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RESEARCH ARTICLE

Hyperelliptic Gorenstein curves and logarithmic differentials

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Abstract
We produce a flexible tool for contracting subcurves of logarithmic hyperelliptic curves, which is local around the subcurve and commutes with arbitrary base-change. As an application, we prove that a hyperelliptic multiscale differential determines a sequence of Gorenstein contractions of the underlying nodal curve, such that each meromorphic piece of the differential descends to generate the dualising bundle of one of the Gorenstein contractions. This is the first piece of evidence for a more general conjecture about limits of meromorphic differentials.

Contents

Introduction

Moduli spaces of differentials η on Riemann surfaces C have undergone wide and deep investigation at the interface between dynamics, topology and algebraic geometry [EMM15, Fil16]. Various questions in Teichmüller theory can be interpreted in intersection-theoretic terms on compactifications of these moduli spaces [Mir07, CMSZ20].

In order to compactify strata of differentials, the curve C should be allowed to degenerate: in the limit, a smooth curve can become nodal, and the differential η can vanish on a subcurve $C_{<0}$. Rescaling the differential appropriately, though, it is possible to extract more information – namely, a meromorphic differential $\eta_{<0}$ on the subcurve $C_{<0}$ (possibly vanishing on a subcurve $C_{<-1}$, and so on). By assigning each vertex of the dual graph of C the generic vanishing order of the differential, and each half-edge of the dual graph a slope from the order of zeroes or poles of the various meromorphic differentials, we define a conewise-linear function λ with integer slopes on the dual graph. This is, roughly speaking, the data of a *generalised multiscale differential* [BCG+19]. See also [Gen18, FP18] for different approaches to compactifying strata of differentials.

Moduli spaces of generalised multiscale differentials are typically 'too large' of a compactification in the sense that they contain more than the limits of differentials on smooth curves. The locus of these smoothable differentials has been characterized in terms of the so-called *global residue condition*: a zero-sum condition on residues of the differential at poles belonging to different irreducible components of the curve, which are connected through components with greater values of λ [BCG+18].

The compactification is intrinsically logarithmic [CC19, CGH+22]. The conewise linear function λ is indeed a section of the characteristic sheaf of a log structure on the curve. It is a tropical canonical differential in the sense that it belongs to the tropical canonical linear series. Even for these purely combinatorial data, the moduli space is not in general irreducible (nor pure-dimensional); the locus of smoothable (*realisable*) tropical differentials has been described explicitly in [MUW21].

With the logarithmic approach providing a purely algebraic point of view on multiscale differentials, identifying the 'main component' parametrising smoothable differentials is the only outstanding problem towards a characteristic-free understanding of moduli spaces of differentials. We state a conjecture, originally due to D. Ranganathan and J. Wise, relating smoothable differentials and Gorenstein singularities:

Conjecture G (≈ Conjecture 5.1). *Let (C, η) be a generalised multiscale/rubber differential (up to scaling), and let $\bar{\lambda}$ denote its tropicalization. Then η is smoothable if and only if*

(i) *every level truncation $\bar{\lambda}_i$ of $\bar{\lambda}$ (as in §1.6) is a realisable tropical differential [MUW21];*
(ii) *there exists a logarithmic modification $\tilde{\sigma}\colon \widetilde{C} \to C$ with a natural extension $\tilde{\eta}$ of the pullback of η to $\widetilde{C}$, and a reduced Gorenstein contraction $\sigma\colon \widetilde{C} \to \overline{C}_i$ such that $\sigma^*\omega_{\overline{C}_i}(\bar{\lambda}_i) = \tilde{\sigma}^*\omega_C$, and*
(iii) *the differential $\tilde{\eta}_i$ at level i descends to a local generator of $\omega_{\overline{C}_i}$.*

The conjecture is motivated by work on stable maps [RSPW19a, RSPW19b, BCM20, BNR21, Zhe21, BC23, BC22]. The connection between Gorenstein singularities and the (algebraic and tropical) geometry of differentials was first evidenced in [Bat22]; see also [Bat24]. It appears from these works on curves of genus one and two that Brill–Noether theory, intended as the study of special linear series on curves, plays a key role in the construction of alternative compactifications of the moduli space of curves, both abstract and embedded. In this paper, we explore this connection in the more general framework of hyperelliptic curves and study Conjecture G in this special case. In forthcoming work, we will present applications of our construction to the birational geometry of the moduli space of hyperelliptic curves [Smy13, Smy11, Fed14, Bat22, BKN23, BM21, BB22].

Strata of differentials are known to have at most three connected components [KZ03]. One of them parametrises *hyperelliptic differentials* (i.e., differentials on hyperelliptic curves that are anti-invariant under the hyperelliptic involution). Even after compactifying, this component is already irreducible [CC19, §5]; hence, the above conjecture postulates that every hyperelliptic multiscale differential should come from a Gorenstein contraction. This is indeed what we prove; the bulk of the paper consists of the construction of such a contraction. The combinatorial data we need is a cutoff of the tropicalisation of the hyperelliptic differential, which we call a *contraction datum*. Note that these tropical differentials come from the target of the admissible cover and are therefore automatically realisable, as the target is rational. We prove the following:

Theorem A (≈ Theorem 2.5). *Let* $(\psi\colon C \to P, \lambda)$ *be a family of log hyperelliptic admissible covers of genus* g *with a contraction datum over a base log scheme* (S, M_S). *There exists a commutative diagram in the category of schemes over* S

$$\begin{array}{ccc} C & \xrightarrow{\sigma} & \overline{C} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\overline{\psi}} \\ P & \xrightarrow{\tau} & \overline{P} \end{array}$$

such that

(i) τ *is a contraction to a family of rational, reduced, Cohen–Macaulay curves* $\overline{P}$;
(ii) $\overline{C}$ *is a family of (not-necessarily reduced) Gorenstein curves of genus* g *such that* $\sigma^*\omega_{\overline{C}} = \omega_C(\lambda)$;
(iii) $\overline{\psi}$ *is a two-to-one cover, specifically the quotient by a hyperelliptic involution* $\bar{\iota}\colon \overline{C} \to \overline{C}$.

Moreover, the construction commutes with any strict base-change in (S, M_S).

Note that constructing the contraction[1] $\sigma\colon C \to \overline{C}$ directly is problematic: the naive strategy of taking $\mathrm{Proj}(\omega_C(\lambda))$ does not behave well under base-change and does not in general produce a flat family. One solution has been put forward in [Boz21], by presenting $\mathcal{O}_{\overline{C}}$ directly in terms of logarithmic data. Here, we pursue a similar strategy, by first contracting the target of the admissible cover – which, being rational, does not present any issues – and then reconstructing $\overline{C}$ as a double cover of the rational, not necessarily Gorenstein curve $\overline{P}$: the cover 'cures' the failure of $\overline{P}$ to be Gorenstein by building in the structure sheaf all the superabundant differentials. This suggests perhaps that, although restricting to Gorenstein curves is very helpful with deformation theory, more general Cohen–Macaulay curves may appear quite naturally when looking at covers and other types of maps from curves, see for instance [HSS21].

In §3, we perform a local study of the singularities arising from our construction, including their explicit equations and their dualising bundle.

Next, we proceed to show that our construction is very general: indeed, we have the following:

Theorem B (≈ Theorem 4.3). *All smoothable, in particular all reduced, hyperelliptic Gorenstein curves arise from the above construction.*

Finally, in Section 5, we explain the conjectural relation between smoothable differentials and Gorenstein curves. In the presence of a multiscale differential, there is a natural way to log modify the curve C, so that a twist of the dualising bundle is trivial on higher levels. This has another benefit – namely, it avoids nonreduced components arising in the Gorenstein contractions. In the special case of hyperelliptic log differentials, we prove that they always descend to the Gorenstein contractions associated to the cutoffs of their tropicalisations.

[1]A contraction is a surjective morphism of curves $\phi\colon C \to D$ which is an isomorphism outside an exceptional subcurve of C, whose connected components are contracted to curve singularities of the same genus in D [Smy13, §2.2]. Thus, technically, σ is not a contraction when $\overline{C}$ is not reduced.

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Corollary C (≈ Proposition 5.3). *Conjecture G holds when C is a hyperelliptic curve and η is anti-invariant with respect to the hyperelliptic involution.*

Recently, a complete proof of the conjecture has been given by D. Chen and Q. Chen [CC24].

Conventions

We work throughout over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$. We use the theory of fs logarithmic schemes in the sense of [Kat89]; we recommend readers new to this theory consult [Kat89] or the more extensive textbook [Ogu18]. Given a logarithmic scheme (X, M_X), we denote by $\underline{X}$ its underlying scheme, especially when we need to endow it with different logarithmic structures. We also use the language of tropical geometry as developed in [CCUW20], and twisted curves in the sense of [AV02]. We refer the reader to §1, where all the relevant concepts are introduced and precise references are provided.

1. Logarithmic hyperelliptic curves and differentials

1.1. Log (twisted) curves

We will be interested in enriching families of curves with some combinatorial data; for this reason, we adopt the language of logarithmic and tropical geometry. Let (S, M_S) be a logarithmic scheme. F. Kato defined *log curves* over S as integral, saturated and logarithmically smooth morphisms $\pi\colon (C, M_C) \to (S, M_S)$ with one-dimensional geometric fibres. He also proved the following more explicit characterisation [Kat96, Theorem 1.3], which we may take as a definition.

Definition-Proposition 1.1. A family of log curves is a morphism of logarithmic schemes

$$\pi\colon (C, M_C) \to (S, M_S)$$

such that

1. the underlying morphism of schemes is a flat family of nodal curves;
2. the log structure admits the following description: for each geometric point p of C, there exists an étale local neighbourhood with a strict étale morphism over S to one of the following:

   smooth point $\mathbb{A}^1_S$ with the log structure pulled back from the base $\pi^* M_S$;
   marking $\mathbb{A}^1_S$ with the log structure generated by the zero section and $\pi^* M_S$;
   node $\mathcal{O}_S[[x, y]]/(xy = t)$ for some $t \in \mathcal{O}_S$, with semistable log structure induced by the multiplication map $\mathbb{A}^2_S \to \mathbb{A}^1_S$ and $t\colon S \to \mathbb{A}^1$.

In the last case, $\log(t)$ (a local section of M_S) and t are called smoothing parameters of the node.

An *n-pointed family of log curves* consists of a family of log curves $\pi : (C, M_C) \to (S, M_S)$ and a tuple of disjoint sections $\sigma_i : \underline{S} \to \underline{C}$, $i = 1, \ldots, n$ (not logarithmic), such that for each $p \in C$ with the log structure of a marking, the zero section of $\mathbb{A}^1_S$ agrees with exactly one of the sections σ_i on the étale local neighborhood of p in the definition above. A family of n-pointed log curves is *stable* if the same is true of its underlying morphism of schemes.

The stacks $\mathfrak{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ of prestable (resp. stable) n-pointed curves of genus g admit log structures such that they *represent* the moduli of prestable (resp. stable) families of log curves over the category of logarithmic schemes [Gil12]. The log structure in each case is easily described: it is the log structure associated to the divisor of singular curves. In particular, the stalk of the characteristic sheaf $\overline{M}_{\mathfrak{M}_{g,n}}$ at a strict geometric point $s \to \mathfrak{M}_{g,n}$ is isomorphic to $\mathbb{N}^{\#E(s)}$, where $E(s)$ is the set of nodes of the nodal curve C_s associated to $s \to \mathfrak{M}_{g,n}$. Given a family of prestable log curves $\pi : (C, M_C) \to (S, M_S)$, the logarithmic structure on $\underline{S}$ obtained by pulling back the log structure of $\mathfrak{M}_{g,n}$ along the classifying morphism $\overline{\pi} : S \to \mathfrak{M}_{g,n}$ is called the *minimal* logarithmic structure and denoted by M_S^{can}. It has the following universal property: there exists a unique log curve $\pi^{\mathrm{can}}\colon (\underline{C}, M_C^{\mathrm{can}}) \to (\underline{S}, M_S^{\mathrm{can}})$ and a

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unique morphism of log schemes $\mu\colon (\underline{S}, M_S) \to (\underline{S}, M_S^{\mathrm{can}})$ covering $\mathrm{id}_{\underline{S}}$, such that π is the pullback of π^{can} along μ.

Generalising Kato's result, M. Olsson [Ols07] proved that there is an equivalence of categories between (balanced) *twisted curves* in the sense of [AV02] and log smooth curves $\pi\colon (C, M_C) \to (S, M_S)$ with a Kummer extension $M_S \hookrightarrow M'_S$ and a choice of an integer a_i for every marking p_i of C. We recall that a map of fs monoids $h\colon Q \to P$ is called *Kummer* if it is injective and for every $p \in P$, there exists a $b \in \mathbb{N}$ such that $bp = h(q)$ for some $q \in Q$ (see, for instance, [Ill02]). The Kummer extension $M_S \hookrightarrow M'_S$ induces a Kummer log ètale morphism $S' \to S$ of Deligne–Mumford stacks, roughly speaking, when S is log smooth over a point with trivial log structure, an iterated *root stack* introducing roots of the smoothing parameters $\log(t) = b\log(s)$. By pulling back C to S' and taking a further root stack of $C \times_S S'$ along components over the nodal divisor, we arrive at a twisted curve C' with local model:

$$[\mathrm{Spec}(\mathcal{O}_{S'}[[u, v]]/(uv - s))/\mu_b],$$

where $x = u^b, y = v^b$ and μ_b acts with weights $(1, -1, 0)$ on (x, y, s). Similarly, C' entails an a_i-root stack of $C \times_S S'$ along the marking p_i.

The stack of twisted curves $\mathfrak{M}^{\mathrm{tw}}_{g,n}$ admits a canonical locally free log structure and a Kummer log ètale map to $\mathfrak{M}_{g,n}$.

1.2. Tropicalisation and conewise-linear functions

To a family of log smooth curves $\pi\colon C \to S$ one can associate a family of tropical curves $\mathrm{trop}(\pi)\colon \Gamma \to S$[CCUW20, Section 7.2]. Over a geometric point s, the tropical curve Γ_s is the dual graph of C_s metrised in $\overline{M}_{S,s}$, where the length of an edge e_q, corresponding to the node $q \in C_s$, is the class of the smoothing parameter $\log(t_q)$ in the stalk of the characteristic sheaf $\overline{M}_{S,s}$. For each specialisation $s_1 \rightsquigarrow s_0$, there is an associated map $\Gamma_{s_0} \to \Gamma_{s_1}$ which applies the induced map $\overline{M}_{S,s_0} \to \overline{M}_{S,s_1}$ to each edge length and contracts the edges whose length goes to 0. Thus, Γ_s can be thought of as a family of (standard, $\mathbb{R}_{\geq 0}$ metrised) tropical curves over the dual cone $\sigma_s = \mathrm{Hom}(\overline{M}_{S_s}, \mathbb{R}_{\geq 0})$. These cones σ_s can be glued together into the *tropicalisation* of S, as we proceed to explain next, and Γ can be equivalently thought of as a family of tropical curves over $\mathrm{trop}(S)$.

Indeed, tropical geometry can be embedded in algebraic geometry by means of the Artin fan machinery [CCUW20, Section 6.3]: an Artin fan is a (relative) 0-dimensional Artin stack with log structure, admitting a strict ètale cover by Artin cones. These are quotients (in the sense of stacks) of affine toric varieties by their dense tori: $\mathcal{A}_\sigma = [\mathrm{Spec}(R[\sigma^\vee \cap \overline{M}^{\mathrm{gp}}])/\mathrm{Spec}(R[\overline{M}^{\mathrm{gp}}])]$. In fact, the 2-category of Artin fans is equivalent to the 2-category of stacks over rational polyhedral cones. Artin fans provide a cover of the Olsson stack of logarithmic structures [Ols03], with the property that every logarithmic scheme admits a universal strict morphism to an Artin fan $X \to \mathcal{A}_X$ encoding all the combinatorial (and none of the geometric) information about X. Thanks to the above-mentioned equivalence, we may think of $\mathcal{A}_X$ as a cone stack, which we call the tropicalisation of X, and sometimes denote by $\mathrm{trop}(X)$.

Let C be a log smooth curve over a geometric point, and let $q \in C$ be a node. The groupification of the characteristic monoid at q can be identified with

$$\overline{M}^{\mathrm{gp}}_{C,q} \simeq \{(\overline{\gamma}_1, \overline{\gamma}_2) \in \overline{M}^{\mathrm{gp},\oplus 2}_{S,\pi(q)} \mid \overline{\gamma}_2 - \overline{\gamma}_1 \in \mathbb{Z}\overline{\log(t_q)}\}.$$

This allows for an identification of global sections of the characteristic group on C with *conewise linear* (CL) functions[2] on the tropicalisation Γ with values in $\overline{M}^{\mathrm{gp}}_S$ and integral slopes along the edges:

[2]The traditional terminology is *piecewise linear*. We use the term ‘conewise’ to stress the fact that $\overline{\gamma}$ is required to be linear on the edges of Γ, and not only on some unspecified subdivision. However, to be precise, these functions should be called conewise affine.

$$H^0(C,\overline{M}_C^{\mathrm{gp}}) = \{\mathbb{Z} - \text{CL functions on } \Gamma \text{ with values in } \overline{M}_S^{\mathrm{gp}}\};$$

see [CCUW20, Remark 7.3]. Moreover, the fundamental exact sequence

$$0 \to \mathcal{O}_C^\times \to M_C^{\mathrm{gp}} \to \overline{M}_C^{\mathrm{gp}} \to 0,$$

and its long exact sequence in cohomology show that to every section $\overline{\gamma} \in H^0(C,\overline{M}_C^{\mathrm{gp}})$, there is an associated $\mathcal{O}_C^\times$-torsor of lifts of $\overline{\gamma}$ to M_C^{gp}; this can be filled into a line bundle in two ways, and we choose the convention such that for $\overline{\gamma} \in H^0(C,\overline{M}_C)$ (i.e., $\overline{\gamma} \geq 0$ in the partial order on $\overline{M}_C^{\mathrm{gp}}$ induced by $\overline{M}_C$), the log structure induces a section $\mathcal{O}_C \to \mathcal{O}_C(\overline{\gamma})$. The restriction of $\mathcal{O}_C(\overline{\gamma})$ to the component C_v of C corresponding to a vertex v of Γ is made explicit in [RSPW19a, Proposition 2.4.1]:

$$\mathcal{O}_{C_v}(\overline{\gamma}) \simeq \mathcal{O}_{C_v}\Big(\sum s(\overline{\gamma},e_q)q\Big) \otimes \pi^*\mathcal{O}_S(\overline{\gamma}(v)), \tag{1.1}$$

where $s(\overline{\gamma},e_q)$ denotes the outgoing slope of $\overline{\gamma}$ along the edge corresponding to q.

1.3. Rubber and multiscale differentials

Multiscale differentials were introduced to compactify strata of differentials over the Deligne–Mumford compactification of the moduli space of curves [BCG+18]. We will focus on an equivalent approach based on logarithmic geometry [CGH+22]. On a smooth curve, a holomorphic differential up to scaling is encoded in the location of its $2g-2$ zeroes, counted with multiplicity. Hence, we may think of strata of differentials as codimension g substacks of $\mathcal{M}_{g,n}$. This naive approach fails over reducible curves, where the distribution of the markings does not reflect the multidegree of the dualising bundle (or any other bundle we may be interested in representing as a weighted sum of the markings). A possible solution is to twist by some components of the nodal curve (a vertical divisor which will be represented by a CL function on the dual graph up to translation), but even this may only work after restricting to a substack of a birational modification of $\overline{\mathcal{M}}_{g,n}$. The following moduli problems were introduced in [MW20] in order to extend the classical Abel-Jacobi section $\mathrm{aj}\colon \mathcal{M}_{g,n} \to \mathbf{Pic}_{g,n}$ beyond the compact-type locus. We fix an n-tuple $\mu = (m_1,\ldots,m_n)$ of integers summing to $2g-2$.

Definition 1.2. $\mathbf{Div}_{g,\mu}$ is the stack parametrising families of log smooth curves $C \to S$ of genus g with the choice of a global section α of the characteristic group $\overline{M}_C^{\mathrm{gp}}$ up to translations by $\overline{M}_S^{\mathrm{gp}}$, such that the slope of α along the i-th leg is m_i. The conewise-linear function α is said to be *aligned* if the values of α at the vertices of Γ are totally ordered in $\overline{M}_S^{\mathrm{gp}}$ (with the partial order induced by $\overline{M}_S$). The logarithmic subfunctor of $\mathbf{Div}_{g,\mu}$ consisting of pairs (C,α) such that α is aligned is denoted by $\mathbf{Rub}_{g,\mu}$.[3]

The following is one of the main results of [MW20]:

Theorem 1.3. *$\mathbf{Rub}_{g,\mu}$ and $\mathbf{Div}_{g,\mu}$ are represented by algebraic stacks with a log smooth log structure. In fact, they are both birational log ètale modifications of the stack of prestable curves:*

$$\mathbf{Rub}_{g,\mu} \to \mathbf{Div}_{g,\mu} \to \mathfrak{M}_{g,n}.$$

The Abel-Jacobi section extends to a finite and unramified morphism:

$$\mathrm{aj}\colon \mathbf{Div}_{g,\mu} \to \mathbf{Pic}_{g,n}.$$

[3]An extra requirement is needed in order to ensure that **Rub** is smooth – namely, that the log modification of C induced by α is itself a log curve; we refer the reader to [CGH+22, Definition 2.1] for the details. Notice that the logarithmic subfunctor $\mathbf{Rub} \subseteq \mathbf{Div}$ is represented by a birational modification of underlying stacks.

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Definition 1.4. The moduli space of *rubber differentials* $\mathcal{H}^\dagger(\mu)$ is defined via the Cartesian diagram:

$$\begin{array}{ccc} \mathcal{H}^\dagger(\mu) & \longrightarrow & \overline{\mathcal{M}}_{g,n} \\ \Big\downarrow & \square & \Big\downarrow{\scriptstyle \omega} \\ \mathbf{Rub}_{g,\mu} & \xrightarrow[\text{aj}]{} & \mathbf{Pic}^{2g-2}_{g,n}, \end{array}$$

where the right vertical map associates to a stable curve C its dualising bundle ω_C.

Spelling out the definition, a rubber differential consists of a stable curve $C \to S$ and a conewise-linear function $\bar{\lambda}$ on $\Gamma = \mathrm{trop}(C) \to S$ up to translations by $\overline{M}_S$, whose values at the vertices of Γ are totally ordered, together with a specified isomorphism

$$\mathcal{O}_C(\bar{\lambda}) = \omega_C, \text{ or equivalently } \omega_C(\lambda) = \mathcal{O}_C, \tag{1.2}$$

where we have set $\lambda = -\bar{\lambda}$.

We may thus think of Γ as an ordered (or *level*) graph. Equation (1.2) may be thought of as a section of ω_C (i.e., a differential), vanishing on all but the top level. In fact, it is proved in [CGH+22] that more information can be extracted from $\bar{\lambda}$ – namely, a collection $(\eta_i)_{i=0,\dots,-N}$ of meromorphic differentials, one on each level subcurve of C. The orders of the η_i at markings and at the preimages of the nodes are determined by the slopes of $\bar{\lambda}$ at the corresponding legs and edges. Rubber differentials turn out to be equivalent to the generalised multiscale differentials from [BCG+19]. To resemble the conventions adopted in the literature on multiscale differentials, we may fix a lift of $\bar{\lambda}$ to an honest CL function by setting $\max(\bar{\lambda}) = 0$ (equivalently, $\min(\lambda) = 0$) and identify the values of $\bar{\lambda}$ with the set $\{0, -1, \dots, -N\}$.

The vertical maps in the diagram of Definition 1.4 are strict; thus, a rubber differential on $C \to S$ induces the same minimal logarithmic structure on S as its tropicalisation $\bar{\lambda}$. This is a locally free log structure parametrising the differences between consecutive values of $\bar{\lambda}$. The moduli space of stable rubber differentials $\mathcal{H}^\dagger(\mu)$ is represented by a separated DM stack with log structure and is finite (in particular, representable) over the Hodge bundle over **Rub** (cf. [CC19, Corollary 3.2]). Typically, $\mathcal{H}^\dagger(\mu)$ is *not* irreducible. Limits of differentials on smooth curves are identified in [BCG+18] by means of an analytic condition called *global residues*.

Remark 1.5 (Tropical differentials). We briefly recall the theory of divisors on metric graphs, in parallel with that of Riemann surfaces. A divisor D on Γ is a finite formal sum of points of Γ with integer coefficients. For an integral conewise-linear function f (up to translation) on Γ, let $\mathrm{div}(f) = \sum_{e \multimap v} s(f, e \multimap v)v$ be the principal divisor on Γ associated to f, where $s(f, e \multimap v)$ denotes the slope of f along e in the outgoing direction from v. We see that $\mathbb{Z}$-CL functions on Γ play the same role as rational functions on algebraic curves; thus, they are also denoted by $\mathrm{Rat}(\Gamma)$. The *tropical linear series* of a divisor D on Γ refers either to $R(D) = \{f \in \mathrm{Rat}(D) | D - \mathrm{div}(f) \geq 0\}$ or to $|D|$, consisting of the same functions taken up to translation.

Let K_Γ denote the canonical divisor of the tropicalisation of C – that is, the divisor on Γ with $2g(v) - 2 + \mathrm{val}(v)$ chips on the vertex v (here the valence of a vertex is the number of edges of finite length that are adjacent to it). Then, the tropicalisation of a rubber differential is a member of the canonical linear series $|K_\Gamma|$ (i.e., a tropical differential). The same is true for the restriction of $\bar{\lambda}$ to a subcurve (i.e., if we truncate Γ up to a certain level of $\bar{\lambda}$).

1.4. Hyperelliptic admissible covers via roots and logs

Admissible covers were introduced by Harris and Mumford in their work on the Kodaira dimension of the moduli space of curves of high genus [HM82] in order to compactify the locus of smooth k-gonal curves. Anticipating the developments of relative Gromov–Witten theory, their moduli were studied by S. Mochizuki by introducing logarithmic structures [Moc95], and by D. Abramovich, A. Corti and A. Vistoli by introducing orbifold structures [ACV03]. The two approaches are essentially equivalent in

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view of Olsson's work on log twisted curves that has been recalled above. We refer the reader to the paper [SvZ20] (resp. the book [BR11]) for a concise (resp. extensive) overview of the subject.

In this paper, we will focus on the hyperelliptic case. Fix a genus $g \geq 1$ and a number of markings $n \geq 0$ (later on, markings will record the zeroes of a differential). We first recall the orbifold point of view:

Definition 1.6. A family of twisted hyperelliptic covers over a scheme S consists of the following data: $(P^{\mathrm{tw}} \to S, \Sigma \subseteq P^{\mathrm{tw}}, \phi\colon P^{\mathrm{tw}} \to \mathcal{B}(\mathbb{Z}/2\mathbb{Z}))$ where

- $P^{\mathrm{tw}} \to S$ is a twisted curve with relative coarse moduli $P \to S$, a rational nodal curve,
- $\Sigma \to S$ is a collection of $2g+2$ gerbes banded by $\mathbb{Z}/2\mathbb{Z}$ (unlabelled) together with n sections of $P^{\mathrm{tw}} \to S$ (labelled), all disjoint, and
- $\phi\colon P^{\mathrm{tw}} \to \mathcal{B}(\mathbb{Z}/2\mathbb{Z})$ is a representable morphism of orbifolds to the classifying stack of $\mathbb{Z}/2\mathbb{Z}$.

Pulling back the universal $\mathbb{Z}/2\mathbb{Z}$-cover $*$ over $\mathcal{B}(\mathbb{Z}/2\mathbb{Z})$, we obtain one $\psi'\colon C \to P^{\mathrm{tw}}$ over the twisted curve. Since ϕ is representable and $*$ is a scheme, so is $C \to S$: in fact, it is an ordinary nodal curve of genus g [ACV03, Lemma 2.2.1], and $\psi\colon C \to P$ is a $\mathbb{Z}/2\mathbb{Z}$-admissible cover ramified exactly at the preimage of the gerby markings and nodes [ACV03, §4].

We may think of the necessity of gerby nodes as follows: when P is smooth, $\psi_*\mathcal{O}_C = \mathcal{O}_P \oplus \mathcal{O}_P(-\frac{1}{2}\mathbf{b})$ makes sense because $\deg(\mathbf{b}) = 2g+2$ is even and $\mathrm{Pic}(P) = \mathbb{Z}$. When P is reducible, though, a problem occurs if $\mathbf{b}$ has odd degree on some components of P: in fact, removing any node q of P separates the latter into two connected components, and we call q even (resp. odd) if so is the degree of $\mathbf{b}$ on either of the two. Twisting along the odd nodes will allow us to find the root of $\mathcal{O}_P(-\mathbf{b})$, which we call $\mathcal{F}$; see, for instance, [Fed14, §2]. Summing up, we have

$$C \xrightarrow{\psi'} P^{\mathrm{tw}} \xrightarrow{\rho} P, \qquad \psi = \rho \circ \psi'\colon C \to P$$

where ψ' is finite ètale with $\psi'_*\mathcal{O}_C = \mathcal{O}_{P'} \oplus \mathcal{F}$, and ψ is finite but not flat with $\psi_*\mathcal{O}_C = \mathcal{O}_P \oplus \rho_*\mathcal{F}$. Indeed, the character of the line bundle $\mathcal{F}$ at an odd node is nontrivial, and correspondingly, $\rho_*\mathcal{F}$ is a torsion-free, rank-one sheaf on P that is not locally free at the odd nodes.

Remark 1.7. With an eye towards motivating our construction in §2, we observe that $\psi'_*\omega_C = \omega_P \oplus (\omega_P \otimes \mathcal{F}^{-1})$ by duality, and the morphism induced by adjunction $(\psi')^*(\omega_P \otimes \mathcal{F}^{-1}) \to \omega_C$ is an isomorphism, generalising the smooth case.

Now, following [Moc95, Definition 3.5], we will consider families of hyperelliptic admissible covers adapted to the setting of log schemes. Parallel to [Moc95, Definition 3.4], we observe that the stack $\mathfrak{M}^{\log}_{g,n_1+n_2}$ of prestable log curves of genus g with n_1+n_2 markings admits a natural action of the symmetric group on n_1 letters $\mathfrak{S}_{n_1}$ permuting the first n_1 points. The stack quotient $\mathfrak{M}\mathfrak{S}^{\log}_{g,n_1|n_2} := [\mathfrak{M}^{\log}_{g,n_1+n_2}/\mathfrak{S}_{n_1}]$ parametrises prestable log curves C with a simple divisor $\mathbf{r}$ of n_1 'symmetrised markings', and n_2 ordered markings $u_1, \ldots, u_{n_2}$ (collectively $\mathbf{u}$).

Definition 1.8. A family of n-marked *log hyperelliptic covers* of genus g over a log scheme S consists of the following data:

1. families $(C, \mathbf{r}, \mathbf{u}) \in \mathfrak{M}\mathfrak{S}^{\log}_{g,2g+2|2n}(S)$ and $(P, \mathbf{b}, \mathbf{v}) \in \mathfrak{M}\mathfrak{S}^{\log}_{0,2g+2|n}(S)$,
2. a $\mathbb{Z}/2\mathbb{Z}$-Galois–Kummer log étale morphism $\psi\colon C \to P$ over S,

satisfying the following conditions:

- ψ is ramified at $\mathbf{r}$ and possibly some nodes of C, and $\psi^{-1}(\mathbf{b}) = 2\mathbf{r}$;
- writing $u_{1,1}, u_{1,2}, \ldots, u_{n,1}, u_{n,2}$ for the markings making up $\mathbf{u}$ in C and $v_1, \ldots, v_n$ for the markings making up $\mathbf{v}$ in P, we have that ψ maps $u_{i,1}, u_{i,2}$ to v_i for each $i = 1, \ldots, n$;
- there is a *hyperelliptic involution* $\iota\colon C \to C$ over ψ fixing $\mathbf{r}$ and swapping $u_{i,1}$ with $u_{i,2}$.

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Remark 1.9. More concretely, in keeping with Kato's local description of log curves, the map $\psi\colon C \to P$ takes one of the following forms strict étale locally in P:

1. (unramified points) A strict, trivial double cover of a neighborhood of a smooth point, a marked point v_i or a node with the usual log structure;
2. (ramification over a point of $\mathbf{b}$) The map $\operatorname{Spec} \mathcal{O}_S[\mathbb{N}] \to \operatorname{Spec} \mathcal{O}_S[\mathbb{N}]$ induced by the multiplication by 2 map from $\mathbb{N} \to \mathbb{N}$;
3. (ramification over a node) The map

$$\operatorname{Spec} \mathcal{O}_S[x, y]/(xy - t) \to \operatorname{Spec} \mathcal{O}_S[z, w]/(zw - t^2)$$

induced by $z \mapsto x^2, w \mapsto y^2$ and $\log(z) \mapsto 2\log(x), \log(w) \mapsto 2\log(y)$ on log structures.

Remark 1.10. The two definitions above are almost equivalent in view of Olsson's work on log twisted curves. While P is the schematic quotient of C by the hyperelliptic involution ι, the twisted curve P^{tw} can be recovered as the stack quotient $[C/\iota]$. A minor difference between the two definitions is that the latter entails an individual labelling of the $2n$ preimages in C of the n markings of P (or P^{tw}). As it will be apparent, this is not needed for our construction.

We say that a log hyperelliptic admissible cover is stable if $(P, \mathbf{b}, \mathbf{v})$ is Deligne–Mumford stable as a rational pointed curve. Notice that $(C, \mathbf{r}, \mathbf{u})$ will be as well. We denote by $\mathcal{H}_{g,n}$ the moduli space of genus g, n-marked, stable log hyperelliptic admissible covers. Then, $\mathcal{H}_{g,n}$ is represented by a proper DM stack with log structure, which is furthermore (log) smooth [Moc95, §3.22].

Indeed, given a family of log hyperelliptic admissible covers over S, there is an associated minimal logarithmic structure on S, satisfying a similar universal property to the minimal log structure for pointed log curves. When ψ is stable, this is the same as the log structure pulled back from $\mathcal{H}_{g,n}$ along the classifying map. More explicitly, it is a Kummer extension of the minimal log structure of P as a log smooth curve, introducing square roots of the smoothing parameters corresponding to the nodes over which ψ is ramified. Indeed, ψ can be factored as $C \to P^{\mathrm{tw}} \to P$, where the first map is strict étale, and the second one is Kummer log étale and birational – albeit it is only representable by DM stacks. The minimal log structure for ψ is the same as that for P^{tw} as a log twisted curve, which is recalled in §1.1.

1.5. Hyperelliptic differentials

The paper [CC19] developed the theory of log differentials without imposing an alignment (so, replacing the stack **Rub** with the stack **Div** in Definition 1.4), focusing on the hyperelliptic and spin components. Here, we shall revisit [CC19, Definition 5.3]. Unlike [CC19], we do impose an alignment.

Definition 1.11. A *hyperelliptic rubber differential* over a log scheme S is the datum of

(i) a log hyperelliptic admissible cover $\psi\colon (C, \mathbf{r}, \mathbf{u}) \to (P, \mathbf{b}, \mathbf{v})$ over S, and
(ii) a rubber differential η on C,

such that

$$\iota^*(\eta) = -\eta, \tag{1.3}$$

where ι is the hyperelliptic involution.

Remark 1.12. Let C be a smooth hyperelliptic curve. Then, every (global holomorphic) differential on C is ι-anti-invariant. This follows, for instance, from the well-known fact that if an affine patch of C is written as $\{y^2 = p(x)\} \subseteq \mathbb{A}^2$ (with p a square-free polynomial of degree $2g + 2$), then ι acts as $(x, y) \mapsto (x, -y)$, and a basis of the space of differentials on C is given by $\{\frac{\mathrm{d}x}{y}, \dots, x^{g-1}\frac{\mathrm{d}x}{y}\}$ (compatibly with the Riemann–Hurwitz formula $\omega_C = \psi^*\omega_P(\frac{1}{2}\mathbf{b})$). Alternatively, any ι-invariant

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differential descends to $\mathbb{P}^1$ and must therefore be trivial. We will see a generalisation of this statement in Remark 4.6. It follows that limits of Abelian differentials on smooth hyperelliptic curves are anti-invariant.

Given the latter condition, the vanishing orders of η at the conjugate points $u_{i,1}$ and $u_{i,2}$ are the same. We may therefore denote the vanishing order of η by an n-tuple $\mu = (m_1, \dots, m_n) \in \mathbb{N}^n_{g-1}$ of nonnegative integers summing to $g-1$. We restrict our attention to strata of hyperelliptic differentials with no zeroes at the Weierstrass points.[4]

1.6. Contraction data

Consider a hyperelliptic rubber differential $(\psi : C \to P, \bar{\lambda})$. Denote by Γ, T' and T the tropicalizations of C, P^{tw} and P, respectively. As $\rho : P^{\mathrm{tw}} \to P$ is a root stack, the induced map $T' \to T$ is also a root stack, introducing halves of lengths of the edges of T corresponding to branching nodes of P. By condition (1.3), the CL function $\bar{\lambda}$ descends to a CL function $\bar{\lambda}_T$ on T'. We view $\bar{\lambda}_T$ as a CL function on T, except that it may have half-integral slopes on the branching edges; pulling back to Γ multiplies the slope by 2 precisely along these edges. Equation (1.2) descends then to

$$\omega_{P^{\mathrm{tw}}}(\mathbf{b}/2) = \mathcal{O}_{P^{\mathrm{tw}}}(\bar{\lambda}_T) \text{ or equivalently } \omega_{P^{\mathrm{tw}}}(\mathbf{b}/2 + \lambda_T) = \mathcal{O}_{P^{\mathrm{tw}}}, \tag{1.4}$$

where $\lambda_T = -\bar{\lambda}_T$. Indeed, $\rho^*\omega_P = \omega_{P^{\mathrm{tw}}}$ [Chi08, Proposition 2.5.1], and $\psi^*\omega_P = \omega_C(-\mathbf{r})$ by definition of the ramification divisor. Since ρ induces an isomorphism of Picard groups up to torsion and P is rational, (a multiple of) condition (1.4) can be checked *numerically*. Equation (1.4) becomes

$$\mathrm{val}(v) - 2 + \frac{1}{2}\deg(\mathbf{b})(v) + \mathrm{div}(\lambda_T)(v) = 0, \tag{1.5}$$

for every vertex v of T. Here, val denotes the edge valency, $\deg(\mathbf{b})$ the multidegree (or tropicalization) of the branch divisor, and $\mathrm{div}(\lambda_T)(v)$ the sum of the outgoing slopes of λ_T along all the edges adjacent to v.[5] In keeping with Remark 1.5, we could say that λ_T is in the *log canonical* tropical linear series of T'. The same is true as well for its restriction to any subcurve of T'.

In order to produce a Gorenstein contraction of a log hyperelliptic cover, we do not need the whole data of a log hyperelliptic differential, but only its combinatorial shadow (i.e., its tropicalisation). In fact, if we are happy to allow nonreduced structures along components of the contraction, we do not need to know precisely at which points the zeroes of the differential are located, but only on which components. We extract this combinatorial information in the following:

Definition 1.13. Let $\psi\colon (C, \mathbf{r}, \mathbf{u}) \to (P, \mathbf{b}, \mathbf{v})$ be a log hyperelliptic admissible cover. A *contraction datum* is a CL function λ_T on $T' = \mathrm{trop}(P^{\mathrm{tw}})$ such that $\lambda_T(v) \geq 0$ for every vertex v of T', and λ_T is a member of the log canonical tropical linear series of the support of λ_T (i.e., the coefficient

$$D(v) = \mathrm{val}(v) - 2 + \frac{1}{2}\deg(\mathbf{b})(v) + \mathrm{div}(\lambda_T)(v)$$

is a nonnegative integer for every vertex v such that $\lambda_T(v)$ is strictly positive).

Example 1.14. To illustrate the combinatorics of contraction data, we consider the dual graphs of several log hyperelliptic admissible covers with $g = 2$ and $n = 1$ (i.e., two simple zeroes at conjugate points). We first consider the possible stable hyperelliptic admissible covers where T' has a single edge. We may

[4]Differentials with zeroes of even multiplicity $2m'$ at a Weierstrass point arise in the boundary of these spaces when a marking v of contact order m' bubbles off onto a new rational component together with a single branch point: the node is then also branching, and the differential vanishes there with multiplicity $2m'$.
Notice the slight abuse of notation with μ being half of what it used to be in the previous sections.

[5]We can also modify $\bar{\lambda}_T$ to $\bar{\lambda}'_T$ by making its slope $-\frac{1}{2}$ on all the legs corresponding to $\mathbf{b}$, so that Equation (1.5) becomes $\mathrm{val}(v) - 2 = \mathrm{div}(\bar{\lambda}'_T)(v)$.

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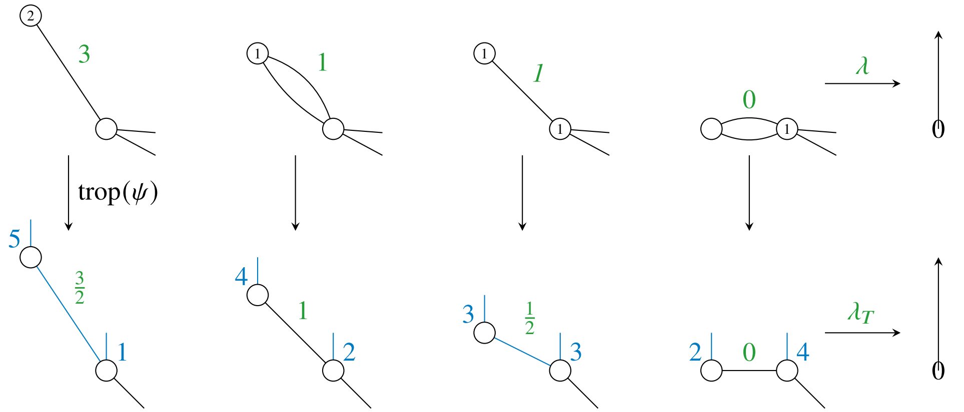

Figure 1. *Contraction data on hyperelliptic admissible covers with one edge.*

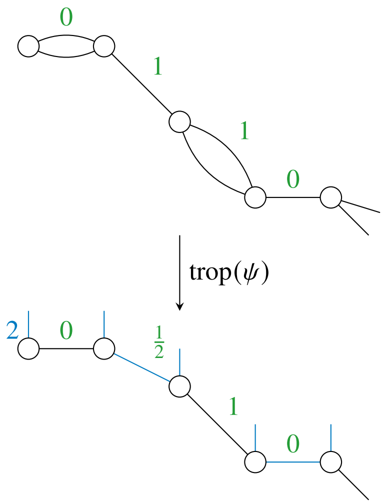

Figure 2. *A contraction datum on a larger log hyperelliptic admissible cover supported on the left 3 vertices. Unnumbered blue legs are single branch legs.*

attempt to construct a contraction datum supported on a single vertex v by computing the required slope of λ_T at v using Equation (1.5), then shifting λ_T so that its minimum value is 0. The possibilities are illustrated in Figure 1. We indicate the number of legs coming from **b** and the branching edges (nodes) in blue, the legs coming from markings in black (λ has slope -1 along them, since our sign convention implies that zeroes of the differential point down, and poles point up), and the positive slopes of λ, λ_T in green. We recall that pulling back λ_T to λ doubles slopes on branching edges. Since Equation (1.5) is stable under generization, we can deduce the slopes of contraction data on genus two admissible covers with more components by generizing to the single-edge graphs. See Figures 1 and 2.

2. Construction of Gorenstein contractions

Let now $(\psi\colon C \to P, \lambda_T)$ be a log hyperelliptic admissible cover over S with a contraction datum. Using λ_T, we will produce a hyperelliptic Gorenstein contraction $\overline{C}$ of C. As we have mentioned in the introduction, the idea is for $\overline{C}$ to be associated with $\omega_C(\lambda)$, but taking Proj directly does not behave well with base-change. Our strategy is to first contract P to $\overline{P}$, which is less problematic since P is rational, and then construct $\overline{C}$ as a double cover of $\overline{P}$ by twisting the (horizontal) branch divisor $\mathbf{b}$ of ψ with the (vertical) CL function λ_T. The dualising sheaf and its twists are seen to play a key role in connection with the Gorenstein condition. We stress that the double cover $\overline{\psi}$ will not be flat whenever the dualising sheaf of $\overline{P}$ is not a line bundle, and the branch locus of $\overline{\psi}$ will only be a generalised divisor. The double cover $\overline{\psi}$ will also fail to be flat at some of the odd nodes of P, but this issue has already been solved in the classical (nodal) case by introducing some orbifold structure.

2.1. Contracting the rational curve

Consider the line bundle $\mathcal{L} := \omega_{P^{\mathrm{tw}}}(\lambda_T)$, whose pullback to C is $\omega_C(\lambda)$. Note that $\mathcal{L}^{\otimes 2}$ descends to a line bundle $L^{\otimes 2} = \omega_P^{\otimes 2}(\mathbf{b}+2\lambda_T)$ of degree $2g-2$ and nonnegative multidegree on P. In particular, since P is rational, L has vanishing higher cohomology, $\pi_{P,*}L^{\otimes 2k}$ is a vector bundle of rank $2k(g-1)+1$ on S for any positive integer k (the relative Proj will therefore be flat by [Sta18, Tag 0D4C]), and $L^{\otimes 2}$ is semiample relative to the base (the contraction map τ is thus defined everywhere). Moreover, the formation of $\pi_{P,*}L^{\otimes 2k}$ commutes with arbitrary base-change by [Har77, Theorem III.12.11]. Let

$$P \xrightarrow{\tau} \overline{P} := \underline{\mathrm{Proj}}_S\left(\bigoplus_{k\geq 0} \pi_{P,*}L^{\otimes 2k}\right)$$

be the resulting contraction. The fibres of τ are connected components of the locus where $L^{\otimes 2}$ is trivial (equivalently, of multidegree $\underline{0}$). In particular, they are connected, rational, nodal curves. The map $\mathcal{O}_P \to \mathcal{O}_{\overline{P}}$ induces an identification of the latter with $\mathrm{R}\tau_*\mathcal{O}_P$, and $\overline{P}$ is a family of rational Cohen–Macaulay curves. Therefore, the singularities appearing in the fibres of $\overline{P}$ are ordinary m-fold points (the singularity of the coordinate axes in $\mathbb{A}^m$): $q \in \overline{P}$ is an m-fold point if $\tau^{-1}(q)$ meets the rest of P in m nodes. These singularities are Gorenstein (even planar) if and only if they are smooth points or nodes (i.e., $m \leq 2$). Let $\mathcal{O}_{\overline{P}}(\mathbb{2})$ denote the line bundle on $\overline{P}$ induced by $L^{\otimes 2}$ on P.

2.2. Square roots of bundles and curves

We will need a square root of $\mathcal{O}_{\overline{P}}(\mathbb{2})$. By mere multidegree considerations, such a line bundle does not always exist on $\overline{P}$ itself, but it does after reintroduction of some stack structure on P and $\overline{P}$. An important observation is that this orbifold structure is only ever needed at nodes of P and $\overline{P}$ away from the exceptional locus of τ.

Lemma 2.1 (cf. [Fed14, Lemma 3.5]). *There is a commutative diagram*

$$\begin{array}{ccc} P' & \xrightarrow{\tau'} & \overline{P}' \\ \downarrow & & \downarrow \\ P & \xrightarrow{\tau} & \overline{P} \end{array}$$

and unique line bundles $\mathcal{L}' \in \mathrm{Pic}(P')$, $\mathcal{O}_{\overline{P}'}(\mathbb{1}) \in \mathrm{Pic}(\overline{P}')$ with the following properties:

1. *P' is a partial coarsening of P^{tw}, while τ' is representable;*
2. *$\mathcal{O}_{\overline{P}'}(\mathbb{1})^{\otimes 2}$ is isomorphic to the pullback of $\mathcal{O}_{\overline{P}}(\mathbb{2})$,*
3. *$(\tau')^*\mathcal{O}_{\overline{P}'}(\mathbb{1}) \cong \mathcal{L}'$, and so $(\mathcal{L}')^{\otimes 2}$ is isomorphic to the pullback of $L^{\otimes 2}$;*
4. *$v_i^*(\mathcal{L}'|_{P^{\mathrm{tw}}}) \cong v_i^*\mathcal{L}$ for each $i = 1, \ldots, n$.*

Proof. Given a node p of P, we say that p is odd (resp. even) for a line bundle B of even degree if B restricts to an odd (resp. even) degree line bundle on the connected components of the normalisation of P at p. Let Z_{odd} be the locus of odd nodes for $\mathcal{L}^{\otimes 2}$ outside of the support of λ_T. Similarly, let Z_{even} be

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the locus of even nodes for $\mathcal{L}^{\otimes 2}$ unioned with the support of λ_T. We may construct the partial coarsening P' by gluing P away from Z_{odd} with P^{tw} away from Z_{even}. Using the fact that τ is an isomorphism away from its exceptional locus, we construct the twisted curve $\overline{P}'$ by gluing $\overline{P}$ away from the image of Z_{odd} with P' away from the exceptional locus of τ.

We begin by considering the case of an individual curve P. Imagine first trying to find a square root of $L^{\otimes 2}$ on P; since P is rational, this is purely a matter of multidegree. Since $\omega_P^{\otimes 2}$ has even degree on every component, a problem may occur only when there is a node p which is odd for $\mathbf{b}$ (i.e., a node of P separating $\mathbf{b}$ into two odd parts). These are precisely the nodes where P^{tw} has nontrivial orbifold structure.[6] The issue is resolved by twisting P at a subset of the odd nodes for $\mathbf{b}$, and the same twisting will guarantee the existence of a square root of $\mathcal{O}_{\overline{P}}(\mathbb{2})$ on $\overline{P}'$. Importantly, the odd nodes in the exceptional locus of τ need no twisting because λ_T acts as a correction factor, as we now show. So let p be an odd node for $\mathbf{b}$, and assume that λ_T has nonzero slope along the edge e_p corresponding to p. Then at least one of the two adjacent vertices is contained in the support of λ_T; call it v. Notice that Equation (1.5) is stable under edge contractions, so in applying it to v, we may as well assume that e_p is the only edge of T. Then we see from Equation (1.5) that λ_T must have half-integral slope along e_p so that it can balance $\mathbf{b}$. It follows that p is even for $L^{\otimes 2}$.

Returning to the case of an arbitrary family, choose $v_i : S \to P^{\mathrm{tw}}$ to be any of the marked points. As in [Fed14, Lemma 3.6], a standard descent argument shows that the square roots on fibers can be glued to a unique line bundle $\mathcal{L}'$ so that $(\mathcal{L}')^{\otimes 2}$ is isomorphic to the pullback of $L^{\otimes 2}$ and $v_i^*(\mathcal{L}'|_{P^{\mathrm{tw}}}) \cong v_i^*\mathcal{L}$. Since $P^{\mathrm{tw}} \to P$ is an isomorphism in the complement of the odd nodes, the rigidification condition $v_i^*(\mathcal{L}'|_{P^{\mathrm{tw}}}) \cong v_i^*\mathcal{L}$ assures that $\mathcal{L}'$ is isomorphic to $\mathcal{L}$ on the complement of the odd nodes, so the analogous rigidification conditions $v_j^*(\mathcal{L}'|_{P^{\mathrm{tw}}}) \cong v_j^*\mathcal{L}$ hold as well.

Moreover, $\mathcal{L}'$ is trivial on the exceptional locus of τ, and it therefore descends to a line bundle $\mathcal{O}_{\overline{P}'}(\mathbb{1})$ which squares to (the pullback of) $\mathcal{O}_{\overline{P}}(\mathbb{2})$. □

Remark 2.2. The arrows $P' \to P$ and $\overline{P}' \to \overline{P}$ are isomorphisms on an open neighborhood of the locus contracted by τ – namely, on the complement of Z_{odd}. Thus, in the following construction, one can ignore the difference between P' and P and $\overline{P}'$ and $\overline{P}$ when working locally near the contracted locus.

2.3. *The double cover: module structure*

We will construct a Gorenstein double cover $\overline{\psi}: \overline{C} \to \overline{P}$ by first constructing the following commutative diagram of curve-line bundle pairs:

$$\begin{array}{ccc} (C, \omega_C(\lambda)) & \xrightarrow{\sigma} & (\overline{C}, \omega_{\overline{C}}) \\ \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \overline{\psi}'} \\ (P', \mathcal{L}') & \xrightarrow{\tau'} & (\overline{P}', \mathcal{O}_{\overline{P}'}(\mathbb{1})). \end{array} \tag{2.1}$$

Then we will compose with the commutative square of Lemma 2.1.

For this, we will first construct an $\mathcal{O}_{\overline{P}'}$-algebra $\mathcal{O}_{\overline{P}'} \oplus \left(\omega_{\overline{P}'} \otimes \mathcal{O}_{\overline{P}}(-\mathbb{1})\right)$, and then set

$$\overline{C} := \underline{\mathrm{Spec}}_{\overline{P}'}\left(\mathcal{O}_{\overline{P}'} \oplus \left(\omega_{\overline{P}'} \otimes \mathcal{O}_{\overline{P}'}(-\mathbb{1})\right)\right).$$

Let us denote $\omega_{\overline{P}'} \otimes \mathcal{O}_{\overline{P}'}(-\mathbb{1})$ by $\overline{\mathcal{F}}$, and let $\overline{F}$ denote its pushforward to $\overline{P}$. The latter is a rank one, torsion-free (i.e., depth one) sheaf on $\overline{P}$, which fails to be a line bundle in two cases:

[6]The representable map $P^{\mathrm{tw}} \to \mathcal{B}\mathbb{Z}/2\mathbb{Z}$ can be thought of as extracting a root of $\omega_P^{\otimes 2}(\mathbf{b})$ – that is, $\omega_{P^{\mathrm{tw}}}$.

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1. when $\overline{P}$ has worse than nodal singularities, because $\omega_{\overline{P}}$ is not locally free there;
2. and at the odd nodes of $\overline{P}$, because both $\omega_{\overline{P}'}$ and $\mathcal{O}_{\overline{P}'}(-\mathbb{1})$ are line bundles over those nodes, but the former comes from $\overline{P}$ and the latter does not, so $\overline{\mathcal{F}}$ has a nontrivial character.

As a consequence, $\overline{\psi}$ will fail to be flat at those points. However, since $\overline{P}$ is flat over the base, then so is $\overline{C}$.

Remark 2.3. Twisting the dualising sheaf of a non-Gorenstein curve with a line bundle is known to produce another irreducible component of the compactified Picard scheme [Kas12].

2.4. The double cover: algebra structure

In order to give $\mathcal{O}_{\overline{C}}$ an $\mathcal{O}_{\overline{P}'}$-algebra structure, we need a cosection $\overline{\mathcal{F}}^{\otimes 2} \to \mathcal{O}_{\overline{P}'}$. Twisting by $\mathcal{O}_{\overline{P}'}(\mathbb{2})$, it is equivalent to find an $\mathcal{O}_{\overline{P}'}$-module map $\omega_{\overline{P}'}^{\otimes 2} \to \mathcal{O}_{\overline{P}'}(\mathbb{2})$. Notice that everything comes from $\overline{P}$, so we could as well work on the coarse curve for this subsection. Since τ' is representable and it only contracts rational curves, its higher direct images vanish, and $\mathrm{R}\tau'_*\mathcal{O}_{P'} = \mathcal{O}_{\overline{P}'}$. A direct application of Grothendieck duality yields

$$\omega_{\overline{P}'} = \mathrm{R}\mathcal{H}om(\mathrm{R}\tau'_*\mathcal{O}_{P'}, \omega_{\overline{P}'}) = \mathrm{R}\tau'_*\,\mathrm{R}\mathcal{H}om(\mathcal{O}_{P'}, (\tau')^!\omega_{\overline{P}'}) = \tau'_*\omega_{P'}.$$

Recall from §1.2 that the fundamental sequence of log geometry associates to every section $\bar{\gamma}$ of the characteristic monoid $\overline{M}^{\mathrm{gp}}$ an $\mathcal{O}^*$-torsor of lifts to the log structure M^{gp}, whose associated line bundle we denote by $\mathcal{O}(-\bar{\gamma})$. Moreover, if $\bar{\gamma} \geq 0$ in the partial order of $\overline{M}^{\mathrm{gp}}$ induced by $\overline{M}$, the log structure map $\alpha\colon M \to \mathcal{O}^*$ endows $\mathcal{O}(-\bar{\gamma})$ with a cosection, making it into a generalised Cartier divisor. Putting together the (vertical) divisor of λ_T with the (horizontal) branch divisor $\mathbf{b}$, adjunction gives us a map

$$(\tau')^*\omega_{\overline{P}'}^{\otimes 2} \to (\tau')^*\tau'_*\omega_{P'}^{\otimes 2} \to \omega_{P'}^{\otimes 2} \to \omega_{P'}^{\otimes 2}(2\lambda_T + \mathbf{b}) = (\mathcal{L}')^{\otimes 2} \tag{2.2}$$

pushing forward to the desired

$$\omega_{\overline{P}'}^{\otimes 2} \to \tau'_*(\tau')^*\omega_{\overline{P}'}^{\otimes 2} \to \mathcal{O}_{\overline{P}'}(\mathbb{2}).$$

Note that there is an involution $\bar{\iota}$ of $\mathcal{O}_C$ over $\mathcal{O}_T$ acting as -1 on sections of $\overline{\mathcal{F}}$.

Also, note that the fibres of $\overline{C}$ fail to be reduced whenever there is a component P_1 of P in the support of λ_T such that the degree of $L^{\otimes 2}$ on P_1 is strictly positive. Indeed, the section $\mathcal{O}_{P'} \to \mathcal{O}_{P'}(2\lambda_T)$ is constantly 0 along such a component, and P_1 is not contracted by τ'; so the algebra structure of $\overline{C}$ over the generic point of P_1 is isomorphic to $\mathbf{k}(t)[\epsilon]/(\epsilon^2)$.

2.5. The Gorenstein property

We argue that $\overline{C}$ is Gorenstein. More precisely, its dualising sheaf $\omega_{\overline{C}}$ can be identified with the line bundle $(\overline{\psi}')^*\mathcal{O}_{\overline{P}'}(\mathbb{1})$. Duality for the finite morphism $\overline{\psi}'$ gives

$$\overline{\psi}'_*\omega_{\overline{C}} = \mathcal{H}om(\overline{\psi}'_*\mathcal{O}_{\overline{C}}, \omega_{\overline{P}'}) = \omega_{\overline{P}'} \oplus \mathcal{H}om(\omega_{\overline{P}'}, \omega_{\overline{P}'})(\mathbb{1}).$$

Now, $\mathcal{O}_{\overline{P}'} \to \mathcal{H}om(\omega_{\overline{P}'}, \omega_{\overline{P}'})$ is an isomorphism because everything is pulled back from the coarse curve $\overline{P}$, and the statement is true there by [Har07, Corollary 1.7]. So we get a morphism $\mathcal{O}_{\overline{P}'}(\mathbb{1}) \to \overline{\psi}'_*\omega_{\overline{C}}$, or equivalently, $(\overline{\psi}')^*\mathcal{O}_{\overline{P}'}(\mathbb{1}) \to \omega_{\overline{C}}$. Since $\overline{\psi}'$ is affine, and therefore $\overline{\psi}'_*$ is exact, it is enough to show that this is an isomorphism after pushforward along $\overline{\psi}'$, which follows from push-pull and the above:

$$\overline{\psi}'_*(\overline{\psi}')^*\mathcal{O}_{\overline{P}'}(\mathbb{1}) = \mathcal{O}_{\overline{P}'}(\mathbb{1}) \oplus \omega_{\overline{P}'} = \overline{\psi}'_*\omega_{\overline{C}}.$$

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2.6. The contraction morphism

We argue that there exists a morphism $\sigma\colon C \to \overline{C}$ covering τ'. Since $\overline{\psi}'$ is affine, it is enough to define an $\mathcal{O}_S$-algebra map $\mathcal{O}_{\overline{C}} \to \sigma_*\mathcal{O}_C$ after pushing forward along $\overline{\psi}'$. Since we want $\overline{\psi}'_*\sigma_* = \tau'_*\psi'_*$, by adjunction, it is enough to define a morphism $(\tau'\psi')^*\overline{\psi}'_*\mathcal{O}_{\overline{C}} \to \mathcal{O}_C$. We focus on $\overline{\mathcal{F}}$ since the pullback of $\mathcal{O}_{\overline{P}'}$ is naturally identified with $\mathcal{O}_C$. The map is induced by $(\tau')^*\omega_{\overline{P}'} \to \omega'_P$ (see 2.4) and by the effective Cartier divisor $\mathbf{r}+\lambda$ on C as follows:

$$(\tau'\psi')^*\omega_{\overline{P}'}(-\mathbb{1}) = (\psi')^*((\tau')^*\omega_{\overline{P}'} \otimes (\mathcal{L}')^{-1}) = (\psi')^*(\tau')^*\omega_{\overline{P}'} \otimes (\omega_C(\lambda))^{-1} \to$$
$$(\psi')^*\omega_P \otimes (\omega_C(\lambda))^{-1} = \omega_C(-\mathbf{r}) \otimes (\omega_C(\lambda))^{-1} = \mathcal{O}_C(-\mathbf{r}-\lambda) \to \mathcal{O}_C.$$

This an isomorphism in the complement of the ramification and contracted loci.

Remark 2.4. Following up on Remark 2.2, we observe that $\overline{C}$ can be constructed by gluing its restriction over the complement of the odd nodes of $\overline{P}$, where no twisting of the base is necessary, together with its restriction to the complement of $\mathrm{Exc}(\tau)$, where it is isomorphic to C over P.

2.7. The arithmetic genus

Finally, we note that the arithmetic genus of $\overline{C}$ is the same as that of C. This follows from smoothing and the following important observation: the construction of Diagram (2.1) – in particular, of the contraction τ, the double cover $\overline{\psi}$ and the contraction σ – commutes with arbitrary base-change.

The arithmetic genus of $\overline{C}$ can also be computed directly. Since $\overline{\psi}'$ is affine, it is enough to compute $h^0(\overline{P}', \overline{\psi}'_*\omega_{\overline{C}}) = h^0(\overline{P}', \mathcal{O}_{\overline{P}'}(\mathbb{1}) \oplus \omega_{\overline{P}'})$. Since $\omega_{\overline{P}'}$ is the pullback of $\omega_{\overline{P}}$ and since $\overline{P}$ is rational, $h^0(\overline{P}', \omega_{\overline{P}'}) = 0$. However, $h^0(\overline{P}', \mathcal{O}_{\overline{P}'}(\mathbb{1})) = h^0(P', \mathcal{L}') = g$, since $\mathcal{L}'$ is a nonnegative line bundle of total degree $g-1$ on the twisted rational curve P'.

Summing up, we have proved the following:

Theorem 2.5. *Let $(\psi\colon C \to P, \lambda_T)$ be a log hyperelliptic admissible cover of genus g with a contraction datum. There exists a contraction (σ,τ) to $(\overline{\psi}\colon \overline{C} \to \overline{P})$ such that*

(i) *$\overline{P}$ is a rational, reduced, Cohen–Macaulay curve;*
(ii) *$\overline{C}$ is a (not-necessarily reduced) Gorenstein curve of genus g such that $\sigma^*\omega_{\overline{C}} = \omega_C(\lambda)$;*
(iii) *$\overline{\psi}$ is the schematic quotient of a hyperelliptic involution $\iota\colon \overline{C} \to \overline{C}$.*

Moreover, the construction commutes with arbitrary base-change.

3. Local computations

In this section, we provide local equations for the singularities of $\overline{C}$ (over a geometric point), including some familiar examples in low genera. From these, we deduce that our singularities can be constructed by gluing A_m-singularities (and ribbons) along specified tangent directions. Then we write down the normalisation of $\overline{C}$ and use this explicit expression to compute its dualising bundle and conductor, verifying once again that $\overline{C}$ is Gorenstein.

3.1. Local equations

Suppose now that p is a point of $\overline{P}$ with ℓ branches. Write s_i for a local parameter along the ith branch of P above p (we shall replace these by unit multiples if needed, taking advantage of $\mathbf{k}$ being algebraically closed). Local equations of $\overline{P}$ at p are

$$A = \hat{\mathcal{O}}_{\overline{P},p} = \mathbf{k}[\![s_1,\ldots,s_l]\!]/(s_is_j : i \neq j).$$

Now write q for the point of $\overline{C}$ above p. We obtain the local ring of $\overline{C}$ at q by adjoining a variable u_i for every local generator of $\overline{F}$ (or $\omega_{\overline{P}}$) at p, subject to the relations expressing the multiplicative structure. Recall that the latter is determined by twisting with $2\lambda_T$ and $\mathbf{b}$. Let m_i be the (positive) slope of $2\lambda_T$ at the ith branch (note that this is *not* the slope of λ at the non-ramified edges of $\mathrm{Supp}(\lambda)$). Nonreduced (double) components, also called *split ribbons*, arise in $\overline{C}$ when a component of $\mathrm{Supp}(\lambda_T)$ is not contracted under τ. We encode this by writing

$$\delta_i = \begin{cases} 0 & \text{if } \lambda_T > 0 \text{ on the } i\text{th branch} \\ 1 & \text{else.} \end{cases}$$

Then, if $\ell \neq 1$, we have the ring

$$\hat{\mathcal{O}}_{\overline{C},q} = A[u_2, \ldots, u_\ell]/I,$$

where $u_2, \ldots, u_\ell$ represent the differentials $\frac{ds_1}{s_1} - \frac{ds_2}{s_2}, \ldots, \frac{ds_1}{s_1} - \frac{ds_\ell}{s_\ell}$, and I is generated by

1. $s_1(u_i - u_j)$ for each $2 \leq i < j \leq \ell$;
2. $s_i u_j$ for each $i \neq j$, $2 \leq i, j \leq \ell$;
3. $u_i^2 - \delta_1 s_1^{m_1} - \delta_i s_i^{m_i}$ for each $i = 2, \ldots, \ell$;
4. $u_i u_j - \delta_1 s_1^{m_1}$ for each $i \neq j$ with $2 \leq i, j \leq \ell$,

where the first two equations describe the A-module structure of $\omega_{\overline{P}}$, and the last two describe the multiplication as in Equation (2.2).

If $\ell = 1$ and p is the image of a contracted component, the formula above needs a correction to account for the difference between $\omega_{\overline{P}}$ (being generated by ds_1) and $\mathcal{O}_{\overline{P}}(\mathbb{1})$ (with ω_P being generated by $\frac{ds_1}{s_1}$). More precisely, if $\tilde{p}$ denotes the node of P over p, the natural map $\tau^*\omega_{\overline{P}} \cong \tau^*\tau_*\omega_P \to \omega_P$ is induced by twisting by $\tilde{p}$. We thus have

$$\hat{\mathcal{O}}_{\overline{C},q} = A[u]/(u^2 - \delta_1 s_1^{m_1+2}),$$

which is either a germ of an A_{m_1+1} singularity (if $\delta_1 \neq 0$) or a ribbon (if $\delta_1 = 0$).

Similarly (but easier), if $\ell = 1$ and p belongs to $\mathbf{b}$, we have

$$\hat{\mathcal{O}}_{\overline{C},q} = A[u]/(u^2 - \delta_1 s_1),$$

either a germ of a ribbon or a point of ramification of the cover; and if $\ell = 1$ and p does not belong to $\mathbf{b}$, we have

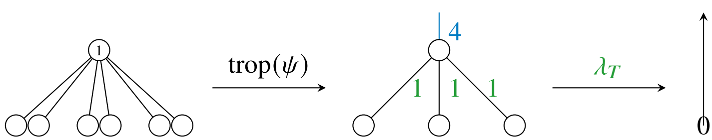
$$\hat{\mathcal{O}}_{\overline{C},q} = A[u]/(u^2 - \delta_1),$$

either a germ of a ribbon or a trivial part of the double cover.

3.2. Examples

We recover some familiar examples of curve singularities of low genus.

Example 3.1. We construct a genus one singularity with six branches, cf. [Smy11, Boz21].



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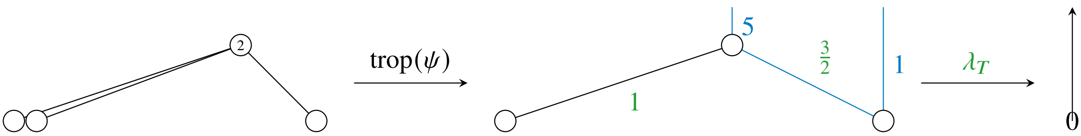
In green, we write the slopes of λ_T, and in blue instead, the number of branch points. The line bundle L is trivial on the central vertex of the tree, so the corresponding component is contracted into an ordinary (rational) 3-fold point. The sheaf $\omega_{\overline{P}} \otimes \mathcal{O}_{\overline{P}}(-1)$ has two generators u_2 and u_3, corresponding to the generators $(\frac{\mathrm{d}s_1}{s_1}, -\frac{\mathrm{d}s_2}{s_2}, 0)$ and $(\frac{\mathrm{d}s_1}{s_1}, 0, -\frac{\mathrm{d}s_3}{s_3})$. The resulting singularity has local equations of the form

$$\hat{\mathcal{O}}_{\overline{C},q} = \mathbf{k}[[s_1, s_2, s_3]][u_2, u_3]/(s_i s_j, u_2^2 - s_1^2 - s_2^2, u_3^2 - s_1^2 - s_3^2, s_1(u_2 - u_3), s_2 u_3, s_3 u_2, u_2 u_3 - s_1^2),$$

which is isomorphic to $\mathbf{k}[[x_1, \bar{x}_1, x_2, \bar{x}_2, x_3]]/I_6$ from [Smy11, Proposition A.3] via

$$s_1 = x_1 - \bar{x}_1, s_2 = x_2 - \bar{x}_2, s_3 = \frac{1}{2}x_3, u_2 = x_1 + \bar{x}_1 - (x_2 + \bar{x}_2), u_3 = x_1 + \bar{x}_1 - x_3.$$

Example 3.2. We construct a genus two singularity 'of type I' with three branches; cf. [Bat22].



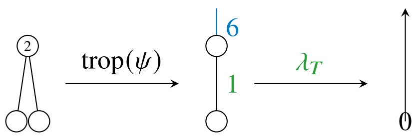
The line bundle L is trivial on the middle component of the chain, which is contracted to a node. The dualising sheaf $\omega_{\overline{P}}$ is itself a line bundle, with local generator u. We obtain

$$\hat{\mathcal{O}}_{\overline{C},q} = \mathbf{k}[[s_1, s_2]][u]/(s_1 s_2, u^2 - s_1^2 - s_2^3),$$

which is isomorphic to $\mathbf{k}[[x_1, \bar{x}_1, x_2]]/(x_1 x_2 - \bar{x}_1 x_2, x_1 \bar{x}_1 - x_2^3)$ [Bat22, Equation (4) on p.11] via

$$s_1 = \frac{1}{2}(x_1 - \bar{x}_1), s_2 = x_2, u = \frac{1}{2}(x_1 + \bar{x}_1).$$

Example 3.3. We construct a (2)-tailed ribbon of genus two; cf. [BC23, Definition 2.21].



In this case, $\overline{P}$ is isomorphic to P. Local equations:

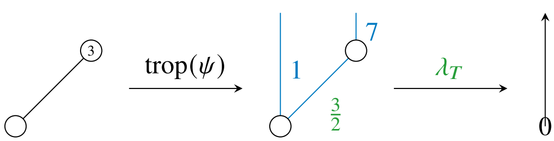
$$\hat{\mathcal{O}}_{\overline{C},q} = \mathbf{k}[[s_1, s_2]][u]/(s_1 s_2, u^2 - s_1^2),$$

which is isomorphic to $\mathbf{k}[[x_1, x_2, y]]/(x_1 x_2, (x_1 - x_2)y)$ [BC23, Example 2.20] via

$$x_1 = u + s_1, x_2 = u - s_1, y = s_2.$$

Example 3.4. Isolated Gorenstein singularities of genus three are classified in [Bat24]. The hyperelliptic ones are identified with the ones described in the present paper.

Example 3.5. We construct a nonreduced singularity of genus three.



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Again, $\overline{P}$ is isomorphic to $P = L \cup R$. Local equations:

$$\hat{\mathcal{O}}_{\overline{C},q} = \mathbf{k}[[s_1, s_2]][u]/(s_1 s_2, u^2 - s_1^3).$$

The double structure on the right component R is given by the line bundle $\mathcal{O}_R(-\frac{1}{2}\mathbf{b} - \lambda_T) = \mathcal{O}_R(-2)$; hence, the ribbon $\overline{R}$ has genus 1. We can obtain the curve $\overline{C}$ by gluing the cuspidal curve $\overline{L}$ with the ribbon $\overline{R}$ along a length two subscheme, which checks out to give $p_a(\overline{C}) = 3$.

Example 3.6. Classical A_k and D_k singularities are recovered by setting $\ell = 1, m_1 = k - 1$ and $\ell = 2, m_1 = 1, m_2 = k - 2$, respectively.

3.3. Gluing

The preimage in $\overline{C}$ of a branch of $\overline{P}$ is either a double curve or an A_m singularity. Here, we explain how to recover $\overline{C}$ by gluing them along some tangent directions.

The subscheme cut out by s_i for $i \neq 1$ and $u_i - u_j$ for $i < j$ is isomorphic to

$$\mathbf{k}[\![s_1]\!][u_1]/(u_1^2 - \delta_1 s_1^{m_1}), \tag{3.1}$$

where u_1 is the common image of $u_2, \ldots, u_m$. It is either a germ of a ribbon or an A_{m_1-1} singularity.

Similarly, for each $j = 2, \ldots, \ell$, the subscheme cut out by s_i for $i \neq j$ and u_i for $i \neq j$ is isomorphic to

$$\mathbf{k}[\![s_j]\!][u_j]/(u_j^2 - \delta_j s_j^{m_j}),$$

which is again either a germ of a ribbon or an A_{m_j-1} singularity.

If we only restrict down to the subscheme cut out by s_1, we find that we get

$$\mathbf{k}[\![s_2, \ldots, s_\ell]\!][u_2, \ldots, u_\ell]/(s_i s_j, s_i u_j, u_i^2 - \delta_i s_i^{m_i} : i \neq j, 2 \leq i, j \leq \ell), \tag{3.2}$$

the transverse union of the singularities for $j = 2, \ldots, \ell$ above.

Our next claim is that the singularity at q is the result of gluing the tangent vector $\frac{\partial}{\partial u_1}$ of Spec of (3.1) with the tangent vector $\sum_{i=2}^{\ell} \frac{\partial}{\partial u_i}$ of Spec of (3.2).

To see this, consider the sequence

$$0 \to \hat{\mathcal{O}}_{\overline{C},q} \to \frac{\mathbf{k}[\![s_1]\!][u_1]}{(u_1^2 - \delta_1 s_1^{m_1})} \times \frac{\mathbf{k}[\![s_2, \ldots, s_\ell]\!][u_2, \ldots, u_\ell]}{(s_i s_j, s_i u_j, u_i^2 - \delta_i s_i^{m_i} : i \neq j, 2 \leq i, j \leq \ell)} \to Q \to 0.$$

To see that the first map is injective, observe that the kernel is contained in $(s_1) \cap (s_2, \ldots, s_\ell) = 0$. Note that both $\langle s_1, 0 \rangle$ and $\langle 0, s_i \rangle$ for $i = 2, \ldots, \ell$ are in the image of the first map, so Q is supported on $V(s_1, \ldots, s_\ell)$. Restricting to this vanishing, we find

$$0 \to k[u_2, \ldots, u_\ell]/(u_2, \ldots, u_\ell)^2 \to k[u_1]/u_1^2 \times \frac{k[u_2, \ldots, u_\ell]}{(u_2, \ldots, u_\ell)^2} \to Q \to 0.$$

The first map clearly admits a retract, so we conclude $Q \cong k[\epsilon]/\epsilon^2$. This yields the claim.

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3.4. Normalisation

The normalisation $\overline{C}^\nu$ of the germ of $\overline{C}$ at q can be computed as follows. Consider

$$\begin{array}{ccccccc} \overline{C}^\nu & \longrightarrow & \overline{C}_{\overline{P}^\nu,\mathrm{red}} & \hookrightarrow & \overline{C}_{\overline{P}^\nu} & \longrightarrow & \overline{C} \\ & & & & \downarrow & \square & \downarrow \\ & & & & \overline{P}^\nu & \longrightarrow & \overline{P}. \end{array}$$

From this, we see that it is enough to understand the normalisation of the A_m-singularities and of ribbons, and put these formulae together. Assume that $\delta_1 = 1$ and m_1 is even; the other cases are left to the avid reader. Renumbering $\{2, \dots, \ell\}$, we may assume that

- for $i = 2, \dots, h$, we have $\delta_i = 1$ and m_i even,
- for $i = h+1, \dots, k$, we have $\delta_i = 1$ and m_i odd,
- for $i = k+1, \dots, \ell$, we have $\delta_i = 0$.

The normalisation is then given by the ring

$$\prod_{i=1}^{h} \mathbf{k}[\![a_i]\!] \times \mathbf{k}[\![b_i]\!] \times \prod_{i=h+1}^{k} \mathbf{k}[\![c_i]\!] \times \prod_{i=k+1}^{l} \mathbf{k}[\![d_i]\!]$$

with ring homomorphism

$$s_i \mapsto \begin{cases} (a_i, b_i) & \text{for } i = 1, \dots, h \\ c_i^2 & \text{for } i = h+1, \dots, k \\ d_i & \text{for } i = k+1, \dots, \ell \end{cases} \qquad u_i \mapsto \begin{cases} (a_1^{m_1/2}, -b_1^{m_1/2}, a_i^{m_i/2}, -b_i^{m_i/2}) & \text{for } i = 2, \dots, h \\ (a_1^{m_1/2}, -b_1^{m_1/2}, c_i^{m_i}) & \text{for } i = h+1, \dots, k \\ (a_1^{m_1/2}, -b_1^{m_1/2}, 0) & \text{for } i = k+1, \dots, \ell \end{cases} \tag{3.3}$$

Notice that $\frac{m_i}{2}$ (resp. m_i) is precisely the slope of λ on the edge corresponding to q_i, $i = 1, \dots, h$ (resp. $i = h+1, \dots, k$).

3.5. Differentials

For this section, we assume that $\overline{C}$ is reduced. Recall that the conductor ideal of the normalisation $\nu\colon C^\nu \to \overline{C}$ is $\mathfrak{c} = \mathrm{Ann}(\nu_* \hat{\mathcal{O}}_{C^\nu,q} / \hat{\mathcal{O}}_{\overline{C},q})$; it is the largest ideal of $\hat{\mathcal{O}}_{C,q}$ that is also an ideal of $\nu_* \hat{\mathcal{O}}_{C^\nu,q}$. It follows from the explicit parametrisation in the previous section that

$$\mathfrak{c} = \left(a_i^{m_i/2+1}, b_i^{m_i/2+1}, c_i^{m_i+1}\right). \tag{3.4}$$

From this, we can verify that $\overline{C}$ is Gorenstein in a second way.

Corollary 3.7. *$\overline{C}$ is a Gorenstein curve.*

Proof. The following criterion is due to Serre (see, for instance, [AK70, Proposition VIII.1.16]): $\overline{C}$ is Gorenstein if and only if $\dim_{\mathbf{k}}(\mathcal{O}_{C^\nu}/\mathfrak{c}) = 2\delta$. Now,

$$\delta = g(q) + 2h + k - 1, \tag{3.5}$$

where $2h + k$ is the number of branches of q, and $g(q)$ is the genus of the singularity.

The latter is the same as the genus of the subcurve of C contracted to it. This corresponds to a connected component of the support of λ_T. Since the following formulae are stable under edge contraction, we

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may as well assume that there is a single vertex v in the support of λ_T, resp. irreducible component C_v of C contracting to q. The genus of this component is determined by the Riemann–Hurwitz formula

$$2g(C_v) + 2 = b + k, \tag{3.6}$$

where b is the number of branch points supported on C_v, and k the number of odd nodes (for $\mathbf{b}$) adjacent to v. However, balancing λ_T at v as in Equation (1.5), we find

$$\mathrm{div}(\lambda_T) = \sum_{i=1}^{h+k} \frac{m_i}{2} = \mathrm{val}(v) - 2 + \frac{b}{2}. \tag{3.7}$$

Finally, from the above formula for the conductor, we find that

$$\begin{aligned}
\dim_{\mathbf{k}}(\mathcal{O}_{C^\nu}/\mathfrak{c}) &= \sum_{i=1}^{h+k} m_i + 2h + k && \text{by eq. (3.4)}\\
&= 2\mathrm{val}(v) - 4 + b + 2h + k && \text{by eq. (3.7)}\\
&= b + k - 4 + 2(2h + k) && \text{by a simple manipulation}\\
&= 2(g + 2h + k - 1) = 2\delta. && \text{by eq. (3.6) and eq. (3.5).}
\end{aligned}$$

□

We can also describe the dualising bundle of $\overline{C}$ more explicitly.

Corollary 3.8. *A local generator of $\omega_{\overline{C}}$ at q is given by*

$$\begin{aligned}
\eta &= \frac{\mathrm{d}s_1}{us_1} - \sum_{i=2}^{k} \frac{\mathrm{d}s_i}{u_i s_i}\\
&= \frac{\mathrm{d}a_1}{a_1^{m_1/2+1}} - \frac{\mathrm{d}b_1}{b_1^{m_1/2+1}} - \sum_{i=2}^{h}\left(\frac{\mathrm{d}a_i}{a_i^{m_i/2+1}} - \frac{\mathrm{d}b_i}{b_i^{m_i/2+1}}\right) - \sum_{i=h+1}^{k} \frac{\mathrm{d}c_i}{c_i^{m_i+1}},
\end{aligned}$$

and, if we write $(C^\nu, q_i, \bar{q}_i, q_j)_{\substack{i=1,\dots,h\\ j=h+1,\dots,k}}$ for the pointed normalisation of $\overline{C}$ at q, then

$$\nu^*\omega_{\overline{C}} = \omega_{C^\nu}\left(\sum_{i=1}^{h}\left(\frac{m_i}{2} + 1\right)(q_i + \bar{q}_i) + \sum_{i=h+1}^{k}(m_i + 1)q_i\right).$$

Proof. Recall Rosenlicht's theorem [AK70, Proposition VIII.1.16]: for a reduced curve $\overline{C}$, sections of the dualising sheaf $\omega_{\overline{C}}$ can be identified with meromorphic differentials η on the normalisation C^ν such that, for all regular functions f on $\overline{C}$, one has

$$\sum_{q_i \in \nu^{-1}(q)} \mathrm{Res}_{q_i}(f\eta) = 0. \tag{3.8}$$

This implies that the order of vanishing of the conductor at q_i is an upper bound for the order of pole of sections of $\omega_{\overline{C}}$ at q_i: if t is a local parameter of C^ν at q_i, and t^μ is a section of $\mathfrak{c}$ (and in particular of $\hat{\mathcal{O}}_{\overline{C},q}$), no meromorphic differential with pole order $\mu + 1$ or higher at q_i can ever descend to $\omega_{\overline{C}}$. Under this condition, Equation (3.8) is automatically satisfied for all $f \in \mathfrak{c}$.

Since $\hat{\mathcal{O}}_{\overline{C},q}/\mathfrak{c}$ is generated by $\left\langle 1, s_i, \dots, s_i^{m_i/2}, u_j\right\rangle_{\substack{i=1,\dots,k;\\ j=2,\dots,k}}$ as a $\mathbf{k}$-vector space, it is easy to check that the meromorphic differential η from the statement descends to a local section of $\omega_{\overline{C}}$.

Moreover, since the latter is a line bundle and η has the highest possible pole order at every q_i, it follows that η is indeed a generator: pick a local generator η', and write $\eta = g\eta'$ for some $g \in \hat{\mathcal{O}}_{C,q}$; then the order of pole of η at q_i is lower than that of η' (or η vanishes on the entire branch containing q_i, which it does not), so they have to be equal, so g has to be an invertible scalar.

The second claim follows from Noether's formula [Cat82, Proposition 1.2]: $\omega_{C^\nu} = \nu^*\omega_{\overline{C}}(\mathfrak{c})$. □

4. Classification of Gorenstein hyperelliptic curves

In this section, we prove a partial converse to our previous result – namely, that most Gorenstein hyperelliptic curves arise from our construction. We focus on the unmarked case for notational simplicity. We start by specifying what exactly we mean by a Gorenstein hyperelliptic curve.

Definition 4.1. We say that $\overline{\psi}\colon \overline{C} \to \overline{P}$ is a *Gorenstein hyperelliptic cover* if $\overline{P}$ is a rational, reduced, Cohen–Macaulay projective curve; $\overline{\psi}$ is a finite (not necessarily flat) cover of degree two over every irreducible component of $\overline{P}$; $\overline{C}$ is a Gorenstein (not necessarily reduced) curve; there is a hyperelliptic involution $\bar{\iota}$ on $\overline{C}$ with quotient $\overline{P}$.

Remark 4.2. Every nonreduced component of $\overline{C}$ is a *split* ribbon: recall that a ribbon is called split when it admits a projection to its underlying reduced curve $R_{\mathrm{red}} \hookrightarrow R \to R_{\mathrm{red}}$ [BE95, §1].

Theorem 4.3. *Every smoothable Gorenstein hyperelliptic cover arises from the construction of §2.*

Corollary 4.4. *Every reduced Gorenstein hyperelliptic cover arises from the construction of §2.*

In the presence of a $G = \mathbb{Z}/2\mathbb{Z}$-action on $\overline{C}$, we may split the structure (in fact, any equivariant) sheaf into eigenspaces for the G-action (on every G-stable open). We can thus write

$$\overline{\psi}_*\mathcal{O}_{\overline{C}} = \mathcal{O} \oplus \overline{F},$$

where $\mathcal{O}$ denotes the 1-eigenspace, and $\overline{F}$ the -1. By assumption, $\bar{\iota}$-invariant functions descend to $\overline{P}$, whence we can identify $\mathcal{O}$ with $\mathcal{O}_{\overline{P}}$. In particular, the finite cover $\overline{\psi}$ admits a trace map even when it is not flat.

We know that $\overline{F}$ is some sheaf of pure rank one on $\overline{P}$. Our next goal is to show that $\overline{F}$ is a twist of $\omega_{\overline{P}}$ by a line bundle. We recall that, in his study of generalised divisors, Hartshorne has introduced a generalisation of reflexivity for sheaves which is useful when the base scheme is Cohen–Macaulay but not Gorenstein. Denote by $-^{\omega}$ the functor $\mathcal{H}om(-,\omega)$, i.e. ω-dualisation. A sheaf $\mathcal{G}$ is *ω-reflexive* if $\mathcal{G} \to \mathcal{G}^{\omega\omega}$ is an isomorphism. This implies that $\mathcal{G}$ is torsion-free [Har07, Lemma 1.4].

Remark 4.5. It follows from Grothendieck's duality for a finite morphism $f\colon X \to Y$ that

$$(f_*\mathcal{G})^{\omega} = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{G}, \omega_Y) = f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, f^!\omega_Y) = f_*(\mathcal{G}^{\omega}).$$

In particular, $\overline{\psi}_*\omega_{\overline{C}} = \mathcal{H}om_{\mathcal{O}_{\overline{P}}}(\overline{\psi}_*\mathcal{O}_{\overline{C}}, \omega_P) = \omega_{\overline{P}} \oplus \overline{F}^{\omega}$.

Remark 4.6. Since $\overline{P}$ is rational, $\omega_{\overline{P}}$ has no global section (by Serre duality). It follows that (global regular) sections of the dualising sheaf on $\overline{C}$ can be identified with sections of $\overline{F}^{\omega}$ on $\overline{P}$; in particular, they are all ι-anti-invariant. This generalises Remark 1.12 beyond the case of smooth curves.

Since $\mathcal{O}_{\overline{C}}$ and $\mathcal{O}_{\overline{P}}$ are both ω-reflexive, we may conclude that the same holds true for $\overline{F}$.

Lemma 4.7. *$\overline{F}$ is a rank-one, ω-reflexive sheaf.*

Lemma 4.8. *$\overline{F}^{\omega}$ is a line bundle, except where $\overline{\psi}$ maps a node to a node with ramification.*

Proof. We may work locally around a closed point p of $\overline{P}$. If $\overline{P}$ is smooth at p, then $\overline{\psi}$ is flat by 'miracle flatness', so $\overline{F}$ is itself a line bundle, and $\overline{F}^{\omega}$ is as well.

If p is a node, we consider two cases: either $\overline{\psi}$ is flat over p, in which case we can conclude as before[7]; or $\overline{\psi}$ is not flat. In this case, we claim that $\overline{C}$ has a node at the preimage n of p, and $\overline{\psi}$ is ramified at n on both branches. To show this, we are going to normalise $\overline{P}$ and $\overline{C}$ simultaneously. Indeed, $\overline{F}$ is not a line bundle, but there is a line bundle $\overline{F}'$ on the normalisation $\nu\colon \overline{P}' \to \overline{P}$ such that $\overline{F} = \nu_*\overline{F}'$ [OS79, Proposition 10.1]. Consider the following exact sequence:

$$0 \to \mathcal{O}_{\overline{P}} \oplus \overline{F} \to \nu_*(\mathcal{O}_{\overline{P}'} \oplus \overline{F}') \to \mathbf{k}_p \to 0. \tag{4.1}$$

We may endow the second term with a $\nu_*\mathcal{O}_{\overline{P}'}$-algebra structure induced by the one of $\mathcal{O}_{\overline{C}}$. Indeed, there is always a map

$$(\nu_*\overline{F}')^{\otimes 2} \to \nu_*(\overline{F}'^{\otimes 2}),$$

which, in this case, a local computation shows to be surjective. In fact, we may as well replace $(\nu_*\overline{F}')^{\otimes 2}$ by $\mathrm{Sym}^2(\nu_*\overline{F}')$. We get the desired multiplication map by lifting

$$\begin{array}{ccc} \mathrm{Sym}^2\,\overline{F} & \xrightarrow{\mu} & \mathcal{O}_{\overline{P}} \\ \downarrow & & \downarrow \\ \nu_*(\overline{F}'^{\otimes 2}) & \overset{\mu'}{\dashrightarrow} & \nu_*\mathcal{O}_{\overline{P}'}. \end{array}$$

Explicitly, if $\overline{F}'$ is generated as $\mathcal{O}_{\overline{P}'}$-module by an element (x, y) (and its pushforward along ν is generated as an $\mathcal{O}_{\overline{P}}$-module by two elements $x = (1, 0)\cdot(x, y)$ and $y = (0, 1)\cdot(x, y)$), its square $\overline{F}'^{\otimes 2}$ is generated by (x^2, y^2) as an $\mathcal{O}_{\overline{P}'}$-module, and by x^2 and y^2 as an $\mathcal{O}_{\overline{P}}$-module. The $\mathcal{O}_{\overline{P}}$-module $\mathrm{Sym}^2\,\overline{F}$ has an extra generator xy. However, locally, $\mathcal{O}_{\overline{P}} \simeq \mathbf{k}[s, t]/(st)$ (while $\mathcal{O}_{\overline{P}'} \simeq \mathbf{k}[s] \oplus \mathbf{k}[t]$), and $sy = tx = 0$ implies that $\mathfrak{m}_p \cdot xy = 0$; hence, the multiplication map $\mu\colon\ \mathrm{Sym}^2\,\overline{F} \to \mathcal{O}_{\overline{P}}$ must send this element to 0. It follows that the multiplication map $\mu'\colon \nu_*(\overline{F}'^{\otimes 2}) \to \nu_*\mathcal{O}_{\overline{P}'}$ is well-defined. Moreover, it clearly lifts to a map of $\mathcal{O}_{\overline{P}'}$-modules. We thus get the desired double cover $\overline{C}' \to \overline{P}'$, together with a birational morphism $\overline{C}' \to \overline{C}$. Since $\overline{P}'$ is smooth (disconnected), the former map is flat and $\overline{C}'$ is smooth, so the latter map is the normalisation of $\overline{C}$. Equation (4.1) shows that $\overline{C}$ has δ-invariant 1 (and at least two branches) at n, so n must be a node, and moreover, the cover is ramified at n on both branches.

Finally, if $\overline{P}$ is not Gorenstein at p, we may argue as follows. Let q be the point of $\overline{C}$ over p (if there were two, $\overline{\psi}$ would be a local isomorphism, contradicting the fact that $\overline{C}$ is Gorenstein). The group G acts on $\omega_{\overline{C}}$. By assumption, $\omega_{\overline{C}}$ admits a single generator at q that we will call η. Consider the eigenspace decomposition $\eta = \eta_1 + \eta_{-1}$. If $\eta_{-1} = 0$, then $\omega_{\overline{P}}$ is generated by η_1 as an $\mathcal{O}_{\overline{P}}$-module, which is a contradiction. Since η generates $\omega_{\overline{C}}$ and η_{-1} is itself a section of $\omega_{\overline{C}}$, we can write $\eta_{-1} = f\eta$. We claim that $f(q) \neq 0$, so we can as well take η_{-1} as a generator of $\omega_{\overline{C}}$. Decomposing f and η into their homogeneous pieces, we write

$$\eta_{-1} = f_{-1}\eta_1 + f_1\eta_{-1}.$$

Since $\overline{\iota}^* f_{-1}(q) = -f_{-1}(q)$, which implies $f_{-1} \in \mathfrak{m}_q$, we have to check that $f_1(q) \neq 0$. Were $f_1(q) = 0$, then $1 - f_1$ would be a unit, and we could write

$$\eta_{-1} = \frac{f_{-1}}{1 - f_1(q)}\eta_1,$$

so we could take η_1 as a generator of $\omega_{\overline{C}}$, which is a contradiction as above. This shows that the generator of $\omega_{\overline{C}}$ can be assumed to be of pure weight -1; hence, $\overline{F}^{\omega}$ has a single generator as an $\mathcal{O}_{\overline{P}}$-module. $\square$

[7]All singularities of the form $\mathbf{k}[[x, y, z]]/(xy, z^2 - x^\alpha - y^\beta)$ fall under this category (e.g., D_k-singularities when $\alpha = 1$).

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Remark 4.9. As in the previous section, the failure of $\overline{F}^{\omega}$ to be a line bundle can be cured by introducing orbifold structures at the odd nodes. By abuse of notation, we assume that this has been done and that $\overline{F}^{\omega}$ therefore *is* a line bundle on $\overline{P}$.

Lemma 4.10. $\overline{\psi}^{*}\overline{F}^{\omega} \simeq \omega_{\overline{C}}$.

Proof. By adjunction, there exists a morphism

$$\overline{\psi}^{*}\overline{F}^{\omega} \to \overline{\psi}^{*}\overline{\psi}_{*}\omega_{\overline{C}} \to \omega_{\overline{C}}.$$

Since $\overline{\psi}$ is finite, it is enough to check that the composite is an isomorphism after pushing forward along $\overline{\psi}$. Since $\overline{F}^{\omega}$ is a line bundle, we may apply the projection formula to compute

$$\overline{\psi}_{*}\overline{\psi}^{*}\overline{F}^{\omega} = \overline{F}^{\omega} \otimes \overline{\psi}_{*}\mathcal{O}_{\overline{C}} = \overline{F}^{\omega} \otimes (\mathcal{O}_{\overline{P}} \oplus \overline{F}).$$

Since $\overline{F}^{\omega}$ is a line bundle, $\overline{F}^{\omega} \otimes \overline{F}$ is also a rank-one torsion-free; hence, we have a short exact sequence

$$0 \to \overline{F}^{\omega} \otimes \overline{F} \xrightarrow{\text{ev}} \omega_{\overline{P}} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is a torsion sheaf. By taking ω-duals, we get

$$0 \to \mathcal{H}om(\omega_{\overline{P}}, \omega_{\overline{P}}) = \mathcal{O}_{\overline{P}} \to \mathcal{H}om(\overline{F}^{\omega} \otimes \overline{F}, \omega_{\overline{P}}) \to \mathcal{E}xt^{1}(\mathcal{Q}, \omega_{\overline{P}}) \to 0.$$

Since $\overline{F}^{\omega}$ is a line bundle, the first arrow is an isomorphism, which shows that $\mathcal{Q}$ vanishes. We conclude that

$$\overline{\psi}_{*}\overline{\psi}^{*}\overline{F}^{\omega} = \overline{F}^{\omega} \oplus \overline{F}^{\omega} \otimes \overline{F} = \overline{F}^{\omega} \oplus \omega_{\overline{P}} = \overline{\psi}_{*}\omega_{\overline{C}}.$$ □

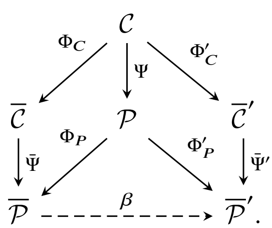
Proof of Theorem 4.3. Consider a smoothing $\bar{\Psi}\colon \overline{\mathcal{C}} \to \overline{\mathcal{P}}$ of $\overline{\psi}$ over Δ, and mark the generic fibre $\overline{\mathcal{P}}_{\eta}$ with the branch divisor of $\bar{\Psi}$. After a finite base change if necessary, let $(\mathcal{P}, \mathcal{B})$ be the unique limit of $(\overline{\mathcal{P}}_{\eta}, \overline{\mathcal{B}}_{\eta})$ as a stable curve with unordered markings. Let $\Psi\colon \mathcal{C} \to \mathcal{P}$ be the associated hyperelliptic admissible cover with the minimal log structure. Let $\Phi_P\colon \mathcal{P} \to \overline{\mathcal{P}}$ denote the contraction, and similarly, Φ_C.

Since $\mathcal{C}$ is a normal surface, and by reflexivity of the sheaves involved, we notice that $\omega_{\mathcal{C}/\Delta}$ and $\Phi_C^{*}\omega_{\overline{\mathcal{C}}/\Delta}$ differ only by a vertical divisor, supported on the central fibre. We may hence write

$$\Phi_C^{*}\omega_{\overline{\mathcal{C}}/\Delta} = \omega_{\mathcal{C}/\Delta}(\lambda),$$

for some conewise-linear function $\lambda \in H^0(\mathcal{C}, \overline{M}_{\mathcal{C}})$, a priori only with the divisorial log structure of $\mathcal{C}$ with respect to its central fibre. However, let $\mathcal{P}^{\text{tw}}$ denote the orbicurve $[\mathcal{C}/\iota]$. Since $\omega_{\overline{C}} = \overline{\psi}^{*}\overline{F}^{\omega}$, and $\omega_C = \psi^{*}\omega_{P^{\text{tw}}}(\mathbf{b}/2)$, we deduce that their difference is also pulled back from P^{tw}. Hence, λ is pulled back from λ_T on T' with its divisorial log structure.

We may now apply our construction to (Ψ, λ_T), thus obtaining a Gorenstein hyperelliptic curve $\bar{\Psi}'\colon \overline{\mathcal{C}}' \to \overline{\mathcal{P}}'$, fitting in the following diagram:



Observe that $\overline{\mathcal{P}}$ and $\overline{\mathcal{P}}'$ are normal surfaces, and the exceptional loci of Φ_P and Φ'_P are the same, so we may find an isomorphism $\beta\colon \overline{\mathcal{P}} \simeq \overline{\mathcal{P}}'$ commuting with the Φ_P's by birational rigidity [Deb01, Lemma 1.15]. Moreover, $\beta^*\mathcal{O}_{\overline{\mathcal{P}}'}(\mathbb{1}) \simeq \overline{F}^{\omega}$.

Now, by ω-reflexivity, we recover $\overline{F} \simeq \beta^*\omega_{\overline{\mathcal{P}}}(-\mathbb{1})$, and therefore, $\bar{\Psi}_*\mathcal{O}_{\overline{\mathcal{C}}} = \beta^*(\bar{\Psi}'_*\mathcal{O}_{\overline{\mathcal{C}}'})$. The branch divisor is determined by the image of $\mathcal{B}$, and by the components of the central fibre that are contained in the support of λ without being contracted by Φ (by Riemann–Hurwitz); hence, we conclude that there is also an isomorphism $\alpha\colon \overline{\mathcal{C}} \simeq \overline{\mathcal{C}}'$ covering β. □

Proof of Corollary 4.4. We are left to show that $\overline{\psi}$ can be smoothed out when $\overline{C}$ is reduced. We will proceed step by step by showing that the various ingredients of this moduli problem are unobstructed.

The reduced rational curve $\overline{P}$ is smoothable; see [Har10, Example 29.10.2].

The line bundle $\overline{F}^{\omega}$ can be smoothed out since the relative Picard scheme of a curve is unobstructed. Consequently, the structure sheaf of $\overline{C}$ can be smoothed out by taking its ω-dual $\overline{F}$.

Finally, the multiplication map μ is a cosection of $\overline{F}^{\otimes 2}$. Consider the pairing

$$H^0(\overline{F}^{\otimes -2}) \times Hom(\overline{F}^{\otimes -2}, \omega_{\overline{P}}) \to H^0(\omega_{\overline{P}}) = 0$$

by composition. Since $\overline{C}$ is reduced by assumption, μ does not vanish generically on any component of $\overline{P}$. It follows that every section of $\mathcal{H}om(\overline{F}^{\otimes -2}, \omega_{\overline{P}})$ must vanish generically, and since this sheaf is torsion-free, it is zero *tout court*. By Serre duality, $h^1(\overline{F}^{\otimes -2}) = 0$; hence, deformations of μ are also unobstructed. □

Remark 4.11. We expect the result to hold for all Gorenstein hyperelliptic curves, but we have not been able to prove the smoothability of nonreduced curves yet. However, these curves can be dispensed with as far as our application to differentials is concerned.

5. The differential descent conjecture

5.1. *Abelian differentials in general*

The moduli space of Abelian differentials has at most three connected components (depending on the multiplicity μ of the zeroes) [KZ03]. In general, connected components of the space of multiscale differentials are not irreducible. The *global residue condition* (GRC) was introduced in [BCG+18] to single out the *smoothable* differentials. Roughly speaking, it says that the sum of the residues at poles of level i that are joined by a connected subcurve at level $i+1$ must vanish, despite the possibility that the corresponding nodes belong to different subcurves at level i. The proof of necessity goes by cutting the generic fibre of a smoothing along the vanishing cycle corresponding to these nodes, and applying Stokes' theorem to compute the integral of the abelian differential on the resulting surface with boundary. The proof of sufficiency is more complicated and based on a refined *plumbing* construction. With the logarithmic understanding of the moduli space of *generalised* multiscale differentials reached in [CC19, CGH+22], the GRC remains the only ingredient of [BCG+19] relying on transcendental techniques. A purely algebraic description of smoothable differentials is contained in the following conjecture, originally due to Ranganathan and Wise.

Conjecture 5.1 (Gorenstein curves and smoothable differentials). *Let (C, η) be a logarithmic rubber differential with tropicalisation λ. Then η is smoothable if and only if*

(i) *for every level i, the truncation λ_i of λ (as in §1.6) is a realisable tropical differential;*
(ii) *there exists a logarithmic modification $\widetilde{C} \to C$, a natural extension $\tilde{\eta}$ of the pullback of η to $\widetilde{C}$, and a reduced Gorenstein contraction $\sigma\colon \widetilde{C} \to \overline{C}_i$ such that $\sigma^*\omega_{\overline{C}_i} = \omega_C(\lambda_i)$, and*
(iii) *the differential $\tilde{\eta}_i$ at level i descends to a local generator of $\omega_{\overline{C}_i}$.*

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Here, $\widetilde{C}_i$ is determined by η as follows, in order to ensure that the twist of the canonical bundle be trivial on the upper levels, and to avoid nonreduced components in the contractions $\overline{C}_i$. Indeed, ribbons appear when $\omega_C(\lambda_i)$ has positive degree on the support of λ_i. This happens precisely when at least one zero of order $m \geq 1$ is contained in the support of λ_i. In this case, since we have a nontrivial logarithmic structure of marking type at the zero, we can subdivide the corresponding leg at level i; classically, this means sprouting a new semistable rational component at the marking. In the natural coordinates $[x_0 : x_1]$ with respect to the two special points, the differential η can be extended uniquely to the new component $\tilde{v}$ by setting $\eta_{\tilde{v}} = x_0^m\, \mathrm{d}x_0$; the choice of a nonzero scalar is compensated by the automorphisms of the underlying curve. Notice that this differential does not contribute to the GRC, since $m + 2 > 1$. The mere existence of the nonzero differentials at levels higher than i guarantees that the twist of the canonical bundle by λ_i will be trivial (not just numerically). We provide the following ad hoc example in the hope of acquainting the reader with the log modification procedure.

Example 5.2. Let (C, η) be a generalised multiscale differential, where C consists of two components C_0 and C_{-1} joined at a single node q. Assume that C_0 is a curve of genus two, and η_0 is a holomorphic differential with simple zeroes at q and its conjugate point $\bar{q}$, which in particular is a marking of C (note that C is not hyperelliptic in the sense of admissible covers, although C_0 is; the specific C_1 will be immaterial for this discussion). In a general one-parameter smoothing, C_0 will have negative self-intersection; in particular, it can be contracted by general principles (Artin's criterion). The resulting singularity is formally isomorphic to $\mathbf{k}[[t^3, t^4, t^5]]$, which is not Gorenstein. Indeed, since $\omega_{C_0} = \mathcal{O}_{C_0}(q + \bar{q})$, twisting by a multiple of C_0 will never make the relative dualising bundle of the family trivial on C_0. Instead, we are going to modify C by log blowing up C_0 at $\bar{q}$, and *then* contract, which results into a locally planar singularity of type A_5, whose dualising bundle is generated by a meromorphic differential with poles of order three on either branch.

5.2. Hyperelliptic differentials

The connected component consisting of hyperelliptic differentials is already irreducible [CC19, Proposition 5.16]. This is proved by identifying the moduli space of hyperelliptic differentials with a moduli space of quadratic differentials on rational curves. We therefore view the following result as a first proof of concept for Conjecture 5.1.

Proposition 5.3. *Let $(\psi\colon C \to P, \eta)$ be a log rubber hyperelliptic differential with tropicalisation $\bar{\lambda}$. The differential η_i at level i descends to a generator of the dualising sheaf of the Gorenstein contraction associated to $(\psi\colon \widetilde{C}_i \to P, \lambda_i)$ as in §2.*

Proof. Let η_i denote the collection of differentials on components at level $\leq i$. Then η_i is a section of the restriction of $\omega_C(\lambda_i)$ to $C_{\leq i}$ (i.e., a meromorphic differential with poles along the level $[i, i+1]$-nodes, whose order of pole is determined by the slopes of $\bar{\lambda}$ plus one). Since η_i is ι-anti-invariant, it descends to a section of the odd part of $\psi_*\omega_C(\lambda_i)$ on $P_{\leq i}$, which is the restriction of L. Since the latter is trivial on $P_{>i}$, this section extends uniquely to P. We can therefore identify it with a section of $\mathcal{O}_{\overline{P}}(\mathbb{1})$ on $\overline{P}$, and in turn with an anti-invariant section of $\omega_{\overline{C}}$. In fact, up to scaling, it can be identified with the local generator given in Corollary 3.8. □

Remark 5.4. Anti-invariance under ι implies that residues at conjugate (resp. Weierstrass) points are opposite (resp. zero). In particular, ι-anti-invariance implies that the Global Residue Condition holds. Although every holomorphic differential on a hyperelliptic curve is ι-anti-invariant (Remark 4.6) (and so are their limits), generalised multiscale differentials are only meromorphic on lower levels of the curve; hence, the above is a proof of Conjecture 5.1 not for any differential on a hyperelliptic curve C (in the sense of admissible covers), but only for the ι-anti-invariant ones. See Example 5.6.

Example 5.5. Let C be a nodal curve consisting of two hyperelliptic components C_0, of genus g_0, and C_{-1}, joined at a single node q, which is Weierstrass on both. Let η be an anti-invariant multiscale differential on C, such that η_0 has a single zero of multiplicity $2g_0 - 2$ at q. Then λ has slope $2g_0 - 1$

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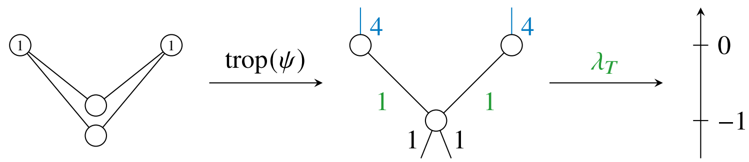
along the corresponding edge. The meromorphic differential η_1 has a pole of order $2g_0$ at q; notice that the GRC is automatically satisfied by the Residue Theorem. The contraction $\overline{C}_{-1}$ has an A_{2g_0}-singularity (of genus g_0) at q, and $\eta_{-1} \approx \frac{dt}{t^{2g_0}}$ descends to a generator of $\omega_{\overline{C}_{-1}}$.

Example 5.6. Let (C, η) be a genus 3 hyperelliptic multiscale differential whose level graph is the following:



Let η restrict to dz on the two elliptic curves. Choose coordinates on the rational curves R_i at level -1 in such a way that the nodes are 0 and ∞, and the zeroes a and b. Then η restricts to

$$\alpha_i(t-a)(t-b)\frac{dt}{t^2}$$

on the rational curve R_i, $i = 1, 2$. Here, ι-anti-invariance forces $\alpha_1 = -\alpha_2$. Contracting the subcurves at level 0, we obtain two rational curves joined at two tacnodes. There is a linear condition for a meromorphic differential with poles of order two on the pointed normalisation to descend to the tacnode (cf. [Smy11, §2.2]) which is analogous to the condition $\alpha_1 = -\alpha_2$ from above. However, any choice of α_i gives rise to a generalised multiscale differential.

Remark 5.7. If $a = -b$, the residues are zero. By varying the α_i, we thus get an example of a multiscale differential which satisfies the GRC but is not anti-invariant. This will be the limit of differentials on smooth, non-hyperelliptic curves. There are of course even more examples of non-hyperelliptic differentials on a hyperelliptic curve if we do not impose that the sets of zeroes and poles are invariant under the hyperelliptic involution.

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