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A boundedness theorem for nearby slopes of holonomic $\mathcal{D}$-modules

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Abstract

Using twisted nearby cycles, we define a new notion of slopes for complex holonomic $\mathcal{D}$-modules. We prove a boundedness result for these slopes, study their functoriality and use them to characterize regularity. For a family of (possibly irregular) algebraic connections $\mathcal{E}_t$ parametrized by a smooth curve, we deduce under natural conditions an explicit bound for the usual slopes of the differential equation satisfied by the family of irregular periods of the $\mathcal{E}_t$. This generalizes the regularity of the Gauss–Manin connection proved by Griffiths, Katz and Deligne.

1. Introduction

Let V be a smooth algebraic variety over a finite field of characteristic $p > 0$, and let U be an open subset in V such that $D := V\backslash U$ is a normal crossing divisor. Let ℓ be a prime number different from p. Using restriction to curves, Deligne defined [Del11] a notion of ℓ-adic local system on U with bounded ramification along D. Such a definition is problematic for treating functoriality questions: the direct image of a local system is not a local system any more, and duality does not commute with restriction in general. In this paper, we investigate the characteristic 0 aspect of this problem, that is, the following question.

Question 1. Let X be a complex manifold. Can one define a notion of holonomic $\mathcal{D}_X$-module with bounded irregularity which has good functoriality properties?

In dimension 1, to bound the irregularity number of a $\mathcal{D}$-module with given generic rank amounts to bounding its slopes. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and let Z be a hypersurface of X. Mebkhout [Meb90] showed that the *irregularity complex* $\mathrm{Irr}_Z\,\mathcal{M}$ of $\mathcal{M}$ along Z is a perverse sheaf endowed with an $\mathbb{R}_{>1}$ increasing locally finite filtration by sub-perverse sheaves $(\mathrm{Irr}_Z\,\mathcal{M})(r)$. If the support of the rth graded piece of $((\mathrm{Irr}_Z\,\mathcal{M})(r))_{r>1}$ is not empty, we say that $1/(r-1)$ is an *analytic slope of* $\mathcal{M}$ *along* Z.[1]

The existence of a uniform bound in Z is not clear a priori. We thus formulate the following conjecture.

Conjecture 1. Locally on X, the set of analytic slopes of a holonomic $\mathcal{D}_X$-module is bounded.

This statement means that for a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, one can find for every point in X a neighbourhood U and a constant $C > 0$ such that the analytic slopes of $\mathcal{M}$ along any

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[1] Note that this terminology differs from that of Mebkhout by the transformation $r \longrightarrow 1/(r-1)$, so that in dimension 1, analytic slopes correspond to the classical slopes defined via Newton polygons.

germ of hypersurface in U are less than or equal to C. The main obstacle to the proof of Conjecture 1 lies in the behaviour of analytic slopes with respect to proper push-forward. On the other hand, Laurent defined *algebraic slopes* using his theory of micro-characteristic varieties [Lau87]. From work by Laurent and Mebkhout [LM99], we know that the set of analytic slopes of a holonomic $\mathcal{D}$-module $\mathcal{M}$ along Z is equal to the set of algebraic slopes of $\mathcal{M}$ along Z. Since micro-characteristic varieties are invariant by duality, we deduce that analytic slopes are invariant by duality.

For a germ $\mathcal{M}$ of $\mathcal{D}_{\mathbb{C}}$-module at $0 \in \mathbb{C}$, the set of analytic slopes of $\mathcal{M}$ at 0 is also the set of slopes of the Newton polygon [SV00] of the formal differential module $\hat{\mathcal{M}} := \mathbb{C}((x)) \otimes_{\mathbb{C}\{x\}} \mathcal{M}$, where $\mathbb{C}\{x\}$ stands for the space of germs at 0 of holomorphic functions. We will simply call these slopes the *slopes of* $\mathcal{M}$ *at* 0.

The aim of this paper is to define a third notion of slopes and to investigate some of its properties. The main idea lies in the observation that for a germ $\mathcal{M}$ of $\mathcal{D}_{\mathbb{C}}$-module at $0 \in \mathbb{C}$, *the slopes of* $\mathcal{M}$ *at* 0 *are encoded in the vanishing of certain nearby cycles*. We show in Proposition 3.3.1 that $r \in \mathbb{Q}_{\geqslant 0}$ is a slope for $\mathcal{M}$ at 0 if and only if one can find a germ N of meromorphic connection at 0 with slope r such that $\psi_0(\mathcal{M} \otimes N) \neq 0$. We thus introduce the following definition.

Definition. Let X be a complex manifold and let $\mathcal{M}$ be an object of the derived category $\mathcal{D}^b_{\mathrm{hol}}(X)$ of complexes of $\mathcal{D}_X$-modules with bounded and holonomic cohomology. Let $f \in \mathcal{O}_X$ be non-constant. We denote by ψ_f the nearby cycle functor[2] associated to f. We define the *nearby slopes of* $\mathcal{M}$ *associated to* f to be the set $\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{M})$ which is the complement in $\mathbb{Q}_{\geqslant 0}$ of the set of rationals $r \geqslant 0$ such that for every germ N of meromorphic connection at $0 \in \mathbb{C}$ with slope r, we have

$$\psi_f(\mathcal{M} \otimes f^+ N) \simeq 0. \tag{1.0.1}$$

Let us observe that the left-hand side of (1.0.1) depends on N via $\mathbb{C}((x)) \otimes_{\mathbb{C}\{x\}} N$, and that nearby slopes are sensitive to the non-reduced structure of div f, whereas analytic and algebraic slopes only see the support of div f.

Twisted nearby cycles appear for the first time in the algebraic context in [Del07]. Deligne proves in [Del07] that for a given function f, the set $\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{M})$ is finite. The main result of this paper is an affirmative answer to Conjecture 1 for nearby slopes, stated in the following theorem.

Theorem 1. *Locally on* X*, the set of nearby slopes of a holonomic* $\mathcal{D}$*-module is bounded.*

This statement means that for a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, one can find for every point in X a neighbourhood U and a constant $C > 0$ such that the nearby slopes of $\mathcal{M}$ associated to any $f \in \mathcal{O}_U$ are less than or equal to C. For flat meromorphic connections with good formal structure, we show the following refinement.

Theorem 2. *Let* $\mathcal{M}$ *be a flat meromorphic connection with good formal structure. Let* D *be the pole locus of* $\mathcal{M}$ *and let* $D_1, \dots, D_n$ *be the irreducible components of* D*. We denote by* $r_i(\mathcal{M}) \in \mathbb{Q}_{\geqslant 0}$ *the highest generic slope of* $\mathcal{M}$ *along* D_i*. Then, the nearby slopes of* $\mathcal{M}$ *are less than or equal to* $r_1(\mathcal{M}) + \cdots + r_n(\mathcal{M})$*.*

[2] For general references on the nearby cycle functor, let us mention [Kas83, Mal83, MS89, MM04].

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The main tool used in the proof of Theorem 1 is a structure theorem for formal flat meromorphic connections first conjectured in [CS89], studied by Sabbah [Sab00] and proved by Kedlaya [Ked10, Ked11] in the context of excellent schemes and analytic spaces, and independently by Mochizuki [Moc09, Moc11b] in the algebraic context.

Let us give some details on the strategy of the proof of Theorem 1. A dévissage carried out in § 4.1 allows one to suppose that $\mathcal{M}$ is a flat meromorphic connection. Using the Kedlaya–Mochizuki theorem, one reduces further to the case where $\mathcal{M}$ has good formal structure. We are thus left to prove Theorem 2. We resolve the singularities of $Z := \operatorname{div} f$. The problem that occurs at this step is that a randomly chosen embedded resolution $p : \tilde{X} \longrightarrow X$ will increase the generic slopes of $\mathcal{M}$ in a way that cannot be controlled. We show in Proposition 4.2.2 that a fine version of embedded resolution [BM89] allows us to control the generic slopes of $p^{+}\mathcal{M}$ in terms of the sum $r_1(\mathcal{M}) + \cdots + r_n(\mathcal{M})$ and the multiplicities of $p^{*}Z$. A crucial tool for this is a theorem [Sab00, I 2.4.3] proved by Sabbah in dimension 2 and by Mochizuki [Moc11a, 2.19] in any dimension relating the good formal models appearing at a given point with the generic models on the divisor locus. Using a vanishing criterion (Proposition 3.4.1), one finally proves (1.0.1) for $r > r_1(\mathcal{M}) + \cdots + r_n(\mathcal{M})$.

Let $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)$ and let us denote by $\mathbb{D}\mathcal{M}$ the dual complex of $\mathcal{M}$. Nearby slopes satisfy the following functorialities.

Theorem 3. (i) *For every $f \in \mathcal{O}_X$, we have*

$$\mathrm{Sl}^{\mathrm{nb}}_f(\mathbb{D}\mathcal{M}) = \mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{M}).$$

(ii) *Let $p : X \longrightarrow Y$ be a proper morphism and let $f \in \mathcal{O}_Y$ such that $p(X)$ is not contained in $f^{-1}(0)$. Then*

$$\mathrm{Sl}^{\mathrm{nb}}_f(p_{+}\mathcal{M}) \subset \mathrm{Sl}^{\mathrm{nb}}_{fp}(\mathcal{M}).$$

Let us observe that (ii) is a direct application of the compatibility of nearby cycles with proper direct image [MS89].

It is an interesting problem to try to compare nearby slopes and analytic slopes. This question will not be discussed in this paper, but we characterize regular holonomic $\mathcal{D}$-modules using nearby slopes.

Theorem 4. *A complex $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)$ is regular if and only if for every quasi-finite morphism $\rho : Y \longrightarrow X$ with Y a complex manifold, the set of nearby slopes of $\rho^{+}\mathcal{M}$ is contained in $\{0\}$.*

For an other characterization of regularity (harder to deal with in practice) using derived endomorphisms, we refer to [Tey16].

Let us give an application of the preceding results. Let U be a smooth complex algebraic variety and let $\mathcal{E}$ be an algebraic connection on U. We denote by $H^k_{\mathrm{dR}}(U, \mathcal{E})$ the kth de Rham cohomology group of $\mathcal{E}$, and by $\mathcal{V}$ the local system of horizontal sections of $\mathcal{E}^{\mathrm{an}}$ on U^{an}. If $\mathcal{E}$ is regular, Deligne proved [Del70] that the canonical comparison morphism

$$H^k_{\mathrm{dR}}(U, \mathcal{E}) \longrightarrow H^k(U^{\mathrm{an}}, \mathcal{V}) \tag{1.0.2}$$

is an isomorphism. If $\mathcal{E}$ is the trivial connection, this is due to Grothendieck [Gro66]. In the irregular case, (1.0.2) is no longer an isomorphism. It can happen that $H^k_{\mathrm{dR}}(U, \mathcal{E})$ is non-zero and $H^k(U^{\mathrm{an}}, \mathcal{V})$ is zero, which means that there are not enough topological cycles in U^{an}. The *rapid decay homology* $H^{\mathrm{rd}}_k(U, \mathcal{E}^{*})$ needed to remedy this problem appears in dimension 1 in [BE04] and in higher dimension in [Hie07, Hie09]. It includes cycles drawn on a compactification of U^{an} taking into account the asymptotic at infinity of the solutions of the dual connection $\mathcal{E}^*$. By Hien duality theorem, we have a perfect pairing

$$\int : H^k_{\mathrm{dR}}(U,\mathcal{E}) \times H^{\mathrm{rd}}_k(U,\mathcal{E}^*) \longrightarrow \mathbb{C}. \tag{1.0.3}$$

For $\omega \in H^k_{\mathrm{dR}}(U,\mathcal{E})$ and $\gamma \in H^{\mathrm{rd}}_k(U,\mathcal{E}^*)$, we call $\int_\gamma \omega$ a *k-period for $\mathcal{E}$*.[3]

Let $f : X \longrightarrow S$ be a proper and generically smooth morphism, where X denotes an algebraic variety and S denotes a neighbourhood of 0 in $\mathbb{A}^1_{\mathbb{C}}$. Let U be the complement of a normal crossing divisor D of X such that for every $t \neq 0$ close enough to 0, D_t is a normal crossing divisor of X_t. Let $\mathcal{E}$ be an algebraic connection on U. Let us denote by $D_1, \ldots, D_n$ the irreducible components of D meeting $f^{-1}(0)$ and let $r_i(\mathcal{E})$ be the highest generic slope of $\mathcal{E}$ along D_i.

As an application of Theorem 2, we prove the following result.

THEOREM 5. *If $\mathcal{E}$ has good formal structure along D and if the fibres X_t, $t \neq 0$, of f are non-characteristic at infinity*[4] *for $\mathcal{E}$, then the k-period vectors of the family $(\mathcal{E}_t)_{t\neq 0}$ are the analytic solutions of the system of differential equations associated to $\mathcal{H}^k f_+\mathcal{E}$. The slopes at 0 of this system are less than or equal to $r_1(\mathcal{E}) + \cdots + r_n(\mathcal{E})$.*

In the case where $\mathcal{E}$ is the trivial connection, we recover that the periods of a proper generically smooth family of algebraic varieties are solutions of a regular singular differential equation with polynomial coefficients [Gri68, Kat70, Del70].

The role played in this paper by nearby cycles has Verdier specialization [Ver83] and moderate nearby cycles as ℓ-adic counterparts. For a discussion of the problems arising in the ℓ-adic case, we refer to [Tey15a].

2. Notation

We collect here a few definitions used all throughout this paper. The letter X will denote a complex manifold.

2.1. For a morphism $f : Y \longrightarrow X$ with Y a complex manifold, we denote by $f^+ : D^b_{\mathrm{hol}}(\mathcal{D}_X) \longrightarrow D^b_{\mathrm{hol}}(\mathcal{D}_Y)$ and $f_+ : D^b_{\mathrm{hol}}(\mathcal{D}_Y) \longrightarrow D^b_{\mathrm{hol}}(\mathcal{D}_X)$ the inverse image and direct image functors for $\mathcal{D}$-modules. We write $f^\dagger$ for $f^+[\dim Y - \dim X]$.

2.2. Let $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)$ and $f \in \mathcal{O}_X$. From $\mathcal{H}^k\psi_f(\mathcal{M} \otimes f^+N) \simeq \psi_f(\mathcal{H}^k\mathcal{M} \otimes f^+N)$ for every k, we deduce

$$\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{M}) = \bigcup_k \mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{H}^k\mathcal{M}). \tag{2.2.1}$$

Let us define $\mathrm{Sl}^{\mathrm{nb}}(\mathcal{M}) := \bigcup_{f\in\mathcal{O}_X} \mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{M})$. The elements of $\mathrm{Sl}^{\mathrm{nb}}(\mathcal{M})$ are the *nearby slopes* of $\mathcal{M}$. For $S \subset \mathbb{Q}_{\geqslant 0}$, we denote by $\mathcal{D}^b_{\mathrm{hol}}(X)_S$ the full subcategory of $\mathcal{D}^b_{\mathrm{hol}}(X)$ of complexes whose nearby slopes are in S.

[3] This is an abuse of terminology, since there are no natural rational structures on those spaces in general. However, in some cases including exponential modules, there is such a structure.

[4] This is, for example, the case if D is smooth and if the fibres of f are transverse to D.

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2.3. Let us denote by $\mathrm{DR} : D^b_{\mathrm{hol}}(\mathcal{D}_X) \longrightarrow D^b_c(X,\mathbb{C})$ the *de Rham functor*[5] and by $\mathrm{Sol} : D^b_{\mathrm{hol}}(\mathcal{D}_X) \longrightarrow D^b_c(X,\mathbb{C})$ the *solution functor* for holonomic $\mathcal{D}_X$-modules.

2.4. For every analytic subspace Z in X, we denote by $i_Z : Z \hookrightarrow X$ the canonical inclusion. The *local cohomology triangle* for Z and $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)$ reads

$$R\Gamma_{[Z]}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow R\mathcal{M}(*Z) \xrightarrow{+1} \tag{2.4.1}$$

It is a distinguished triangle in $D^b_{\mathrm{hol}}(\mathcal{D}_X)$. The complex $R\Gamma_{[Z]}\mathcal{M}$ is *the local algebraic cohomology* of $\mathcal{M}$ along Z and $R\mathcal{M}(*Z)$ is the *localization* of $\mathcal{M}$ along Z.

2.5. Let $\mathcal{M}$ be a germ of flat meromorphic connection at the origin of $\mathbb{C}^n$. Let D be the pole locus of $\mathcal{M}$. For $x \in D$, we define $\hat{\mathcal{M}}_x := \hat{\mathcal{O}}_{\mathbb{C}^n,x} \otimes_{\mathcal{O}_{\mathbb{C}^n,x}} \mathcal{M}$, where $\hat{\mathcal{O}}_{\mathbb{C}^n,x}$ stands for the completion of $\mathcal{O}_{\mathbb{C}^n,x}$ with respect to its maximal ideal. We say that $\mathcal{M}$ has *good formal structure* if the following statements hold.

(i) D is a normal crossing divisor.

(ii) For every $x \in D$, one can find coordinates $(x_1,\dots,x_n)$ centred at x with D defined by $x_1\cdots x_i = 0$, and an integer $p \geqslant 1$ such that if ρ is the morphism $(x_1,\dots,x_n) \longrightarrow (x_1^p,\dots, x_i^p, x_{i+1},\dots,x_n)$, we have a decomposition

$$\rho^+\hat{\mathcal{M}}_x \simeq \bigoplus_{\varphi\in\mathcal{O}_{\mathbb{C}^n}(*D)/\mathcal{O}_{\mathbb{C}^n}} \mathcal{E}^{\varphi}\otimes\mathcal{R}_{\varphi} \tag{2.5.1}$$

where $\mathcal{E}^{\varphi} = (\hat{\mathcal{O}}_{\mathbb{C}^n,x}(*D), d + d\varphi)$ and $\mathcal{R}_{\varphi}$ is a flat meromorphic connection with regular singularity along D.

(iii) For all $\varphi \in \mathcal{O}_{\mathbb{C}^n}(*D)/\mathcal{O}_{\mathbb{C}^n}$ contributing to (2.5.1), we have $\operatorname{div}\varphi \leqslant 0$, that is, the multiplicities of $\operatorname{div}\varphi$ are negative integers.

Let us remark that classically, one requires condition (iii) to be also true for the differences of two φ intervening in (2.5.1). We will not impose this extra condition in this paper.

2.6. Let $\mathcal{M}$ be a flat meromorphic connection on X such that the pole locus D of $\mathcal{M}$ has only a finite number of irreducible components $D_1,\dots,D_n$. Let $i \in [\![1,n]\!]$. As a consequence of a theorem of Malgrange [Mal96, 3.2.1], $\mathcal{M}$ has a good formal structure at each point of a dense open subset U_i of D_i. Moreover, the order of ρ and the set of $\varphi \in \mathcal{O}_{\mathbb{C}^n}(*D)/\mathcal{O}_{\mathbb{C}^n}$ contributing to (2.5.1) for a given $x \in U_i$ do not depend on x. The pole orders of those φ (computed with a local smooth function defining U_i) are the *generic slopes of $\mathcal{M}$ along D_i*. We denote by $r_{D_i}(\mathcal{M})$ the highest generic slope of $\mathcal{M}$ along D_i and we define the *divisor of highest generic slopes of $\mathcal{M}$* by

$$r_{D_1}(\mathcal{M})D_1 + \cdots + r_{D_n}(\mathcal{M})D_n \in Z(X)_{\mathbb{Q}}.$$

3. Preliminaries on nearby cycles in the case of good formal structure

3.1. Let n be an integer and take $i \in \mathbb{N}^{[\![1,n]\!]}$. The *support of* i is the set of $k \in [\![1,n]\!]$ such that $i_k \neq 0$. If $E \subset [\![1,n]\!]$, we define i_E by $i_{Ek} = i_k$ for $k \in E$ and $i_{Ek} = 0$ if $k \notin E$.

[5] In this paper, we follow Hien's convention [Hie09] according to which for a holonomic module $\mathcal{M}$, the complex $\mathrm{DR}\,\mathcal{M}$ is concentrated in degrees $0,\dots,\dim X$.

3.2. Let R be a regular $\mathbb{C}((t))$-differential module, and take $\varphi \in \mathbb{C}[t^{-1}]$. For every $n \geqslant 1$, we define $\rho : t \longrightarrow t^p = x$ and

$$\mathrm{El}(\rho, \varphi, R) := \rho_+(\mathcal{E}^\varphi \otimes R).$$

If R is the trivial rank 1 module, we will use the notation $\mathrm{El}(\rho, \varphi)$. In general, $\mathrm{El}(\rho, \varphi, R)$ has slope $\operatorname{ord}\varphi/p$. The $\mathbb{C}((x))$-modules of type $\mathrm{El}(\rho, \varphi, R)$ for variable (ρ, φ, R) are called *elementary modules*. From [Sab08, 3.3], we know that every $\mathbb{C}((x))$-differential module can be written as a direct sum of elementary modules.

3.3 Dimension 1

In this subsection, we work in a neighbourhood of the origin $0 \in \mathbb{C}$. Let x be a coordinate on $\mathbb{C}$. Take $p \geqslant 1$ and define $\rho : x \longrightarrow t = x^p$.

Proposition 3.3.1. *Let $\mathcal{M}$ be a germ of holonomic $\mathcal{D}$-module at the origin. Let $r > 0$ be a rational number. The following conditions are equivalent.*

- (i) *The rational r is not a slope for $\mathcal{M}$ at 0.*
- (ii) *For every germ N of meromorphic connection of slope r/p, we have*

$$\psi_\rho(\mathcal{M} \otimes \rho^+ N) \simeq 0.$$

Proof. Since ψ is not sensitive to localization and formalization, one can work formally at 0 and suppose that $\mathcal{M}$ and N are differential $\mathbb{C}((x))$-modules.

Let us prove (2) $\Longrightarrow$ (1) by contraposition. Define $\rho' : u \longrightarrow u^{p'} = x$, $\varphi(u) \in \mathbb{C}[u^{-1}]$ with $q = \operatorname{ord}\varphi(u)$ and R a $\mathbb{C}((u))$-regular module such that $\mathrm{El}(\rho', \varphi(u), R)$ is a non-zero elementary factor (§ 3.2) of $\mathcal{M}$ with slope $r = q/p$. Define

$$N := \rho_+ \mathrm{El}(\rho', -\varphi(u)) = \mathrm{El}(\rho\rho', -\varphi(u)).$$

The module N has slope $q/pp' = r/p$. A direct factor of $\psi_\rho(\mathcal{M} \otimes \rho^+ N)$ is

$$\begin{aligned}
\psi_\rho(\rho'_+(\mathcal{E}^\varphi \otimes R) \otimes \rho^+ N) &\simeq \psi_\rho(\rho'_+(\mathcal{E}^\varphi \otimes R) \otimes \rho^+ \mathrm{El}(\rho\rho', -\varphi(u))) \\
&\simeq \psi_\rho(\rho'_+(\mathcal{E}^\varphi \otimes R \otimes (\rho\rho')^+ \mathrm{El}(\rho\rho', -\varphi(u)))) \\
&\simeq \psi_{\rho\rho'}(\mathcal{E}^\varphi \otimes R \otimes (\rho\rho')^+ \mathrm{El}(\rho\rho', -\varphi(u)))
\end{aligned}$$

where the last identification comes from the compatibility of ψ with proper direct image. By [Sab08, 2.4], we have

$$(\rho\rho')^+ \mathrm{El}(\rho\rho', -\varphi(u)) \simeq \bigoplus_{\zeta^{pp'}=1} \mathcal{E}^{-\varphi(\zeta u)}.$$

So $\psi_{\rho\rho'} R$ is a direct factor of $\psi_\rho(\mathcal{M} \otimes \rho^+ N)$ of rank $np(\mathrm{rg} R) > 0$, and (2) $\Longrightarrow$ (1) is proved.

Let us prove (1) $\Longrightarrow$ (2). Let N be a $\mathbb{C}((t))$-differential module of slope r/p. Then $\rho^+ N$ has slope r. Thus, the slopes of $\mathcal{M} \otimes \rho^+ N$ are greater than 0. Hence, it is enough to show the following lemma.

Lemma 3.3.2. *Let M be a $\mathbb{C}((x))$-differential module whose slopes are greater than 0. Then $\psi_\rho M \simeq 0$.*

By Levelt–Turrittin decomposition, we are left to study the case where M is a direct sum of modules of type $\mathcal{E}^\varphi \otimes R$, where $\varphi \in \mathbb{C}[x^{-1}]$ and where R is a regular $\mathbb{C}((x))$-module. The hypothesis on the slopes of M implies $\varphi \neq 0$, and the expected vanishing is standard. $\square$

3.4 A vanishing criterion

Let $\mathcal{M}$ be a germ of flat meromorphic connection at the origin $0 \in \mathbb{C}^n$. We suppose that $\mathcal{M}$ has good formal structure at 0. Let D be the pole locus of $\mathcal{M}$. Let ρ_p be a ramification of degree p along the components of D as in (2.5.1).

Proposition 3.4.1. *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$. Let us define $Z := \operatorname{div} f$ and suppose that $|Z| \subset D$. Let $r \in \mathbb{Q}_{\geqslant 0}$ such that for every irreducible component E of $|Z|$, we have*

$$r_E(\mathcal{M}) \leqslant r v_E(f).$$

Then for every germ N of meromorphic connection at 0 with slopes greater than r, we have

$$\psi_f(\mathcal{M} \otimes f^+ N) \simeq 0 \tag{3.4.2}$$

in a neighbourhood of 0.

Proof. Let us choose local coordinates $(x_1, \ldots, x_n)$ and $a \in \mathbb{N}^n$ such that f is the function $x \longrightarrow x^a$. Take N with slopes greater than r. Since ψ_f depends on $\mathcal{M} \otimes f^+ N$ only via the formalization of $\mathcal{M} \otimes f^+ N$ along Z, one can always suppose that N is a $\mathbb{C}((t))$-differential module and $p = qk$ where $\rho' : t \longrightarrow t^k$ decomposes N.

The morphism ρ_p is a finite cover away from D, so the canonical adjunction morphism

$$\rho_{p+}\rho_p^+ \mathcal{M} \longrightarrow \mathcal{M} \tag{3.4.3}$$

is surjective away from D. So the cokernel of (3.4.3) has support in D. From [Meb04, 3.6-4], we know that both sides of (3.4.3) are localized along D. So (3.4.3) is surjective. We thus have to prove

$$\psi_{f\rho_p}(\rho_p^+ \mathcal{M} \otimes (f\rho_p)^+ N) \simeq 0. \tag{3.4.4}$$

Since $|Z| \subset D$, we have $f\rho_p = \rho' f \rho_q$. So the left-hand side of (3.4.4) is a direct sum of k copies of

$$\psi_{f\rho_q}(\rho_p^+ \mathcal{M} \otimes (f\rho_p)^+ N). \tag{3.4.5}$$

We thus have to prove that (3.4.5) is 0 in a neighbourhood of 0. We have

$$(f\rho_p)^+ N \simeq (f\rho_q)^+ \rho'^+ N$$

with $\rho'^+ N$ decomposed with slopes greater than rk. The zero locus of $f\rho_q$ is $|Z|$, and if E is an irreducible component of $|Z|$, the highest generic slope of $\rho_p^+ \mathcal{M}$ along E is

$$r_E(\rho_p^+ \mathcal{M}) = p \cdot r_E(\mathcal{M}) \leqslant rk \cdot q \cdot v_E(f) = rk \cdot v_E(f\rho_q).$$

Hence we can suppose that $\rho_p = \mathrm{id}$ and that N is decomposed.

Take

$$N = \mathcal{E}^{P(t)/t^m} \otimes R$$

with $P(t) \in \mathbb{C}[t]$ satisfying $P(0) \neq 0$, with $m > r$ and with R regular. Since again ψ_f is insensitive to formalization, one can suppose that

$$\mathcal{M} = \mathcal{E}^{\varphi(x)} \otimes \mathcal{R}$$

with φ as in (iii) in § 2.5 and $\mathcal{R}$ regular. The Sabbah–Mochizuki theorem ([Sab00, I 2.4.3], [Moc11a, 2.19]) says that φ contributes to the Levelt–Turrittin decomposition of $\mathcal{M}$ at the generic point of an irreducible component D' of D. So the multiplicity of $-\operatorname{div}\varphi$ along such a D' is a generic slope of $\mathcal{M}$ along D'. Thus, one can write $\varphi(x) = g(x)/x^b$ where $g(0) \neq 0$ and where the b_i are such that if $i \in \operatorname{Supp} a$, we have $b_i \leqslant ra_i < ma_i$. We thus have to prove the following lemma.

Lemma 3.4.6. *Take $g, h \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $g(0) \neq 0$ and $h(0) \neq 0$. Let $\mathcal{R}$ be a regular flat meromorphic connection with poles contained in $x_1 \cdots x_n = 0$. Take $a, b \in \mathbb{N}^{[\![1,n]\!]}$ such that $A := \operatorname{Supp} a$ is non-empty and $b_i < a_i$ for every $i \in A$. Then*

$$\psi_{x^a}(\mathcal{E}^{g(x)/x^b + h(x)/x^a} \otimes \mathcal{R}) \simeq 0$$

in a neighbourhood of 0. □

3.5 Proof of Lemma 3.4.6

We define $\mathcal{M} := \mathcal{E}^{g(x)/x^b + h(x)/x^a} \otimes \mathcal{R}$. Since A is not empty, a change of variables allows one to suppose that $h = 1$. If $\operatorname{Supp} b \subset A$, a change of variable shows that Lemma 3.4.6 is a consequence of Lemma 3.6.1. Let $i \in \operatorname{Supp} b$ be an integer such that $i \notin A$. Using x_i, a change of variables allows one to suppose that $g = 1$. Let $p_1, \ldots, p_n \in \mathbb{N}^*$ such that $a_j p_j$ is independent of j for every $j \in A$ and $p_j = 1$ if $j \notin A$. Let ρ_p be the morphism $x \longrightarrow x^p$. As in (3.4.3), we see that

$$\rho_{p+}\rho_p^+\mathcal{M} \longrightarrow \mathcal{M}$$

is surjective. We are thus left to prove that Lemma 3.4.6 holds for multi-indices a such that a_j does not depend on j for every $j \in A$. Let us denote by $\mathbb{1}_A$ the characteristic function of A. From [Sab05, 3.3.13], it is enough to prove that

$$\psi_{x^{\mathbb{1}_A}}(\mathcal{E}^{1/x^b + 1/x^a} \otimes \mathcal{R}) \simeq 0.$$

Using the fact that $\mathcal{R}$ is a successive extension of regular modules of rank 1, one can suppose that $\mathcal{R} = x^c$, where $c \in \mathbb{C}^{[\![1,n]\!]}$. Let

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, hook, "\iota"] \arrow[dr, "x^{\mathbb{1}_A}"'] & \mathbb{C}^n \times \mathbb{C} \arrow[d] \\ & \mathbb{C} \end{tikzcd}$$

be the inclusion given by the graph of $x \longrightarrow x^{\mathbb{1}_A}$. Let t be a coordinate on the second factor of $\mathbb{C}^n \times \mathbb{C}$. We have to prove that

$$\psi_t(\iota_+(x^c\mathcal{E}^{1/x^b + 1/x^a})) \simeq 0.$$

Define $\delta := \delta(t - x^{\mathbb{1}_A}) \in \iota_+(x^c\mathcal{E}^{1/x^b+1/x^a})$ and let $(V_k)_{k\in\mathbb{Z}}$ be the Kashiwara–Malgrange filtration on $\mathcal{D}_{\mathbb{C}^n\times\mathbb{C}}$ relative to t, that is,

$$V_k := \{P \in \mathcal{D}_{\mathbb{C}^n\times\mathbb{C}}, P((t)^m) \subset (t)^{m-k}\ \forall m \in \mathbb{Z}\}.$$

For $d \in \mathbb{N}^{[\![1,n]\!]}$ such that $x^d = 0$ is the pole locus of $x^c\mathcal{E}^{1/x^b+1/x^a}$, the family of sections x^d generates $x^c\mathcal{E}^{1/x^b+1/x^a}$. For such d, the family $s := x^d\delta$ generates $\iota_+(x^c\mathcal{E}^{1/x^b+1/x^a})$. We are left to prove $s \in V_{-1}s$. One can always suppose that $1 \in A$. We have

$$x_1\partial_1 s = (d_1 + c_1)s - \frac{b_1}{x^b}s - \frac{a_1}{x^a}s - x^{\mathbb{1}_A}\partial_t s.$$

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We define $M \in \mathbb{N}^{[\![1,n]\!]}$ by $M_k = \max(a_k, b_k)$ for every $k \in [\![1,n]\!]$. We thus have

$$x^M x_1 \partial_1 s = (d_1 + c_1) x^M s - b_1 x^{M-b} s - a_1 x^{M-a} s - x^M x^{\mathbb{1}_A} \partial_t s. \tag{3.5.1}$$

We have $M = a + b_{A^c} = \mathbb{1}_A + (a - \mathbb{1}_A) + b_{A^c} = \mathbb{1}_A + b + m$ with $m \in \mathbb{N}^{[\![1,n]\!]}$. So

$$x^{M-b} s = x^m t s \in V_{-1} s.$$

Moreover, we have

$$x^M x_1 \partial_1 s = x_1 \partial_1 x^M s - M_1 x^M s = x_1 \partial_1 x^{m+b} t s - M_1 x^{m+b} t s \in V_{-1} s$$

and

$$x^M x^{\mathbb{1}_A} \partial_t s = x^{m+b} \partial_t x^{2 \times \mathbb{1}_A} s = x^{m+b} \partial_t t^2 s = 2 x^{m+b} t s + x^{m+b} t (t \partial_t) s \in V_{-1} s.$$

So (3.5.1) gives

$$x^{M-a} s \in V_{-1} s. \tag{3.5.2}$$

Recall that i was chosen at the beginning of the proof such that $i \notin A$ and $i \in \operatorname{Supp} b$. In particular, $(M-a)_i = b_i \neq 0$ and $\partial_i \delta = 0$. Applying $x_i \partial_i$ to (3.5.2), we obtain

$$(d_i + c_i + b_i) x^{M-a} s - b_i \frac{x^{M-a}}{x^b} s \in V_{-1} s,$$

so from (3.5.2) we deduce $x^{M-a-b} s \in V_{-1} s$. We have $M - a - b = -b_A$, so by multiplying $x^{M-a-b} s$ by x^{b_A}, we get $s \in V_{-1} s$.

3.6. The aim of this subsection is to prove the following lemma.

LEMMA 3.6.1. *Let $\alpha, a \in \mathbb{N}^{[\![1,n]\!]}$ such that $\operatorname{Supp} \alpha$ is not empty and $\operatorname{Supp} \alpha \subset \operatorname{Supp} a$. Let $\mathcal{R}$ be a regular flat meromorphic connection with poles contained in $x_1 \cdots x_n = 0$. We have*

$$\psi_{x^\alpha}(\mathcal{E}^{1/x^a} \otimes \mathcal{R}) \simeq 0.$$

Proof. Let $p_1, \ldots, p_n$ be integers such that $\alpha_i p_i$ does not depend on i for every $i \in \operatorname{Supp} \alpha$ (we denote such an integer by m) and $p_i = 1$ if $i \neq \operatorname{Supp} \alpha$. Let ρ_p be the morphism $x \longrightarrow x^p$. As in (3.4.3), the morphism $\rho_{p+} \rho_p^+ \mathcal{M} \longrightarrow \mathcal{M}$ is surjective. We are left to prove Lemma 3.6.1 for α such that α_i does not depend on i for every $i \in \operatorname{Supp} \alpha$. From [Sab05, 3.3.13], one can suppose that $\alpha_i = 1$ for every $i \in \operatorname{Supp} \alpha$. So $\alpha \leqslant a$. One can suppose that $\mathcal{R} = x^b$ where $b \in \mathbb{N}^{[\![1,n]\!]}$. Let

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, hook, "\iota"] \arrow[dr, "x^\alpha"'] & \mathbb{C}^n \times \mathbb{C} \arrow[d] \\ & \mathbb{C} \end{tikzcd}$$

be the inclusion given by the graph of $x \longrightarrow x^\alpha$. Let t be a coordinate on the second factor of $\mathbb{C}^n \times \mathbb{C}$. We have to show that

$$\psi_t(\iota_+(x^b \mathcal{E}^{1/x^a})) \simeq 0.$$

Define $\delta := \delta(t - x^\alpha) \in \iota_+(x^b \mathcal{E}^{1/x^a})$. For $c \in \mathbb{N}^{[\![1,n]\!]}$ such that $\operatorname{Supp} c \subset \operatorname{Supp} a \cup \operatorname{Supp} b$, the family of sections x^c generates $x^b \mathcal{E}^{1/x^a}$. For such c, the family $s := x^c \delta$ generates $\iota_+(x^b \mathcal{E}^{1/x^a})$. It is thus enough to show $s \in V_{-1} s$. Let us choose $i \in \operatorname{Supp} \alpha$. We have

$$x_i \partial_i s = (c_i + b_i) s - \frac{a_i}{x^a} s - x^\alpha \partial_t s.$$

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We have $\alpha \leqslant a$. Define $a = \alpha + a'$. From

$$x^{\alpha} x_i \partial_i s = x_i \partial_i x^{\alpha} s - x^{\alpha} s = x_i \partial_i ts - ts \in V_{-1} s$$

we deduce that $a_i s + x^{a'} x^{2\alpha} \partial_t s \in V_{-1} s$. We also have $x^{2\alpha} \partial_t s = \partial_t x^{2\alpha} s = \partial_t t^2 s = 2ts + t(t\partial_t) s \in V_{-1} s$. Since $a_i \neq 0$, we deduce $s \in V_{-1} s$ and Lemma 3.6.1 is proved. □

4. Proof of Theorem 1

4.1 Dévissage to the case of flat meromorphic connections

Suppose that Theorem 1 is true for flat meromorphic connections for every choice of ambient manifold. Let us show that Theorem 1 is true for $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)$. We argue by induction on $\dim X$. The case where X is a point is trivial. Let us suppose that $\dim X > 0$. We define $Y := \operatorname{Supp} \mathcal{M}$ and argue by induction on $\dim Y$.

Let us suppose that Y is a strict closed subset of X. We denote by $i : Y \longrightarrow X$ the canonical inclusion. Let $\pi : \tilde{Y} \longrightarrow Y$ be a resolution of the singularities of Y [AHV75] and $p := i\pi$. The regular locus $\operatorname{Reg} Y$ of Y is a dense open subset in Y and π is an isomorphism above $\operatorname{Reg} Y$. By Kashiwara's theorem, we deduce that the cone $\mathcal{C}$ of the adjunction morphism

$$p_+ p^{\dagger} \mathcal{M} \longrightarrow \mathcal{M}$$

has support in $\operatorname{Sing} Y$, with $\operatorname{Sing} Y$ a strict closed subset in Y. Let $x \in X$ and let B be a neighbourhood of x with compact closure $\overline{B}$. Then, $p^{-1}(\overline{B})$ is compact. Since $\dim \tilde{Y} < \dim X$, Theorem 1 is true for $p^{\dagger}\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(\tilde{Y})$. Let (U_i) be a finite family of open sets in $\tilde{Y}$ covering $p^{-1}(\overline{B})$ and such that for every i, the set $\mathrm{Sl}^{\mathrm{nb}}((p^{\dagger}\mathcal{M})_{|U_i})$ is bounded by a rational r_i. Define $R = \max_i r_i$.

By the induction hypothesis applied to $\mathcal{C}$, one can suppose at the cost of taking a smaller B containing x that the set $\mathrm{Sl}^{\mathrm{nb}}(\mathcal{C}_{|B})$ is bounded by a rational R'. Take $f \in \mathcal{O}_B$. We have a distinguished triangle

$$\psi_f(p_+ p^{\dagger} \mathcal{M} \otimes f^+ N) \longrightarrow \psi_f(\mathcal{M} \otimes f^+ N) \longrightarrow \psi_f(\mathcal{C} \otimes f^+ N) \xrightarrow{+1}. \tag{4.1.1}$$

By the projection formula and compatibility of ψ with proper direct image, (4.1.1) is isomorphic to

$$p_+ \psi_{fp}(p^{\dagger} \mathcal{M} \otimes (pf)^+ N) \longrightarrow \psi_f(\mathcal{M} \otimes f^+ N) \longrightarrow \psi_f(\mathcal{C} \otimes f^+ N) \xrightarrow{+1}.$$

So we have the desired vanishing on B for $r > \max(R, R')$.

We are left with the case where $\dim \operatorname{Supp} \mathcal{M} = \dim X$. Let Z be a hypersurface containing $\operatorname{Sing} \mathcal{M}$. We have a triangle

$$R\Gamma_{[Z]} \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}(*Z) \xrightarrow{+1}.$$

By applying the induction hypothesis to $R\Gamma_{[Z]}\mathcal{M}$, we are left to prove Theorem 1 for $\mathcal{M}(*Z)$. The module $\mathcal{M}(*Z)$ is a flat meromorphic connection, which concludes the reduction step.

4.2 The case of flat meromorphic connections

Let D be the pole locus of $\mathcal{M}$. At the cost of taking an open cover of X, let $\pi : \tilde{X} \longrightarrow X$ be an embedded resolution of the singularities of D. Since π is an isomorphism above $X\backslash D$, the cone of

$$\pi_+\pi^+\mathcal{M} \longrightarrow \mathcal{M} \tag{4.2.1}$$

has support in D. From [Meb04, 3.6-4], the left-hand side of (4.2.1) is localized along D. So (4.2.1) is an isomorphism. We thus have a canonical isomorphism

$$\pi_+\psi_{f\pi}(\pi^+\mathcal{M}\otimes(f\pi)^+N) \simeq \psi_f(\mathcal{M}\otimes f^+N).$$

Since π is proper, we see as in 4.1 that we are left to prove Theorem 1 for $\pi^+\mathcal{M}$. We can thus suppose that D has normal crossing.

Let $p : \tilde{X} \longrightarrow X$ be a resolution of the turning points for $\mathcal{M}$ as given by the Kedlaya–Mochizuki theorem. Again p is proper and induces an isomorphism above $X\backslash D$. So we are left to prove Theorem 1 for $p^+\mathcal{M}$. So we can suppose that $\mathcal{M}$ has a good formal structure.

At the cost of taking an open cover, we can suppose that D has only a finite number of irreducible components. Let S be the divisor of highest generic slopes (§ 2.6) of $\mathcal{M}$. Let $S_1, \dots, S_m$ be the irreducible components of S. Let us prove that $\mathrm{Sl}^{\mathrm{nb}}(\mathcal{M})$ is bounded by the sum $\deg S$ of the multiplicities of the S_i in S. This is a local statement. Let $f \in \mathcal{O}_X$ and define $Z := \operatorname{div} f$. Let us denote by $|Z|$ (respectively, $|S|$) the support of Z (respectively, S) and let us assume for a moment the validity of the following proposition.

Proposition 4.2.2. *Locally on X, one can find a proper birational morphism $\pi : \tilde{X} \longrightarrow X$ such that:*

(i) *π is an isomorphism above $X\backslash|Z|$;*
(ii) *$\pi^{-1}(|Z|) \cup \pi^{-1}(|S|)$ is a normal crossing divisor;*
(iii) *for every valuation v_E measuring the vanishing order along an irreducible component E of $\pi^{-1}(|Z|)$,*

$$v_E(S) \leqslant (\deg S)v_E(f).$$

Let us suppose that Proposition 4.2.2 is true. At the cost of taking an open cover, let us take a morphism $\pi : \tilde{X} \longrightarrow X$ as in Proposition 4.2.2. Since condition (i) is true, the cone of the canonical comparison morphism

$$\pi_+\pi^+\mathcal{M} \longrightarrow \mathcal{M} \tag{4.2.3}$$

has support in $|Z|$. Since f^+N is localized along $|Z|$, we deduce that (4.2.3) induces an isomorphism

$$(\pi_+\pi^+\mathcal{M})\otimes f^+N \xrightarrow{\sim} \mathcal{M}\otimes f^+N.$$

Applying ψ_f and using the fact that π is proper, we see that it is enough to prove that

$$\psi_{f\pi}(\pi^+\mathcal{M}\otimes(f\pi)^+N) \simeq 0 \tag{4.2.4}$$

for every germ N of meromorphic connection at the origin with slope $r > \deg S$. Since $(f\pi)^+N$ is localized along $\pi^{-1}(|Z|)$, the left-hand side of (4.2.4) is

$$\psi_{f\pi}((\pi^+\mathcal{M})(*\pi^{-1}(|Z|))\otimes(f\pi)^+N). \tag{4.2.5}$$

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The vanishing of (4.2.5) is a local statement on $\tilde{X}$. Since (ii) and (iii) are true, Proposition 3.4.1 asserts that it is enough to show that for every irreducible component E of $\pi^{-1}(|Z|)$, we have

$$r_E((\pi^+\mathcal{M})(*\pi^{-1}(|Z|))) \leqslant (\deg S)v_E(f\pi).$$

Notice that $v_E(f\pi) = v_E(f)$. Let P be a point in the smooth locus of E. Let φ be as in (2.5.1) for $\mathcal{M}$ at the point $Q := \pi(P)$. For $i = 1, \dots, n$, let $t_i = 0$ be an equation of S_i in a neighbourhood of Q. Modulo a unit in $\mathcal{O}_{X,Q}$, we have $\varphi = 1/t_1^{r_1} \cdots t_n^{r_n}$ where $r_i \in \mathbb{Q}_{\geqslant 0}$. If $u = 0$ is a local equation for E in a neighbourhood of P, we have, modulo a unit in $\mathcal{O}_{\tilde{X},P}$,

$$\varphi\pi = \frac{1}{u^{r_1 v_E(t_1)} \dots u^{r_n v_E(t_n)}}.$$

So the slope of $\mathcal{E}^{\varphi\pi}(*\pi^{-1}(|Z|))$ along E is $r_1 v_E(t_1) + \cdots + r_n v_E(t_n)$. By the Sabbah–Mochizuki theorem, r_i is a slope of $\mathcal{M}$ generically along S_i, so $r_i \leqslant r_{S_i}(\mathcal{M})$. We deduce that

$$r_E(\pi^+\mathcal{M}(*\pi^{-1}(|Z|))) \leqslant \sum_i r_{S_i}(\mathcal{M}) v_E(t_i) = v_E(S) \leqslant (\deg S)v_E(f).$$

This concludes the proof of Theorems 1 and 2.

4.3 Proof of Porposition 4.2.2

At the cost of taking an open cover of X, let us take a finite blow-up sequence

$$\pi_n : X_n \xrightarrow{p_{n-1}} X_{n-1} \xrightarrow{p_{n-2}} \cdots \longrightarrow X_1 \xrightarrow{p_0} X_0 = X \tag{4.3.1}$$

given by [BM89, 3.15 and 3.17] for Z relative to the normal crossing divisor $|S|$. Let $|Z|_i$ be the strict transform of $|Z|$ in X_i and let C_i be the centre of p_i. We define inductively $H_0 = |S|$ and $H_{i+1} = p_i^{-1}(H_i) \cup p_i^{-1}(C_i)$ for $i = 1, \dots, n$, where p_i^{-1} denotes the set theoretic inverse image. In particular, H_{i+1} is a closed subset of X_{i+1}. We will endow it with its canonical reduced structure. Then (4.3.1) satisfies the following conditions.

(i) C_i is a smooth closed subset of $|Z|_i$.
(ii) C_i is nowhere dense in $|Z|_i$.
(iii) C_i and H_i have normal crossing for every i.
(iv) $|Z|_n \cup H_n$ is a normal crossing divisor.

Since C_i and the components of H_i are reduced and smooth, condition (iii) means that locally on X_i, one can find coordinates $(x_1, \dots, x_k)$ such that H_i is given by the equation $x_1 \cdots x_l = 0$ and the ideal of C_i is generated by some x_j for $j = 1, \dots, k$. Using condition (i), we see by induction that $\pi_n^{-1}(|Z|) \cup \pi_n^{-1}(|S|) = |Z|_n \cup H_n$. Proposition 4.2.2 is thus a consequence of the following result.

PROPOSITION 4.3.2. *Let*

$$\pi_n : X_n \xrightarrow{p_{n-1}} X_{n-1} \xrightarrow{p_{n-2}} \cdots \longrightarrow X_1 \xrightarrow{p_0} X_0 = X$$

be a blow-up sequence satisfying (i), (ii) and (iii). For every irreducible component E of $\pi_n^{-1}(|Z|)$, we have

$$v_E(S) \leqslant (\deg S)v_E(f). \tag{4.3.3}$$

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Proof. Let $S_1, \dots, S_m$ be the irreducible components of $|S|$ and let $Z_1, \dots, Z_{m'}$ be the irreducible components of Z. Note that some Z_i can be in $|S|$. We define $a_i = v_{Z_i}(f) > 0$ and let Z_{ji} (respectively, S_{ji}) be the strict transform of Z_j (respectively, S_j) in X_i.

We argue by induction on n. If $n = 0$, E is one of the Z_i and then (4.3.3) is obvious. We suppose that (4.3.3) is true for a composite of n blow-ups and we prove that it is true for a composite of $n+1$ blow-ups.

Let $\mathcal{C}_n$ be the set of irreducible components of

$$\bigcup_{i=0}^{n-1} (p_{n-1} \cdots p_i)^{-1}(C_i).$$

Each element $E \in \mathcal{C}_n$ will be endowed with its reduced structure. Condition (i) implies that the irreducible components of $\pi_n^* Z$ are the Z_{in} and the elements of $\mathcal{C}_n$. Condition (ii) implies that none of the Z_{in} belongs to $\mathcal{C}_n$. Thus, we have

$$\pi_n^* Z = \operatorname{div} f\pi_n = a_1 Z_{1n} + \cdots + a_{m'} Z_{m'n} + \sum_{E \in \mathcal{C}_n} v_E(f) E.$$

On the other hand, we have

$$\pi_n^* S = r_{S_1}(\mathcal{M}) S_{1n} + \cdots + r_{S_m}(\mathcal{M}) S_{mn} + \sum_{E \in \mathcal{C}_n} v_E(S) E.$$

Let us consider the last blow-up $p_n : X_{n+1} \longrightarrow X_n$. Let us denote by P the exceptional divisor of p_n and let E_{n+1} be the strict transform of $E \in \mathcal{C}_n$ in X_{n+1}. We have

$$p_n^* Z_{in} = Z_{in+1} + \alpha_i P \quad \text{with } \alpha_i \in \mathbb{N}.$$

Since

$$H_n = \bigcup_{j=0}^{m} S_{jn} \cup \bigcup_{E \in \mathcal{C}_n} E$$

we deduce from condition (iii) and smoothness of C_n that

$$p_n^* E = E_{n+1} + \epsilon_E P \quad \text{with } \epsilon_E \in \{0, 1\}$$

and

$$p_n^* S_{in} = S_{in+1} + \epsilon_i P \quad \text{with } \epsilon_i \in \{0, 1\}.$$

Hence, we have

$$\pi_n^* Z = \sum a_i Z_{in+1} + \sum_{E \in \mathcal{C}_n} v_E(f) E_{n+1} + \Big(\sum a_i \alpha_i + \sum_{E \in \mathcal{C}_n} \epsilon_E v_E(f)\Big) P$$

and

$$\pi_n^* S = \sum r_{S_i}(\mathcal{M}) S_{in+1} + \sum_{E \in \mathcal{C}_n} v_E(S) E_{n+1} + \Big(\sum r_{S_i}(\mathcal{M}) \epsilon_i + \sum_{E \in \mathcal{C}_n} \epsilon_E v_E(S)\Big) P.$$

Formula (4.3.3) is true for the Z_{in+1}. By the induction hypothesis, formula (4.3.3) is true for E_{n+1}, where $E \in \mathcal{C}_n$. We are left to prove that (4.3.3) is true for P. Conditions (i) and (ii) imply that one of the α_i is non-zero, so

$$\begin{aligned}(\deg S)\Big(\sum a_i \alpha_i + \sum \epsilon_E v_E(f)\Big) &\geqslant (\deg S) + (\deg S) \sum \epsilon_E v_E(f) \\ &\geqslant \sum r_{S_i}(\mathcal{M}) \epsilon_i + \sum \epsilon_E (\deg S) v_E(f) \\ &\geqslant \sum r_{S_i}(\mathcal{M}) \epsilon_i + \sum \epsilon_E v_E(S).\end{aligned}$$

□

5. Duality

We prove Theorem 3(i). Let us denote by $\mathbb{D}$ the duality functor for $\mathcal{D}$-modules. There is a canonical comparison morphism

$$\mathbb{D}(\mathcal{M} \otimes f^{+}N) \longrightarrow \mathbb{D}\mathcal{M} \otimes f^{+}\mathbb{D}N. \tag{5.0.4}$$

On a punctured neighbourhood of $0 \in \mathbb{C}$, the module N is isomorphic to a finite sum of copies of the trivial connection. Thus, there is a neighbourhood U of Z such that the restriction of (5.0.4) to $U\backslash Z$ is an isomorphism. Hence, the cone of (5.0.4) has support in Z. We deduce that

$$(\mathbb{D}(\mathcal{M} \otimes f^{+}N))(*Z) \xrightarrow{\sim} \mathbb{D}\mathcal{M} \otimes f^{+}((\mathbb{D}N)(*0)).$$

We have $(\mathbb{D}N)(*0) \simeq N^{*}$, where $*$ is the duality functor for meromorphic connection. Note that $*$ is a slope preserving involution. Since nearby cycles are insensitive to localization and commute with duality for $\mathcal{D}$-modules, we have

$$\psi_f(\mathbb{D}\mathcal{M} \otimes f^{+}N^{*}) \simeq \mathbb{D}(\psi_f(\mathcal{M} \otimes f^{+}N))$$

and Theorem 3(i) is proved.

6. Regularity and nearby cycles

The aim of this section is to prove Theorem 4.

6.1. We will use the following lemma.

LEMMA 6.1.1. *Let F be a germ of closed analytic subspace at the origin $0 \in \mathbb{C}^n$. Let $Y_1, \dots, Y_k$ be irreducible closed analytic subspaces of $\mathbb{C}^n$ containing 0 and such that $F \cap Y_i$ is a strict closed subset of Y_i for every i. Then there exists a germ of hypersurface Z at the origin containing F and such that $Z \cap Y_i$ has codimension 1 in Y_i for every i.*

Proof. Denote by $\mathcal{I}_F$ (respectively, $\mathcal{I}_{Y_i}$) the ideal sheaf of F (respectively, Y_i). By irreducibility, $\mathcal{I}_{Y_i,0}$ is a prime ideal in $\mathcal{O}_{\mathbb{C}^n,0}$. The hypothesis says that $\mathcal{I}_F \not\subseteq \mathcal{I}_{Y_i}$ for every i. From [Mat80, 1.B], we deduce that

$$\mathcal{I}_F \not\subseteq \bigcup_i \mathcal{I}_{Y_i}.$$

Any function $f \in \mathcal{I}_F$ not in $\bigcup_i \mathcal{I}_{Y_i}$ defines a hypersurface as required. □

6.2. We say that a holonomic module $\mathcal{M}$ is *smooth* if the support $\operatorname{Supp}\mathcal{M}$ of $\mathcal{M}$ is smooth equidimensional and if the characteristic variety of $\mathcal{M}$ is equal to the conormal of $\operatorname{Supp}\mathcal{M}$ in X. We denote by $\operatorname{Sing}\mathcal{M}$ the complement of the smooth locus of $\mathcal{M}$. It is a strict closed subset of $\operatorname{Supp}\mathcal{M}$.

Let $x \in X$ and let us define F as the union of $\operatorname{Sing}\mathcal{M}$ with the irreducible components of $\operatorname{Supp}\mathcal{M}$ passing through x which are not of maximal dimension. Define $Y_1, \dots, Y_k$ to be the irreducible components of $\operatorname{Supp}\mathcal{M}$ of maximal dimension passing through x. From 6.1.1, one can find a hypersurface Z passing through x such that:

(i) $Z \cap \operatorname{Supp}\mathcal{M}$ has codimension 1 in $\operatorname{Supp}\mathcal{M}$;
(ii) the cohomology modules of $\mathcal{H}^k\mathcal{M}$ are smooth away from Z;
(iii) $\dim \operatorname{Supp} R\Gamma_{[Z]}\mathcal{M} < \dim \operatorname{Supp}\mathcal{M}$.

6.3. The direct implication of Theorem 4 is a consequence of the preservation of regularity by inverse image and the following proposition.

Proposition 6.3.1. *We have* $\mathcal{D}^b_{\mathrm{hol}}(X)_{\mathrm{reg}} \subset \mathcal{D}^b_{\mathrm{hol}}(X)_{\{0\}}$.

Proof. Take $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)_{\mathrm{reg}}$. We argue by induction on $\dim X$. The case where X is a point is trivial. By arguing on $\dim \operatorname{Supp} \mathcal{M}$ as in § 4.1, we are left to prove Proposition 6.3.1 in the case where $\mathcal{M}$ is a regular flat meromorphic connection. Let D be the pole locus of $\mathcal{M}$. Take $f \in \mathcal{O}_X$ and let N with slope greater than 0. To prove

$$\psi_f(\mathcal{M} \otimes f^+ N) \simeq 0$$

one can suppose, using embedded desingularization, that $D + \operatorname{div} f$ is a normal crossing divisor. We then conclude with Proposition 3.4.1. □

6.4. To prove the reverse implication of Theorem 4, we argue by induction on $\dim X \geqslant 1$. The case of curves follows from Proposition 3.3.1. We suppose that $\dim X \geqslant 2$ and we take $\mathcal{M} \in \mathcal{D}^b_{\mathrm{hol}}(X)_{\{0\}}$. We argue by induction on $\dim \operatorname{Supp} \mathcal{M}$. The case where $\operatorname{Supp} \mathcal{M}$ is punctual is trivial.

Suppose that $0 < \dim \operatorname{Supp} \mathcal{M} < \dim X$. Since $\operatorname{Supp} \mathcal{M}$ is a strict closed subset of X, one can always locally write $X = X' \times D$ where D is the unit disc of $\mathbb{C}$ and where the projection $X' \times D \longrightarrow X'$ is finite on $\operatorname{Supp} \mathcal{M}$. Let $i : X' \times D \longrightarrow X' \times \mathbb{P}^1$ be the canonical immersion. There is a commutative diagram

$$\begin{tikzcd} \operatorname{Supp} \mathcal{M} \arrow[r] \arrow[dr] & X' \times \mathbb{P}^1 \arrow[d, "p"] \\ & X' \end{tikzcd} \tag{6.4.1}$$

The oblique arrow of (6.4.1) is finite, and p is proper. So the horizontal arrow is proper. Thus, $\operatorname{Supp} \mathcal{M}$ is a closed subset in $X' \times \mathbb{P}^1$. Hence, $\mathcal{M}$ can be extended by 0 to $X' \times \mathbb{P}^1$. We also denote this extension by $\mathcal{M}$. It is an object of $\mathcal{D}^b_{\mathrm{hol}}(X' \times \mathbb{P}^1)_{\{0\}}$ and we have to show that it is regular.

Let Z be a divisor in X' given by the equation $f = 0$ and let $\rho : Y \longrightarrow X'$ be a finite morphism. Since p is smooth, the analytic space Y' making the diagram

$$\begin{tikzcd} Y' \arrow[r, "\rho'"] \arrow[d, "p'"'] & X' \times \mathbb{P}^1 \arrow[d, "p"] \\ Y \arrow[r, "\rho"'] & X' \end{tikzcd}$$

cartesian is smooth. Moreover, ρ' is finite. By base change [HTT00, 1.7.3], the projection formula and compatibility of ψ with proper direct image, we have for every germ N of meromorphic connection with slope greater than 0,

$$\begin{aligned} \psi_f(\rho^+ p_+ \mathcal{M} \otimes f^+ N) &\simeq \psi_f(p'_+ \rho'^+ \mathcal{M} \otimes f^+ N) \\ &\simeq \psi_f(p'_+(\rho'^+ \mathcal{M} \otimes (fp')^+ N)) \\ &\simeq p'_+ \psi_{fp'}(\rho'^+ \mathcal{M} \otimes (fp')^+ N) \\ &\simeq 0. \end{aligned}$$

By the induction hypothesis $p_+\mathcal{M}$ is regular. Let $Y_1, \dots, Y_n$ be the irreducible components of $\operatorname{Supp}\mathcal{M}$ with maximal dimension. Since $\operatorname{Sing}\mathcal{M} \cap Y_i$ is a strict closed subset of Y_i and since a finite morphism preserves dimension, $p(\operatorname{Sing}\mathcal{M}) \cap p(Y_i)$ is a strict closed subset of the irreducible closed set $p(Y_i)$. In a neighbourhood of a given point of $p(\operatorname{Sing}\mathcal{M})$, one can find from § 6.2 a hypersurface Z containing $p(\operatorname{Sing}\mathcal{M})$ such that $Z \cap p(Y_i)$ has codimension 1 in $p(Y_i)$ for every i. So $p^{-1}(Z)$ contains $\operatorname{Sing}\mathcal{M}$ and

$$\dim p^{-1}(Z) \cap Y_i = \dim Z \cap p(Y_i) = \dim p(Y_i) - 1 = \dim Y_i - 1.$$

Since Irr^*_Z is compatible with proper direct image [Meb04, 3.6-6], we have

$$\mathrm{Irr}^*_Z\, p_+\mathcal{M} \simeq Rp_*\, \mathrm{Irr}^*_{p^{-1}(Z)}\, \mathcal{M} \simeq 0.$$

Since p is finite over $\operatorname{Supp}\mathcal{M}$, we have

$$Rp_*\, \mathrm{Irr}^*_{p^{-1}(Z)}\, \mathcal{M} \simeq p_*\, \mathrm{Irr}^*_{p^{-1}(Z)}\, \mathcal{M}.$$

So for every $x \in p^{-1}(Z)$, the germ of $\mathrm{Irr}^*_{p^{-1}(Z)}\, \mathcal{M}$ at x is a direct factor of the complex $(p_*\, \mathrm{Irr}^*_Z\, p_+\mathcal{M})_{p(x)} \simeq 0$. Thus $\mathrm{Irr}^*_{p^{-1}(Z)}\, \mathcal{M} \simeq 0$. From [Meb04, 4.3-17], We deduce that $\mathcal{M}(*p^{-1}(Z))$ is regular.

To show that $\mathcal{M}$ is regular, we are left to prove that $R\Gamma_{[p^{-1}(Z)]}\mathcal{M}$ is regular. From § 6.3, the nearby slopes of all quasi-finite inverse images of $\mathcal{M}(*p^{-1}(Z))$ are contained in $\{0\}$. Thus, this is also the case for $R\Gamma_{[p^{-1}(Z)]}\mathcal{M}$. By construction of Z,

$$\dim \operatorname{Supp} R\Gamma_{[p^{-1}(Z)]}\mathcal{M} < \dim \operatorname{Supp}\mathcal{M}.$$

We conclude by applying the induction hypothesis to $R\Gamma_{[p^{-1}(Z)]}\mathcal{M}$.

Let us suppose that $\operatorname{Supp}\mathcal{M}$ has dimension $\dim X$, and let Z be a hypersurface as in § 6.2. Then $\mathcal{M}(*Z)$ is a flat meromorphic connection with poles along Z. Let us show that $\mathcal{M}(*Z)$ is regular. By [Meb04, 4.3-17], it is enough to prove regularity generically along Z. Hence, one can suppose that Z is smooth. By Malgrange's theorem [Mal96], one can suppose that Z is smooth and that $\mathcal{M}(*Z)$ has good formal structure along Z. Let $(x_1, \dots, x_n, t)$ be coordinates centred at $0 \in Z$ such that Z is given by $t = 0$ and let $\rho : (x, u) \longrightarrow (x, u^p)$ be as in § 2.5 for $\mathcal{M}(*Z)$. Let $\mathcal{E}^{g(x,u)/u^k} \otimes \mathcal{R}$ be a factor of $\rho^+(\hat{\mathcal{M}}_0(*Z))$ where $g(0,0) \neq 0$ and where $\mathcal{R}$ is a flat regular meromorphic connection with poles along Z. For a choice of kth root in a neighbourhood of $g(0,0)$, we have

$$\psi_{u/\sqrt[k]{g}}(\rho^+\mathcal{M} \otimes (u/\sqrt[k]{g})^+\mathcal{E}^{-1/u^k}) \simeq 0.$$

Since nearby cycles commute with formalization, we deduce that

$$\psi_u(\rho^+(\hat{\mathcal{M}}_0(*Z)) \otimes \mathcal{E}^{-g/u^k}) \simeq \psi_u(\rho^+\hat{\mathcal{M}}_0 \otimes \mathcal{E}^{-g/u^k}) \simeq 0.$$

Thus $\psi_u\mathcal{R} \simeq 0$, so $\mathcal{R} \simeq 0$. Hence, the only possibly non-zero factor of $\rho^+(\hat{\mathcal{M}}_0(*Z))$ is the regular factor. So $\mathcal{M}(*Z)$ is regular. We obtain that $\mathcal{M}$ is regular by applying the induction hypothesis to $R\Gamma_{[Z]}\mathcal{M}$.

7. Slopes and irregular periods

7.1. The main reference for what follows is [Sab00, II]. Let X be a smooth complex manifold of dimension d and let D be a normal crossing divisor in X. Define $U := X\backslash D$ and let $j : U \longrightarrow X$ be the canonical inclusion. Let $\mathcal{M}$ be a flat meromorphic connection on X with poles along D. We denote by $p : \tilde{X} \longrightarrow X$ the real blow-up of X along D and by $\tilde{\imath} : U \longrightarrow \tilde{X}$ the canonical inclusion.

Let $\mathcal{A}_{\tilde{X}}^{\leq D}$ be the sheaf of differentiable functions on $\tilde{X}$ whose restriction to U is holomorphic and whose asymptotic development along $p^{-1}(D)$ is zero, and let $\mathcal{A}_{\tilde{X}}^{\mathrm{mod}}$ be the sheaf of differentiable functions on $\tilde{X}$ whose restriction to U is holomorphic with moderate growth along $p^{-1}(D)$. We define the *de Rham complex with rapid decay* by

$$\mathrm{DR}_{\tilde{X}}^{\leq D}\,\mathcal{M} := \mathcal{A}_{\tilde{X}}^{\leq D} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}\,\mathrm{DR}_X\mathcal{M}$$

and the *moderate de Rham complex* by

$$\mathrm{DR}_{\tilde{X}}^{\mathrm{mod}}\,\mathcal{M} := \mathcal{A}_{\tilde{X}}^{\mathrm{mod}} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}\,\mathrm{DR}_X\mathcal{M}.$$

7.2. With the notation in § 7.1, if $\mathcal{M}$ has good formal structure along D, we define [Hie09, Proposition 2]

$$H_k^{\mathrm{rd}}(X, \mathcal{M}) := H^{2d-k}(\tilde{X}, \mathrm{DR}_{\tilde{X}}^{\leq D}\,\mathcal{M}).$$

The left-hand side is the space of *cycles with rapid decay for* $\mathcal{M}$. For a topological description justifying the terminology, we refer to [Hie09, 5.1].

7.3 Proof of Theorem 5

We first prove the assertion concerning the slopes of $\mathcal{H}^k f_+\mathcal{E}$. We denote by $j : U \longrightarrow X$ the canonical immersion, $d := \dim X$ and $\mathrm{Sl}_0(\mathcal{H}^k f_+\mathcal{E})$ the slopes of $\mathcal{H}^k f_+\mathcal{E}$ at 0. We will also use the letter f for the restriction of f to U. From [HTT00, 4.7.2], we have a canonical identification

$$(f_+\mathcal{E})^{\mathrm{an}} \simeq (f_+(j_+\mathcal{E}))^{\mathrm{an}} \xrightarrow{\sim} f_+^{\mathrm{an}}(j_+\mathcal{E})^{\mathrm{an}}. \tag{7.3.1}$$

We deduce that

$$\mathrm{Sl}_0(\mathcal{H}^k f_+\mathcal{E}) = \mathrm{Sl}_0(\mathcal{H}^k f_+^{\mathrm{an}}(j_+\mathcal{E})^{\mathrm{an}}).$$

Let x be a local coordinate on S centred at the origin. From Proposition 3.3.1, we have

$$\mathrm{Sl}_0(\mathcal{H}^k f_+^{\mathrm{an}}(j_+\mathcal{E})^{\mathrm{an}}) = \mathrm{Sl}_x^{\mathrm{nb}}(\mathcal{H}^k f_+^{\mathrm{an}}(j_+\mathcal{E})^{\mathrm{an}}).$$

Since $\mathrm{Sl}_x^{\mathrm{nb}}(\mathcal{H}^k f_+^{\mathrm{an}}(j_+\mathcal{E})^{\mathrm{an}}) \subset \mathrm{Sl}_x^{\mathrm{nb}}(f_+^{\mathrm{an}}(j_+\mathcal{E})^{\mathrm{an}})$, we deduce from Theorems 2 and 3 that

$$\mathrm{Sl}_0(\mathcal{H}^k f_+\mathcal{E}) \subset \mathrm{Sl}_{f(x)}^{\mathrm{nb}}((j_+\mathcal{E})^{\mathrm{an}}) \subset [0, r_1 + \cdots + r_n].$$

We are thus left to relate $\mathrm{Sol}(\mathcal{H}^k f_+^{\mathrm{an}}(j_+\mathcal{E})^{\mathrm{an}})$ to the periods of $\mathcal{E}_t$, for $t \neq 0$ close enough to 0. Such a relation appears for a special type of rank 1 connections in [HR08]. We prove more generally the following proposition.

Proposition 7.3.2. *For every k, we have a canonical isomorphism*

$$R^k f_*^{\mathrm{an}}\,\mathrm{Sol}(j_+\mathcal{E})^{\mathrm{an}} \xrightarrow{\sim} R^k(f^{\mathrm{an}}p)_*\,\mathrm{DR}_{\tilde{X}}^{\leq D}(j_+\mathcal{E}^*)^{\mathrm{an}}. \tag{7.3.3}$$

For $t \neq 0$ close enough to 0, the fibre of the right-hand side of (7.3.3) at t is canonically isomorphic to $H_{2d-2-k}^{\mathrm{rd}}(U_t, \mathcal{E}_t^*) := H_{2d-2-k}^{\mathrm{rd}}(X_t^{\mathrm{an}}, (j_{t+}\mathcal{E}_t^*)^{\mathrm{an}})$.

Proof. Set $\mathcal{M} := (j_+\mathcal{E}^*)^{\mathrm{an}}$. Hien duality for the De Rham cohomology of $\mathcal{E}$ on U is induced by a canonical isomorphism of sheaves

$$\mathrm{DR}^{<D}_{\widetilde{X^{\mathrm{an}}}}\mathcal{M}^* \simeq R\mathcal{H}om(\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X^{\mathrm{an}}}}\mathcal{M}, \tilde{\imath}_!\mathbb{C}).$$

We thus have

$$\begin{aligned} Rp_*\,\mathrm{DR}^{<D}_{\widetilde{X^{\mathrm{an}}}}\mathcal{M}^* &\simeq Rp_*R\mathcal{H}om(\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X^{\mathrm{an}}}}\mathcal{M}, \tilde{\imath}_!\mathbb{C}) \\ &\simeq R\mathcal{H}om(Rp_*\,\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X^{\mathrm{an}}}}\mathcal{M}, \mathbb{C}) \\ &\simeq R\mathcal{H}om(\mathrm{DR}_{X^{\mathrm{an}}}\mathcal{M}, \mathbb{C}) \\ &\simeq \mathrm{Sol}\,\mathcal{M}. \end{aligned}$$

The second isomorphism comes from Poincaré–Verdier duality and the fact that $\tilde{\imath}_!\mathbb{C}[2\dim X]$ is the dualizing sheaf of $\widetilde{X^{\mathrm{an}}}$. The third isomorphism comes from the projection formula and the canonical identification [Sab00, II 1.1.8]

$$Rp_*\mathcal{A}^{\mathrm{mod}}_{\widetilde{X^{\mathrm{an}}}} \simeq \mathcal{O}_{X^{\mathrm{an}}}(*D).$$

The last isomorphism comes from the duality theorem for $\mathcal{D}$-modules [Meb79, KK81]. By applying Rf^{an}_*, we obtain for every k and every $t \neq 0$ close enough to 0 the following commutative diagram:

$$\begin{tikzcd}
(R^k f^{\mathrm{an}}_*\,\mathrm{Sol}\,\mathcal{M})_t \arrow[r, "\sim"] \arrow[d, "(1)"'] & (R^k(f^{\mathrm{an}}p)_*\,\mathrm{DR}^{<D}_{\tilde{X}}\mathcal{M}^*)_t \arrow[d, "(6)"] \\
H^k(X^{\mathrm{an}}_t, (\mathrm{Sol}\,\mathcal{M})_t) \arrow[d, "(2)"'] & H^k(X^{\mathrm{an}}_t, (\mathrm{DR}^{<D}_{\tilde{X}}\mathcal{M}^*)_t) \arrow[dd, "(7)"] \\
H^k(X^{\mathrm{an}}_t, \mathrm{Sol}\,\mathcal{M}_t) \arrow[d, "(3)"'] & \\
H^{-k}(X^{\mathrm{an}}_t, \mathbb{D}\,\mathrm{Sol}\,\mathcal{M}_t)^* \arrow[d, "(4)"'] & H^k(X^{\mathrm{an}}_t, \mathrm{DR}^{<D_t}_{\widetilde{X_t}}\mathcal{M}^*_t) \arrow[d, "\wr"] \\
H^{2d-2-k}(X^{\mathrm{an}}_t, \mathrm{DR}\,\mathcal{M}_t)^* \arrow[d, "(5)"'] & H^{\mathrm{rd}}_{2d-2-k}(X^{\mathrm{an}}_t, \mathcal{M}^*_t) \arrow[d, "\wr"] \\
H^{2d-2-k}(U_t, \mathrm{DR}\,\mathcal{E}_t)^* \arrow[r, "(8)"'] & H^{\mathrm{rd}}_{2d-2-k}(U_t, \mathcal{E}^*_t)
\end{tikzcd}$$

By the proper base change theorem, morphisms (1) and (6) are isomorphisms. Morphism (2) is an isomorphism by the non-charactericity hypothesis. Morphism (3) is an isomorphism by Poincaré–Verdier duality. Morphism (4) is an isomorphism by the duality theorem for $\mathcal{D}$-modules. Morphism (5) is an isomorphism by Serre's GAGA theorem [Ser56] and exactness of j_{t*} where $j_t : U_t \longrightarrow X_t$ is the inclusion morphism. Morphism (8) is an isomorphism by the Hien duality theorem. We deduce that (7) is an isomorphism. □

Let $\mathbf{e} := (e_1, \dots, e_n)$ be a local trivialization of $\mathcal{H}^k(f_+\mathcal{E})(*0)$ in a neighbourhood of 0. One can suppose that f is smooth above $S^* := S\backslash\{0\}$. Set $U^* := U\backslash\{f^{-1}(0)\}$. From [DMSS00, 1.4], we have an isomorphism of left $\mathcal{D}_S$-modules

$$\mathcal{H}^k(f_+\mathcal{E})_{|S^*} \simeq R^{k+d-1} f_* \operatorname{DR}_{U^*/S^*} \mathcal{E}$$

where the right-hand side is endowed with the Gauss–Manin connection as defined in [KO68]. We deduce that $(\mathbf{e}_t)_{t\neq 0}$ is an algebraic family of bases for the family of spaces $(H^{k+d-1}_{\mathrm{dR}}(X_t, \mathcal{E}_t))_{t\neq 0}$.

At the cost of shrinking S, Kashiwara's perversity theorem [Kas75] shows that the only possibly non-zero terms of the hypercohomology spectral sequence

$$E_2^{pq} = \mathcal{H}^p \operatorname{Sol} \mathcal{H}^{-q}(f_+\mathcal{E})^{\mathrm{an}}_{|S^*} \Longrightarrow \mathcal{H}^{p+q} \operatorname{Sol}(f_+\mathcal{E})^{\mathrm{an}}_{|S^*}$$

sit on the line $p = 0$. Hence, at the cost of shrinking S again, we have

$$\operatorname{Sol} \mathcal{H}^k(f_+\mathcal{E})^{\mathrm{an}}_{|S^*} \simeq \mathcal{H}^0 \operatorname{Sol} \mathcal{H}^k(f_+\mathcal{E})^{\mathrm{an}}_{|S^*} \simeq \mathcal{H}^{-k} \operatorname{Sol}(f_+\mathcal{E})^{\mathrm{an}}_{|S^*}. \tag{7.3.4}$$

Since Sol is compatible with proper direct image, we deduce from (7.3.1) and (7.3.4) that

$$\operatorname{Sol} \mathcal{H}^k(f_+\mathcal{E})^{\mathrm{an}}_{|S^*} \simeq R^{-k+d-1} f_* \operatorname{Sol}(j_+\mathcal{E})^{\mathrm{an}}. \tag{7.3.5}$$

Let $s : \mathcal{H}^k(f_+\mathcal{E})^{\mathrm{an}} \longrightarrow \mathcal{O}_{S^{\mathrm{an}}}$ be a local section of $\operatorname{Sol} \mathcal{H}^k(f_+\mathcal{E})^{\mathrm{an}}$ over an open subset of $S^{*\mathrm{an}}$. From (7.3.5) and Proposition 7.3.2, there exists a unique continuous family $(\gamma_t)_{t\neq 0}$ of elements of the spaces $(H^{\mathrm{rd}}_{2d-2-k}(U_t, \mathcal{E}^*_t))_{t\neq 0}$ inducing s, that is,

$$s(e) : t \longrightarrow \int_{\gamma_t} e_t$$

for every $e \in \mathcal{H}^k(f_+\mathcal{E})_{|S^*}$. Hence, the vector function

$$t \longrightarrow \left(\int_{\gamma_t} e_{1t}, \dots, \int_{\gamma_t} e_{nt}\right)$$

satisfies the system of differential equations corresponding to $\mathcal{H}^k(f_+\mathcal{E})$, and Theorem 5 is proved.

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Jean-Baptiste Teyssier teyssier@zedat.fu-berlin.de
Hebrew University of Jerusalem, Einstein Institute for Mathematics,
Givat Ram, Jerusalem, Israel