A NOTE ON CHARACTERISTIC EQUATION OF TOEPLITZ OPERATORS ON THE SPACES $A_k$

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1. Preliminaries

Let $k$ be any integer, $k \geq 0$. The $k$-th Bergman measure on unit ball $B$ of $\mathbb{C}^n$, $\mu_k$, is given by

$$d\mu_k = \frac{\Gamma(n + k + 1)}{\Gamma(n + 1)\Gamma(k + 1)} (1 - |w|^2)^k dv(w).$$

Note that $\mu_0$ is simply normalized Lebesgue measure on $B$. The $k$-th Bergman space, $A_k$, is defined as the space of analytic functions on $B$ which are square integrable with respect to the measure $\mu_k$. Note that $A_k = H^2(\mu_k)$, where $H^2(\mu_k)$ be the $L^2(\mu_k)$-closure of the ball algebra $A$, and that $A_j \subset A_k$ for $j \leq k$. The standard orthonormal base for $A_k$ is given by

$$e^n_k = c(a, k, n)z^a = c(a, k, n)r_1^{a_1}e^{i\alpha_1} \cdots r_n^{a_n}e^{i\alpha_n}$$

where $c(\alpha, k, n)$ is a constant number such that $c(\alpha, k, n) \|z^n\| = 1$. Let $P_k$ denote the projection of $L^2(\mu_k)$ onto $A_k$. Note that $L^\infty(\mu_k) = L^\infty(B) = \{f : f \text{ is essentially bounded on } B \}$ with respect to Lebesgue measure on $B$. Also $H^\infty(\mu_k)$, the weak* -closure of the polynomials in $z$ in $L^\infty(B)$, is the set $\{f : f \in L^\infty(B) \text{ and } f A_k \subset A_k \} = H^\infty$, the set of bounded analytic functions on $B$. For $f \in L^\infty(B)$, $\|f\|_w$ denotes the essential supremum of $f$ on $B$. For any $\varphi \in L^\infty(B)$ and for any $k \geq 0$, we define a Toeplitz operator $T^{(k)}_\varphi : A_k \rightarrow A_k$ as follows:

$$T^{(k)}_\varphi f = P_k(\varphi f) \quad (f \in A_k).$$

It can be seen easily that

$$T^{(k)}_\varphi f(z) = \int_B \frac{\varphi(\zeta)f(\zeta)}{(1 - \langle z, w \rangle)^{k+n+1}} d\mu_k(\zeta).$$
(consult Rudin [1]). The set of all bounded linear operator on $A_k$ is written as $L(A_k)$, clearly, $T^{(k)}_\varphi \in L(A_k)$. It is well-known that equation $T_z T_T = T$ characterize the Toeplitz operators on Hardy space in one complex variable. A. M. Davie and N. P. Jewell [2] proved that $\sum_{i=1}^{n} T_{\varphi} T_{\varphi} = T$ characterizes Toeplitz operators on Hardy space of several complex variables. D. H. Yu and Sh. H. Sun [3] proved that $T^{*} T = T$ is hold for each inner function $\eta$. In [4], for $n = 1$, N. P. Jewell raised the following:

**Problem.** Is there a set of operator equations which characterize Toeplitz operators on the weighted Bergman spaces of one complex variable?

In next section, we answer negatively the problem.

2. Theorems

**Theorem 1.** Let $B$ be a set of operator equations and $A$ be the set of bounded linear operators on the $k$-th weighted Bergman space $A_k$ which satisfy $B$. If $A$ contains all Toeplitz operators on $A_k$, then $A$ is weak* - dense in $L(A_k)$.

**Proof.** If $A$ is not weak* - dense in $L(A_k)$, there exists a nonzero trace class operator $S$ such that $\text{tr}(ST) = 0$ for any $T \in A$. Then there exist $(f_i)$ in $A_k$ such that $S = \sum_{i=1}^{m} f_i \otimes e_i$, where $(e_i)$ is the orthonormal basis of $A_k$ and $f_i \otimes e_i$ is a 1-rank operator on $A_k$. Without loss of generality, one can assume that $(e_i)_{i=1}^{m} = (e_a)_{a \in \mathbb{Z}^n}$, where $e_a = c(n, k, a)z^a$. For convenience, we replace $f_i$ by $f_a$. Note $S^* = \sum_{a} e_a^k \otimes f_a$, so

$$S^* S = \left( \sum_{a} e_a^k \otimes f_a \right) \left( \sum_{a} f_a \otimes e_a^k \right) = \sum_{a} \| f_a \|_2^2 e_a \otimes e_a^k.$$  

Furthermore, $\| S \|_{c_1} = \text{tr}((S^* S)^{1/2}) = \sum_{a} \| f_a \|_2$. Hence, $\sum_{a} \| f_a \|_2 \leq \infty$, consequently, $\sum_{a} f_a e_a^k \in L^1$. If $A$ contains all Toeplitz operators on $A_k$, then for any $\varphi \in L^\infty(B)$, we have $T_{\varphi}^{(k)} \in A$. Thus

$$\text{tr}(T_{\varphi}^{(k)} S) = \sum_{a \in \mathbb{Z}^n} \langle T_{\varphi}^{(k)} S e_a^k, e_a^k \rangle$$

$$= \sum_{a \in \mathbb{Z}^n} \langle \varphi( \sum_{b \in \mathbb{Z}^n} f_b \otimes e_b^k ) e_a^k, e_a^k \rangle$$

$$= \sum_{a \in \mathbb{Z}^n} \langle \varphi f_a, e_a^k \rangle$$

$$= \sum_{a \in \mathbb{Z}^n} \int_B \varphi f_a e_a^k \, d\mu_k$$

\[ \text{tr}(T_{\varphi}^{(k)} S) = \sum_{a \in \mathbb{Z}^n} \langle T_{\varphi}^{(k)} S e_a^k, e_a^k \rangle \]
\[= \int_B \varphi(\sum_{a \in \mathbb{Z}^{+n}} f_a e^{i\theta}) d\mu_k = 0.\]

Since \(\varphi\) is arbitrary, we easily see that \(\sum_{a} f_a(z) e^{i\theta} = 0\) for any \(z \in B\). Suppose \(f_a\) has series expansion \(f_a = \sum_{a \in \mathbb{Z}^{+n}} a_{a\beta} e^{i\theta}(z)\), then

\[
\sum_{a} f_a(z) e^{i\theta}(z) = \sum_{a} \sum_{\beta} a_{a\beta} e^{i\theta}(z)
\]

\[
= \sum_{a} \sum_{\beta} a_{a\beta} c(n, \alpha, k) c(n, \beta, k) z^a z^\beta
\]

\[
= \sum_{a\beta} a_{a\beta} c(n, \alpha, k) c(n, \beta, k) r^{a+\beta} e^{i(\beta-\alpha)\theta}
\]

\[
= \sum_{a\beta} \left( \sum_{\beta} a_{a\beta} c(n, \alpha, k) c(n, \beta, k) e^{i(\beta-\alpha)\theta} \right) r^t = 0,
\]

where

\[
\theta = (\theta_1, \ldots, \theta_n), 0 \leq \theta_i \leq 2\pi, (\beta - \alpha)\theta = \sum (\beta_i - \alpha_i) \theta_i,
\]

\[
r = (r_1, \ldots, r_n), 0 \leq \|r\| < 1.
\]

So for each \(t \in \mathbb{Z}^{+n}\),

\[
\sum_{a+\beta=t} a_{a\beta} c(n, \alpha, k) c(n, \beta, k) e^{i(\beta-\alpha)\theta} = 0
\]

i.e.

\[
\sum_{a+\beta=t} a_{a\beta} c(n, \alpha, k) c(n, \beta, k) e^{i(\beta-2\alpha)\theta} = 0.
\]

Clearly, \((e^{i(l-2\alpha)\theta})\) is linear independent, so \(a_{a\beta} = 0\) for \(\alpha + \beta = t\). Hence, for any \(\alpha \in \mathbb{Z}^{+n}, \beta \in \mathbb{Z}^{+n}\), we have \(a_{a\beta} = 0\) and so \(S = 0\). It contradicts that \(S \neq 0\). This completes the proof.

Frankfurt [5] proved that no bounded operator \(T\) on \(A_0\) satisfies the operator equation \(B_0^* T B_0 = T\), where \(B_0\) is the Bergman shift on \(A_0(D)\) and \(D\) is the unit disc. We can extend this result to the case \(A_k(B)\). In fact, we have the following.

**Theorem 2.** There isn’t nonzero bounded operator \(T\) on \(A_k(B)\) such that \(\sum_{t=1}^n T_{F_t}^{(k)} T T_{Z_t}^{(k)} = T\).

To prove Theorem 2, we need some lemmas. The proof of Lemma 1 is related to that of Proposition 2.4 in [4].
Lemma 1. Let $M_{z_1} \cdots M_{z_n}$ be multiplication by the coordinate functions on $L^2(B, d\mu)$. If there exists $T \in L(L^2)$ such that $\sum_{i=1}^n M_{z_i}^* T M_{z_i} = T$, then $T$ commutes with $M_{z_i}, M_{z_i}^*$ ($i = 1, \ldots, n$).

Proof. For any positive integer $m$ and $f, g \in L^2$, we have

$$\langle Tf, g \rangle = \sum_{\sum_{i=1}^n k_i = m} \frac{m!}{k_1! \cdots k_n!} \langle TM_{z_1}^{k_1} \cdots M_{z_n}^{k_n} f, M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \rangle$$

by $\sum_{i=1}^n T M_{z_i} = T$. Hence

$$\langle (T M_{z_1} - M_{z_1} T) f, g \rangle$$

$$= \sum_{\sum_{i=1}^n k_i = m} \frac{m!}{k_1! \cdots k_n!} \langle TM_{z_1}^{k_1+1} \cdots M_{z_n}^{k_n} f, M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \rangle$$

$$- \sum_{\sum_{i=1}^n k_i = m} \frac{(m+1)!}{(k_1+1)! k_2! \cdots k_n!} \langle TM_{z_1}^{k_1+1} \cdots M_{z_n}^{k_n} f, M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \rangle$$

$$- \sum_{\sum_{i=1}^n k_i = m+1} \frac{(m+1)!}{k_1! \cdots k_n!} \langle TM_{z_1}^{k_1} \cdots M_{z_n}^{k_n} f, M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \rangle$$

Furthermore

$$\| T \|^{-1} \langle (T M_{z_1} - M_{z_1} T) f, g \rangle$$

$$\leq \sum_{\sum_{i=1}^n k_i = m} \frac{m!}{k_1! \cdots k_n!} \| M_{z_1}^{k_1+1} \cdots M_{z_n}^{k_n} f \| \left( 1 - \frac{m+1}{k_1+1} M_{z_1} M_{z_1}^* \right) M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \|$$

$$+ \sum_{\sum_{i=1}^n k_i = m+1} \frac{(m+1)!}{k_1! \cdots k_n!} \| M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} f \| \| M_{z_1} M_{z_1}^* M_{z_1}^{k_1} \cdots M_{z_n}^{k_n} g \|.$$ 

Note for any $f \in L^2(B, d\mu)$

$$\| (M_{z_1} + \cdots + M_{z_n})^m f \|$$

$$= \| (\sum_{i=1}^n |z_i|^2)^m f \| \to 0 \quad (m \to \infty)$$

and

$$\sum_{\sum_{i=1}^n k_i = m} \frac{m!}{p_1! \cdots p_n!} \| M_{z_1}^{p_1} \cdots M_{z_n}^{p_n} f \|^2.$$
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\[= \sum_{\nu_1, \ldots, \nu_m} \frac{m!}{\nu_1! \cdots \nu_m!} \langle (M_{z_1} M_{z_2})^{\nu_1} \cdots (M_{z_n} M_{z_m})^{\nu_m} f, f \rangle \]

\[= \langle (M_{z_1} M_{z_2} + \cdots + M_{z_n} M_{z_m})^m f, f \rangle \]

\[\leq \| (\sum_{\nu_1, \ldots, \nu_m} \nu^m) f \| \| f \|. \]

By

\[\sum_{\nu_1, \ldots, \nu_m} \frac{m!}{\nu_1! \cdots \nu_m!} \| M_{z_1}^{\nu_1} \cdots M_{z_m}^{\nu_m} f \| \| (1 - \frac{m+1}{k_1+1} M_{z_1} M_{z_2}) M_{z_1}^{k_1} \cdots M_{z_m}^{k_m} \| \]

\[\leq \left[ \sum_{\nu_1, \ldots, \nu_m} \frac{m!}{\nu_1! \cdots \nu_m!} \frac{m+1}{k_1+1} \| M_{z_1}^{k_1+1} \cdots M_{z_m}^{k_m} f \| \right]^\frac{1}{2} \]

\[\leq \left[ \sum_{\nu_1, \ldots, \nu_m} \frac{m!}{\nu_1! \cdots \nu_m!} \frac{k_1+1}{m+1} \| (1 - \frac{m+1}{k_1+1} M_{z_1} M_{z_2}) M_{z_1}^{k_1} \cdots M_{z_m}^{k_m} g \| \right]^\frac{1}{2} \]

and

\[\sum_{\nu_1, \ldots, \nu_m} \frac{m!}{\nu_1! \cdots \nu_m!} \frac{k_1+1}{m+1} \| (1 - \frac{m+1}{k_1+1} M_{z_1} M_{z_2}) M_{z_1}^{k_1} \cdots M_{z_m}^{k_m} g \|^2 \]

\[= \left[ \sum_{\nu_1, \ldots, \nu_m} \frac{m!}{\nu_1! \cdots \nu_m!} \frac{k_1+1}{m+1} \| M_{z_1} \cdots M_{z_m} g \|^2 \right] \]

\[+ 2 \frac{m+1}{k_1+1} \Re \langle M_{z_1}^{k_1} \cdots M_{z_m}^{k_m} g, M_{z_1} M_{z_2} M_{z_1}^{k_1} \cdots M_{z_m}^{k_m} g \rangle \]

\[+ \left( \frac{m+1}{k_1+1} \right)^2 \| M_{z_1} M_{z_2} M_{z_1}^{k_1} \cdots M_{z_m}^{k_m} g \|^2 \]

\[\leq \left[ \sum_{\nu_1, \ldots, \nu_m} \frac{m!}{\nu_1! \cdots \nu_m!} \| M_{z_1}^{\nu_1} \cdots M_{z_m}^{\nu_m} g \|^2 + 2 \| M_{z_1}^{k_1} \cdots M_{z_m}^{k_m} g \|^2 \right] \]

\[+ \frac{m+1}{k_1+1} \| M_{z_1}^{k_1+1} M_{z_2}^{k_2} \cdots M_{z_m}^{k_m} g \|^2 \]

\[\leq 3 \langle \sum_{i=1}^n M_{z_i} M_{z_i} \rangle^m g, g \rangle + \langle (\sum_{i=1}^n M_{z_i} M_{z_i} )^{m+1}, g \rangle \]

\[\leq 3 \| (\sum_{i=1}^n M_{z_i} M_{z_i} )^m g \| \| g \| + \| \sum_{i=1}^n M_{z_i} M_{z_i} )^{m+1} g \| \| g \|, \]

we have

\[TM_{z_1} - M_{z_1} T = 0,\]

i.e.

\[TM_{z_1} = M_{z_1} T.\]
Similarly, $TM_{z_i} = M_{z_i}T$, for $i = 1, 2, \cdots, n$. It shows the lemma.

**Lemma 2.** If $T \in \mathcal{L}(A_k)$ satisfy $\sum_{i=1}^{n} T_{z_i}T{T}_{z_i} = T$. Then there is $S \in \mathcal{L}(L^2)$ with $\|S\| = \|T\|$, $\sum_{i=1}^{n} M_{z_i}SM_{z_i} = S$ and such that $T$ is the compression of $S$ to $A_k$.

**Proof.** It is similar to the proof of Lemma 2.5 in [2]. In fact, we can define $\phi : \mathcal{L}(L^2) \rightarrow \mathcal{L}(L^2)$ by

$$\phi(S) = \sum_{i=1}^{n} M_{z_i}SM_{z_i}$$

then $\|\phi(S)\| \leq \|S\|$. Let $T^-$ be any operator on $L^2$ whose compression is $T$, with $\|T^-\| = \|T\|$, let $S_m = \frac{1}{m} \sum_{i=1}^{m} \phi^j(T^-)$, and let $S$ be a weak operator topology limit point of $\{S_m\}$, then $S$ has the required properties.

**Lemma 3.** If $T \in \mathcal{L}(A_k)$ satisfies $\sum_{i=1}^{n} T_{z_i}^{k}T{T}_{z_i}^{k} = T$, then $T$ is a Toeplitz operator.

**Proof.** If $T$ satisfies the equation, and $S$ is the operator given by Lemma 2, then Lemma 1 shows that $S$ commutes with $M_{z_k}$ and $M_{z_k} (k = 1, \ldots, n)$, so there is $\varphi \in L^\infty$ such that $S = M_{\varphi}$, consequently, $T = T_{\varphi}^{(k)}$.

**Proof of Theorem 2.** If there is $T \in \mathcal{L}(A_k)$ such that $\sum_{i=1}^{n} T_{z_i}^{k}T{T}_{z_i}^{k} = T$, then $T$ is a Toeplitz operator on $A_k$, i.e., there is $L^\infty$, such that $T = T_{\varphi}^{(k)}$. Note

$$\sum_{i=1}^{n} T_{z_i}^{(k)}T_{z_i}^{(k)}T_{z_i}^{(k)} = T_{\varphi}^{(k)}$$

so $T_{\varphi}^{(k)}(z_{i}^{2})_z = T_{\varphi}^{(k)}$, and hence, $T_{\varphi}^{(k)}(z_{i}^{2})_z = 0$. Hence, $\varphi = 0$, consequently, $T = 0$. We complete the proof of Theorem 2.

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**REFERENCES**


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