)$ and the result follows from Theorem 2.

REMARKS. 1. For a large class of Segal algebras which satisfy conditions *(i)* and *(ii)* of the theorem, see [*11*]. Without some conditions of the sort given in Theorem 2, the results are no longer valid; see Gaudry [3]. If $G_1 = \mathbb{T}$, $G_2 = \mathbb{T} \times \mathbb{T}$, $S_i = L_2(G_i)$, $i = 1, 2$, neither

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the theorem nor the corollary holds.

2. The proof given here will hold good on arbitrary locally compact groups if one can show that there is a subspace A of $S \cap C_0$ such that A is invariant under all multipliers of S and A is either dense in S or in C_0.

3. If it is assumed only that Ψ is norm decreasing, $\Psi\delta_a$ is still an isometry (it has a norm decreasing inverse) and the techniques in the proof of Theorem 2 yield a continuous isomorphism of G_1 into G_2. The difficulty now appears to be in showing that this isomorphism is surjective. Rigelhof [14] has proved that the existence of a norm decreasing isomorphism between $M(G_1)$ and $M(G_2)$ implies that G_1 and G_2 are topologically isomorphic.

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