A CHARACTERIZATION OF THE NORMAL AND
WEIBULL DISTRIBUTIONS

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1. Introduction. Let \( X \) and \( Y \) be two independent normal variates each distributed with zero mean and a common variance. Then the quotient \( X/Y \) has the Cauchy distribution symmetrical about the origin. Of particular interest in recent years has been the converse problem and examples of non-normal distributions with a Cauchy distribution for the quotient have been illustrated by Mauldon [9], Laha [2; 3; 4] and Steck [10].

Characterization problems for the normal distribution based on the independence of suitable statistics and the sample mean have also been considered by several authors [1; 5; 6; 7; 8]. In Section 2, we obtain a characterization of the normal distribution by considering the independence of the sum of squares of \( X \) and \( Y \) and their quotient \( X/Y \).

If \( X \) and \( Y \) are independently distributed as normal variates with zero mean and a common variance, we find that not only does the quotient \( X/Y \) follow the Cauchy law but is independent of the random variable \( X^2 + Y^2 \). This property of independence provides the characterization of the normal law. A similar property of independence between \( X^m + Y^m \) and \( X/Y \) for the Weibull distribution is studied in Section 3.

2. A characterization of the normal law. We need the following two theorems for proving the main result about the normal law.

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(a) THEOREM (Lukacs) [7]. Let \( X_1 \) and \( X_2 \) be two non-degenerate and positive random variables such that \( X_1 \) and \( X_2 \) are independent. The random variables \( U = X_1 + X_2 \) and \( V = X_1 / X_2 \) are independently distributed if and only if both \( X_1 \) and \( X_2 \) have the gamma distribution with the same scale parameter.

(b) THEOREM (Laha) [3]. Let \( X \) and \( Y \) be two independently and identically distributed random variables having a common distribution function \( F(x) \). Let the quotient \( w = x/y \) follow the Cauchy law distributed symmetrically about the origin \( w = 0 \). Then \( F(x) \) has the following properties:

(i) it is symmetric about \( x = 0 \);
(ii) it is absolutely continuous and has a continuous density function \( f(x) = F'(x) > 0 \).

THEOREM. Let \( X \) and \( Y \) be two independently and identically distributed random variables with a common distribution function \( F(x) \). Let the quotient \( W = X/Y \) follow the Cauchy law distributed symmetrically about \( W = 0 \), and be independent of \( U = X^2 + Y^2 \). Then the random variables \( X \) and \( Y \) follow the normal law.

Proof. Applying Lukacs' Theorem to the random variables \( X^2 \) and \( Y^2 \), we find that \( X^2 / Y^2 \) is independent of \( (X^2 + Y^2) \) and hence both \( X^2 \) and \( Y^2 \) have the gamma distribution with the same scale parameter \( \alpha \). If \( X^2 \sim G(\lambda_1, \alpha) \) and \( Y^2 \sim G(\lambda_2, \alpha) \) from the fact that \( W = X/Y \) is Cauchy it is clear that \( V = 1/(1 + W^2) \) has the Beta distribution \( (0, 1) \) with parameters \( (1/2, 1/2) \). But \( V = Y^2 / (X^2 + Y^2) \) has the Beta distribution with parameters \( (\lambda_2, \lambda_1) \) and hence \( \lambda_1 = \lambda_2 = 1/2 \). The density function of \( X^2 \) is therefore

\[
g(x^2) = \left( \frac{\alpha}{\sqrt{\pi}} \right) x^{-1} \exp(-\alpha x^2).
\]

From Laha's Theorem, (i) \( F(x) \) is symmetric about \( x = 0 \), and (ii) \( F(x) \) is absolutely continuous with a continuous probability density function \( f(x) = F'(x) > 0 \). Therefore \( f(x) = f(-x) \). From (A), we have

(letting \( G(.) \) denote the distribution function of \( x^2 \) that

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\[ \mathbb{P}(X < \lambda) = \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \mathbb{P}(x < \lambda) = F(\sqrt{\lambda}) - F(-\sqrt{\lambda}) = 2F(\sqrt{\lambda}) - 1. \]

Hence \( g(\lambda) = f(\sqrt{\lambda})/\sqrt{\lambda} \) (\( \lambda > 0 \)) so that \( f(\sqrt{\lambda}) = \sqrt{\lambda} g(\lambda) \). Thus \( f(x) = \frac{1}{\sqrt{\alpha \pi}} \exp(-\alpha x^2) \) which is the normal density function.

3. A characterization of the Weibull distribution. If \( X \) and \( Y \) are independently distributed as gamma variates with parameters \((\lambda_1, \alpha)\) and \((\lambda_2, \alpha)\), we observe that \( X + Y \) is independent of the scale invariant function \( X/Y \). On the other hand if \( X \) and \( Y \) are independent normal variables then \( X^2 + Y^2 \) and \( X/Y \) are independent. We find that for the Weibull distribution given by

\[ p(x) = \theta \lambda x^{\lambda-1} \exp(-\theta x^\lambda), \quad \lambda > 1, \quad \theta > 0, \quad x > 0 \]

it turns out that \( X^\lambda + Y^\lambda \) is independent of the quotient \( X/Y \). This motivates the following characterization of the Weibull law.

**Theorem 2.** Let \( X \) and \( Y \) be two positive and independently distributed random variables such that the quotient \( V = X/Y \) has the p.d.f. given by \( f(v) = \lambda v^{\lambda-1} / (1 + v^\lambda)^2 \), where \( v > 0 \) and \( \lambda > 1 \). The random variables \( X \) and \( Y \) have the Weibull distribution with the same scale parameter if \( X^\lambda + Y^\lambda \) is independent of \( X/Y \).

**Proof.** We apply Lukacs' theorem to the positive and non-degenerate random variables \( X^\lambda \) and \( Y^\lambda \) and note that both \( X^\lambda \) and \( Y^\lambda \) must have the gamma distribution with the same scale. Let the parameters be \((\lambda_1, \theta)\) and \((\lambda_2, \theta)\) respectively. Since the distribution of \( V \) is known we can obtain the distribution of \( W = 1/(1 + V^\lambda) \). It is \( g(w) = 1 \) \((0 < w < 1)\). \( W \) is a Beta variable \((0, 1)\) with parameters \((1, 1)\). Since \( X^\lambda \) and \( Y^\lambda \) are gamma variables \( Y^\lambda/(X^\lambda + Y^\lambda) = W \) has the Beta distribution \((0, 1)\) with parameters \((\lambda_2, \lambda_1)\) and so \( \lambda_1 = \lambda_2 = 1 \). Therefore the distribution of \( X^\lambda \) is \( p(x^\lambda) = \theta \exp(-\theta x^\lambda) \) and the distribution of \( X \) is found to be \( p_1(x) = \theta \lambda x^{\lambda-1} \exp(-\theta x^\lambda) \). The same distribution can be derived for \( Y \) and the proof is complete.
REFERENCES


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