

CONTINUITY PROPERTIES OF ATTRACTORS FOR ITERATED FUZZY SET SYSTEMS

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Abstract

An N -map Iterated Fuzzy Set System (IFZS), introduced in [4] and to be denoted as (\mathbf{w}, Φ) , is a system of N contraction maps $w_i : X \rightarrow X$ over a compact metric space (X, d) , with associated “grey level” maps $\phi_i : [0, 1] \rightarrow [0, 1]$. Associated with an IFZS (\mathbf{w}, Φ) is a fixed point $u \in \mathcal{F}^*(X)$, the class of normalized fuzzy sets on X , $u : X \rightarrow [0, 1]$. We are concerned with the continuity properties of u with respect to changes in the w_i and the ϕ_i . Establishing continuity for the fixed points of IFZS is more complicated than for traditional Iterated Function Systems (IFS) with probabilities since a composition of functions is involved. Continuity at each specific attractor u may be established over a suitably restricted domain of ϕ_i maps. Two applications are (i) animation of images and (ii) the inverse problem of fractal construction.

1. Introduction

We consider a special class of dynamical systems over function spaces which has been introduced in [4] and referred to as Iterated Fuzzy Set Systems (IFZS). An IFZS, to be denoted as (\mathbf{w}, Φ) , is a system \mathbf{w} of N contraction mappings $w_i(x) : X \rightarrow X$ over a compact metric space (X, d) (the “base space”), with an associated system Φ of mappings $\phi_i : [0, 1] \rightarrow [0, 1]$ (the “grey level maps”). The IFZS incorporates, at least in part, the idea of set-valued contraction mappings over a compact metric space, developed by Hutchinson [9] and Barnsley and coworkers [1]–[3], the latter referring to such systems as Iterated Function Systems (IFS). There is, however, a fundamental difference between the IFS and the IFZS. The IFS works with measures and a set of probabilities p_i associated with the w_i which act as multiplicative weights. The IFZS works with functions $u : X \rightarrow [0, 1]$ and functions $\phi_i : [0, 1] \rightarrow [0, 1]$ which are composed with the u . From the viewpoint of image processing or pattern recognition,

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the value $u(x)$ may be interpreted as a nonnegative grey level or brightness value at the point (or pixel) $x \in X$. The IFZS component of the IFZS, a set-valued mapping w which acts on nonempty compact subsets of X , will operate on the level sets of u . As such, it is natural to consider $u \in \mathcal{F}(X)$, the class of fuzzy sets on X ; compactness of the level sets requires that the u be restricted to the subspace $\mathcal{F}^*(X) \subset \mathcal{F}(X)$ of (normalized) upper semicontinuous fuzzy sets with the d_∞ metric [7] involving Hausdorff distances between their level sets. (In the literature, $\mathcal{F}^*(X)$ continues to be referred to as the class of *fuzzy sets* on X because of its historical probabilistic interpretations.) Associated with each IFZS (w, Φ) is a contractive operator T which possesses a unique fixed point u , the *attractor* of the IFZS. The support of u is a subset of the attractor $A \subseteq X$ of the IFS defined by w . The relevance of the IFZS approach to image processing has been discussed in [4]. In addition, our preliminary studies indicate that the IFZS affords a considerable simplification in the treatment of the *inverse problem* of fractal or image construction [1]–[5], [10], [11].

From the viewpoint of image representation, an important advantage of an IFS-type method over function spaces (in this paper, $\mathcal{F}^*(X)$ with the d_∞ -metric) is that it permits a more direct control of grey level or brightness values at individual points $x \in X$. This has been realized in more recent “fractal compression schemes” [2], [10]. In another paper, a complete connection between the IFZS method and these schemes will be provided [8].

In this paper, we are concerned specifically with the continuity properties of invariant sets u for N -map IFZS, particularly with respect to changes in the contraction maps w_i and the grey level maps ϕ_i . Barnsley [1] has shown that the attractor of a contractive IFS varies continuously with respect to parameters in the IFS maps. An obvious application is “animation”, where IFS attractors representing images are deformed in an apparently continuous manner in time. As well, continuity results are clearly important for the inverse problem, where local “fine tuning” of the IFS maps and probabilities is performed to optimize the approximation of sets or measures with IFS attractors or invariant measures. In [6], the (uniform) continuity of IFS attractors and invariant measures with respect to arbitrary variations in the IFS maps and/or associated probabilities was derived, a generalization of the results in [1]. The continuity followed naturally from the rather simple structure of IFS: the metrics employed for the map vectors w and the probability vectors \mathbf{p} are easily and directly related to (i) the Hausdorff metric h on $\mathcal{H}(X)$, the set of nonempty compact subsets of X and (ii) the Hutchinson metric d_H on $\mathcal{M}(X)$, the set of probability measures on $\mathcal{B}(X)$, the σ -algebra of Borel subsets of X .

Establishing continuity of IFZS attractors with respect to IFS or grey level maps is more complicated, primarily because a composition of the ϕ and u functions is involved. From the Banach fixed point theorem, it easily follows that the attractor u is continuous with respect to the IFZS operator T . However, the T operator is not

continuous with respect to the grey level maps ϕ_i . Thus it is *not* possible to use the T operator (unlike the IFS case) to prove that u is continuous with respect to the ϕ_i maps. A main result of this paper is that we may bypass the use of the T operator and establish a “local” continuity of u with respect to the ϕ_i and w_i maps. Uniform continuity is sacrificed, but this is not a great price, since a continuous variation in u can still be produced by means of a controlled variation of the ϕ_i .

The layout of this paper is as follows. In Section 2, some basic properties of IFZS are presented in terms of the IFS and grey-level map components. In Section 3 we derive a basic continuity property of fixed points of contractive maps. We then review the “well-behaved” continuity properties of IFS attractors and invariant measures. The rather trivial continuity property of an IFZS fixed point with respect to the IFS maps w_i is then shown. Finally, we show that the contractive T operator of an IFZS is discontinuous with respect to the grey level maps ϕ_i . In Section 4, we define an appropriate “local” distance function for the ϕ_i maps and derive the continuity of u with respect to a restricted set of ϕ_i . Section 5 is devoted to a further discussion of these results and suggestions for future research. Appendix A contains a brief discussion of invariant measures for IFS to help the general reader understand the IFS continuity results outlined in Section 3.

2. Preliminaries: Iterated Fuzzy Set Systems

In this section we introduce all the relevant notation and outline the salient features of IFZS. The reader is referred to [4] for details. An N -map IFZS, denoted as (\mathbf{w}, Φ) , has an IFS component, \mathbf{w} , on a base space (X, d) and a grey-level component, Φ , on $[0, 1]$, as outlined below.

A. The IFS component: As above, (X, d) will denote a compact metric space. Let \mathbf{w} denote a set of N contraction maps on X , that is, $\mathbf{w} = (w_1, w_2, \dots, w_N)$, $w_i : X \rightarrow X$ such that for all $x, y \in X$,

$$d(w_i(x), w_i(y)) \leq s_i d(x, y), \quad 0 \leq s_i < 1, \quad i = 1, 2, \dots, N. \tag{2.1}$$

We shall refer to s_i as the *contractivity factor* of the map w_i : the contractivity factor of the IFS \mathbf{w} is defined as $s = \max_{1 \leq i \leq N} s_i$.

Let $\mathcal{H}(X)$ denote the set of nonempty compact subsets of X . The Hausdorff distance between two sets $A, B \in \mathcal{H}(X)$ is given by

$$h(A, B) \equiv \max\{D(A, B), D(B, A)\}, \tag{2.2}$$

where

$$D(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y).$$

$(\mathcal{H}(X), h)$ is a complete metric space.

Given an IFS w , define the set-valued mapping $\hat{w} : \mathcal{H}(X) \rightarrow \mathcal{H}(X)$ as follows:

$$\hat{w}(S) = \bigcup_{i=1}^N w_i(S), \quad S \in \mathcal{H}(X), \tag{2.3}$$

where $w_i(S) \equiv \{w_i(x), x \in S\}$. As is well known [1, 9], there exists a unique set $A \in \mathcal{H}(X)$, the *attractor* of the IFS, which is invariant with respect to \hat{w} , that is,

$$A = \hat{w}(A) = \bigcup_{i=1}^N w_i(A). \tag{2.4}$$

This result follows from the fact that \hat{w} is a contraction mapping on $(\mathcal{H}(X), h)$: for $A, B \in \mathcal{H}(X)$,

$$h(\hat{w}(A), \hat{w}(B)) \leq sh(A, B). \tag{2.5}$$

As well, from the contraction mapping principle,

$$h(\hat{w}^n(S), A) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \forall S \in \mathcal{H}(X). \tag{2.6}$$

SIMPLE EXAMPLES. Let $X = [0, 1]$ and $N = 2$, with $w_1(x) = sx$, $w_2(x) = sx + (1 - s)$. (i) For $s \geq 1/2$, $A = [0, 1]$. (ii) For $s = 0$, $A = \{0, 1\}$. (iii) For $0 < s < 1/2$, A is a Cantor-like set on $[0, 1]$. When $s = 1/3$, A is the ternary Cantor set on $[0, 1]$.

B. The grey level component: Let $\mathcal{F}(X)$ denote the class of functions $u : X \rightarrow [0, 1]$. We consider the following subclass of functions [4], $\mathcal{F}^*(X) \subset \mathcal{F}(X) : u \in \mathcal{F}^*(X)$ if and only if

- (i) $u \in \mathcal{F}(X)$,
- (ii) u is upper semicontinuous on (X, d) ,
- (iii) u is normalized, i.e., $u(x_0) = 1$ for some $x_0 \in X$.

Now define the α -level set, $[u]^\alpha$, of $u \in \mathcal{F}^*(X)$ as follows:

$$[u]^\alpha = \{x \in X : u(x) \geq \alpha\}, \text{ for } 0 < \alpha \leq 1,$$

$$[u]^0 = \overline{\{x \in X : u(x) > 0\}} \text{ (the bar denotes closure).}$$

From properties (ii) and (iii) above, the metric space $(\mathcal{H}(X), h)$ contains all α -level sets, $0 \leq \alpha \leq 1$, of all $u \in \mathcal{F}^*(X)$. We consider the following metric on $\mathcal{F}^*(X)$:

$$d_\infty(u, v) = \sup_{0 \leq \alpha \leq 1} \{h([u]^\alpha, [v]^\alpha)\}, \quad \forall u, v \in \mathcal{F}^*(X).$$

The metric space $(\mathcal{F}^*(X), d_\infty)$ is complete [7].

Let $\mathcal{M}^+([0, 1])$ denote the set of all functions $\phi : [0, 1] \rightarrow [0, 1]$ with the following properties:

- (i) ϕ is nondecreasing on $[0, 1]$,
- (ii) ϕ is right continuous on $[0, 1]$,
- (iii) $\phi(0) = 0$,
- (iv) $\phi(t^*) > 0$ for at least one $t^* \in (0, 1]$.

Now given an IFS w , let $\Phi = (\phi_1, \phi_2, \dots, \phi_N)$ denote a set of associated grey level maps which satisfy the following conditions:

- (a) $\phi_i \in \mathcal{M}^+([0, 1])$ for $i \in \{1, 2, \dots, N\}$,
- (b) $\phi_{i^*}(1) = 1$ for at least one $i^* \in \{1, 2, \dots, N\}$.

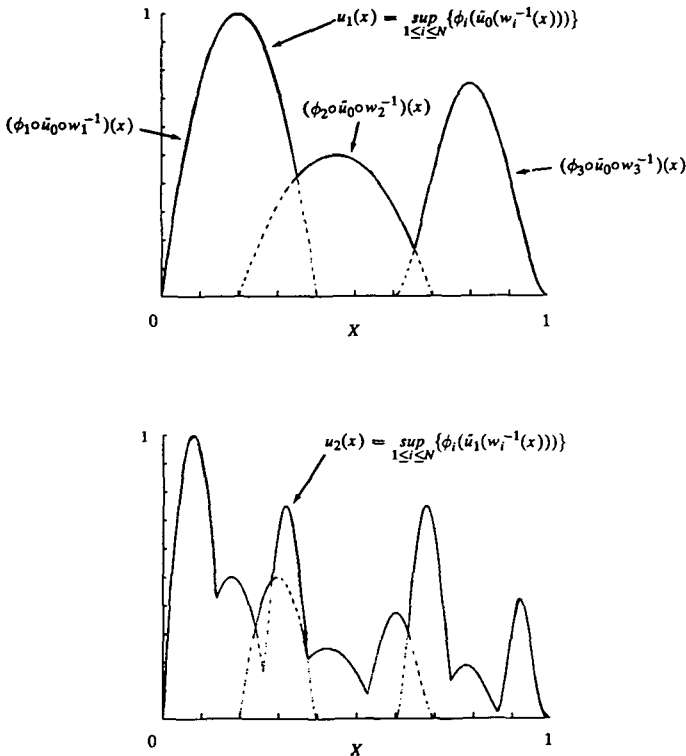


FIGURE 1. Graphs of $u_1(x) = (Tu_0)(x)$ and $u_2(x) = (Tu_1)(x)$, where $u_0(x) = 4x(1 - x)$ and T is the contractive operator for the following three-map IFZS on $X = [0, 1]$:

$$\begin{aligned}
 w_1(x) &= 0.4x, & \phi_1(t) &= t, \\
 w_2(x) &= 0.5x + 0.2, & \phi_2(t) &= 0.5t, \\
 w_3(x) &= 0.4x + 0.6, & \phi_3(t) &= 0.75t^2.
 \end{aligned}$$

The pair of vectors (\mathbf{w}, Φ) constitutes an N -map IFZS. Properties (i), (ii) and (b) will guarantee that the contractive operator T , to be defined below, will map $\mathcal{F}^*(X)$ into itself. Property (iii) is a natural assumption in the consideration of grey level functions: if the grey level of a point (or pixel) $x \in X$ is zero, then it should remain zero after being acted upon by the ϕ_i maps. Property (iv) will be explained below.

We now introduce the following operator³ $T : \mathcal{F}^*(X) \rightarrow \mathcal{F}^*(X)$:

$$(Tu)(x) = \sup_{1 \leq i \leq N} \{ \phi_i(\tilde{u}(w_i^{-1}(x))) \}, \tag{2.7}$$

where, for $B \subset X$,

$$\begin{aligned} \tilde{u}(B) &:= \sup_{y \in B} \{ u(y) \}, \quad \text{if } B \neq \emptyset, \\ \tilde{u}(\emptyset) &:= 0. \end{aligned}$$

The action of the operator T for a simple IFZS on $[0, 1]$ is illustrated in Figure 1.

In [4] it was proved that T is a contraction mapping on the space $(\mathcal{F}^*(X), d_\infty)$:

$$d_\infty(Tu_1, Tu_2) \leq s d_\infty(u_1, u_2) \quad \forall u_1, u_2 \in \mathcal{F}^*(X). \tag{2.8}$$

Thus, there exists a unique function $u \in \mathcal{F}^*(X)$, the *attractor* of the IFZS, such that

$$Tu = u. \tag{2.9}$$

This, in turn, implies that u is the unique solution to the following functional equation in the unknown $v \in \mathcal{F}^*(X)$,

$$v(x) = \sup \{ \phi_1(\tilde{v}(w_1^{-1}(x))), \phi_2(\tilde{v}(w_2^{-1}(x))), \dots, \phi_N(\tilde{v}(w_N^{-1}(x))) \}, \quad \forall x \in X. \tag{2.10}$$

The attractor u for the IFZS of Figure 1 is sketched in Figure 2.

Let \mathcal{T}_N denote the set of all contractive operators T corresponding to N -map IFZS on (X, d) . A noteworthy property of these operators is the following: for $v \in \mathcal{F}^*(X)$,

$$[Tv]^\alpha = \bigcup_{i=1}^N w_i([\phi_i \circ v]^\alpha), \quad 0 \leq \alpha \leq 1. \tag{2.11}$$

Thus, when $v = u = Tu$,

$$[u]^\alpha = \bigcup_{i=1}^N w_i([\phi_i \circ u]^\alpha), \quad 0 \leq \alpha \leq 1, \tag{2.12}$$

which is a generalized self-tiling property of α -level sets for IFZS attractors.

³Note: In [4], this operator was denoted T_s in order to emphasize the fact that the *supremum* function was chosen as the associative operator on $\mathcal{F}^*(X)$. For notational convenience, we have dropped the subscript s in this paper.

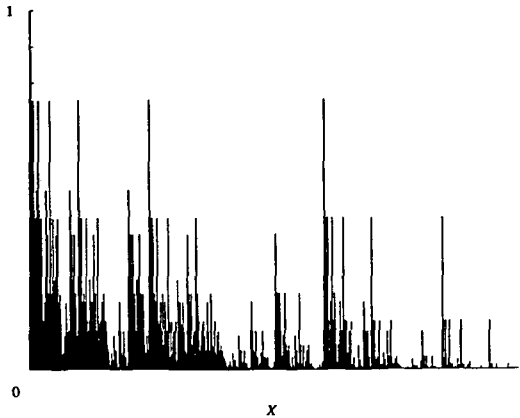


FIGURE 2. Histogram approximation of the attractor $u = Tu$ for the IFZS (w, ϕ) of Figure 1. Note that $u(0) = 1$.

3. Continuity of fixed points and problems with the $d_{Con(\mathcal{F}^*(X))}$ metric

Let (Y, d_Y) be a complete metric space and denote by

$$Con(Y) = \{f : Y \rightarrow Y \mid \exists s_f \in [0, 1) : d_Y(f(x), f(y)) \leq s_f d_Y(x, y), \forall x, y \in Y\}, \tag{3.1}$$

a set of contraction mappings on (Y, d_Y) . Now consider the following natural metric on this function space:

$$d_{Con(Y)}(f, g) \equiv \sup_{y \in Y} d_Y(f(y), g(y)), \quad f, g \in Con(Y). \tag{3.2}$$

Note that the metric space $(Con(Y), d_{Con(Y)})$ itself may not be complete. Now let y_f and y_g be the unique fixed points of the maps f and g , respectively, that is, $f(y_f) = y_f$ and $g(y_g) = y_g$.

PROPOSITION 3.1. *The following inequality holds:*

$$d_Y(y_f, y_g) < \frac{1}{1 - s_f} d_{Con(Y)}(f, g), \tag{3.3}$$

where s_f is the contractivity factor of f .

PROOF.

$$\begin{aligned} d_Y(y_f, y_g) &= d_Y(f(y_f), g(y_g)) \\ &\leq d_Y(f(y_f), f(y_g)) + d_Y(f(y_g), g(y_g)) \\ &< s_f d_Y(y_f, y_g) + d_{Con(Y)}(f, g), \end{aligned}$$

which then yields (3.3).

Inequality (3.3) implies continuity for the fixed point y_f with respect to f , uniformly on the set of contractive maps in $(Con(Y), d_{Con(Y)})$ with contractivity factors $0 \leq s_f \leq s < 1$. Such continuity would be useless when the topology induced on $Con(Y)$ by $d_{Con(Y)}$ is such that every element $f \in Con(Y)$ is isolated, that is, there exists an $\epsilon > 0$ such that $N_\epsilon(f) \setminus \{f\} = \emptyset$, where $N_\epsilon(f) = \{g \in Con(Y) : d_{Con(Y)}(f, g) < \epsilon\}$ denotes the ϵ -neighbourhood of f . This is, in fact, the situation for IFZS attractors as we show below. However, in order that the reader may well appreciate the difficulties posed by the IFZS problem, we first review the basic “well-behaved” continuity results for “traditional” IFS attractors and measures.

3.1. Continuity results for “traditional” IFS attractors and invariant measures

The basic features of IFS with probabilities (IFSP) and their invariant measures are given in Appendix A. In the following discussion, it will be convenient to consider sequences of contractive IFSP, $(\mathbf{w}_k, \mathbf{p}_k)$, $k = 1, 2, \dots$, where

$$\mathbf{w}_k = (w_{k1}, w_{k2}, \dots, w_{kN}), \quad \mathbf{p}_k = (p_{k1}, p_{k2}, \dots, p_{kN}). \tag{3.4}$$

Let s_{kj} denote the contractivity factors of the w_{kj} ; the contractivity factor of the IFS \mathbf{w}_k is then $s_k = \max_{1 \leq j \leq N} s_{kj}$. We then define the following natural metrics for the map and probability vectors, respectively,

$$d_w^N(\mathbf{w}_1, \mathbf{w}_2) \equiv \max_{1 \leq i \leq N} \sup_{x \in X} d(w_{1i}(x), w_{2i}(x)). \tag{3.5}$$

$$d_p^N(\mathbf{p}_1, \mathbf{p}_2) \equiv \max_{1 \leq i \leq N} |p_{1i} - p_{2i}|. \tag{3.6}$$

Using these metrics, the following continuity results were derived in [6].

PROPOSITION 3.2. *Let \mathbf{w}_1 be an N -map contractive IFS with contractivity factor s_1 and attractor A_1 . Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all N -map IFS \mathbf{w}_2 satisfying $d_w^N(\mathbf{w}_1, \mathbf{w}_2) < \delta$, it follows that $h(A_1, A_2) < \epsilon$, where A_2 is the attractor of \mathbf{w}_2 .*

PROPOSITION 3.3. *Let $(\mathbf{w}_1, \mathbf{p}_1)$ be an N -map contractive IFS with probabilities, with contractivity factor s_1 and invariant measure μ_1 . Then for every $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that for all N -map IFSP $(\mathbf{w}_2, \mathbf{p}_2)$ satisfying*

$$d_w^N(\mathbf{w}_1, \mathbf{w}_2) < \delta_1, \quad d_p^N(\mathbf{p}_1, \mathbf{p}_2) < \delta_2,$$

it follows that $d_H(\mu_1, \mu_2) < \epsilon$, where μ_2 is the invariant measure of the IFSP $(\mathbf{w}_2, \mathbf{p}_2)$.

Proposition 3.2, which follows immediately from Proposition 3.1 with $\delta = \epsilon(1 - s_1)$, is a generalization of Barnsley’s result [1, Sect. 3.11] showing continuity of the attractor A with respect to a parameter in the IFS maps. As for the proof of

Proposition 3.3, let $Y = \mathcal{M}(X)$ (see Appendix A) and let $Con(Y)$ be the space of Markov operators corresponding to all contractive N -map IFS with probabilities. Then define the following metric on this space:

$$d_M^N(M_1, M_2) \equiv \sup_{\mu \in \mathcal{M}(X)} d_H(M_1\mu, M_2\mu). \tag{3.7}$$

After suitably defining the δ_i , the result then follows from Proposition 3.1.

3.2. Continuity results for IFZS attractors Now let $(Y, d_Y) = (\mathcal{F}^*(X), d_\infty)$ and $Con(\mathcal{F}^*(X)) = \mathcal{T}_N$, the space of contractive N -map IFZS operators on $(\mathcal{F}^*(X), d_\infty)$. It will be convenient to index the IFZS operators as $T_k \in \mathcal{T}_N, k = 1, 2, 3, \dots$, with action defined by

$$(T_k u)(x) = \sup_{1 \leq i \leq N} \{ \phi_{ki}(\tilde{u}(w_{ki}^{-1}(x))) \}, \quad u \in \mathcal{F}^*(X), \tag{3.8}$$

where the corresponding IFZS has been denoted as (\mathbf{w}_k, Φ_k) ,

$$\mathbf{w}_k = (w_{k1}, w_{k2}, \dots, w_{kN}), \quad \Phi_k = (\phi_{k1}, \phi_{k2}, \dots, \phi_{kN}). \tag{3.9}$$

Also let u_k denote the fuzzy set attractor of (\mathbf{w}_k, Φ_k) , that is, $T_k u_k = u_k$. Note that if $T_k u = T_l u$ for all $u \in \mathcal{F}^*(X)$, then we consider T_k to be equal to T_l . For example, the action of T_k is unchanged when the w_{ki} and associated ϕ_{ki} maps are permuted.

In analogy to the IFS case, cf. (3.7), it appears natural to choose the following distance function on $Con(\mathcal{F}^*(X))$ (see also [7]),

$$d_{Con(\mathcal{F}^*(X))}(T_k, T_l) = \sup_{u \in \mathcal{F}^*(X)} d_\infty(T_k u, T_l u), \tag{3.10}$$

which defines a metric on \mathcal{T}_N . (If $d_{Con(\mathcal{F}^*(X))}(T_1, T_2) = 0$, then $d_\infty(u_1, u_2) = 0$.) Continuity of the fixed points u_k with respect to the T_k operators follows immediate from Proposition 3.1. However, this property contains no information: as in the IFS case, it would be desirable to formulate continuity of the u_k in terms of the IFZS components \mathbf{w}_k and Φ_k . The following result shows that continuity with respect to the IFS maps w_{ki} is trivial.

PROPOSITION 3.4. *Let (\mathbf{w}_1, Φ_1) be an IFZS with contractivity factor s_1 , associated operator $T_1 \in \mathcal{T}_N$ and attractor $u_1 \in \mathcal{F}^*(X)$. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any IFZS (\mathbf{w}_2, Φ_1) satisfying $d_w^N(\mathbf{w}_1, \mathbf{w}_2) < \delta$, it follows that*

$$d_\infty(u_1, u_2) < \epsilon,$$

where $u_2 \in \mathcal{F}^*(X)$ is the attractor of the IFZS (\mathbf{w}_2, Φ_1) .

PROOF. Given an $\epsilon > 0$, let w_2 be such that

$$d_w^N(w_1, w_2) < \epsilon(1 - s_1).$$

Let $T_2 \in \mathcal{T}_N$ be the operator associated with the IFZS (w_2, Φ_1) . Then

$$\begin{aligned} d_{Con(\mathcal{F}^*(X))}(T_1, T_2) &= \sup_{u \in \mathcal{F}^*(X)} d_\infty(T_1 u, T_2 u) \\ &= \sup_{u \in \mathcal{F}^*(X)} \sup_{0 \leq \alpha \leq 1} h([T_1 u]^\alpha, [T_2 u]^\alpha) \\ &= \sup_{u \in \mathcal{F}^*(X)} \sup_{0 \leq \alpha \leq 1} h\left(\bigcup_{1 \leq i \leq N} w_{1i}([\phi_{1i} \circ u]^\alpha), \bigcup_{1 \leq j \leq N} w_{2j}([\phi_{1j} \circ u]^\alpha)\right) \\ &\leq \sup_{u \in \mathcal{F}^*(X)} \sup_{0 \leq \alpha \leq 1} \max_{1 \leq i \leq N} h(w_{1i}([\phi_{1i} \circ u]^\alpha), w_{2i}([\phi_{1i} \circ u]^\alpha)) \\ &\leq d_w^N(w_1, w_2) \\ &< \epsilon(1 - s_1). \end{aligned}$$

Note that we have used the following property of the Hausdorff metric:

$$h\left(\bigcup_{1 \leq i \leq N} A_i, \bigcup_{1 \leq j \leq N} B_j\right) \leq \max_{1 \leq i \leq N} h(A_i, B_i). \tag{3.11}$$

From Proposition 3.1, the desired result then follows.

In other words, with the grey-level maps fixed, the T_k operators are continuous with respect to the IFS maps w_{ki} in the $d_{Con(\mathcal{F}^*(X))}$ metric and continuity of the attractors is implied through Proposition 3.1. However, we now show that the T_k are *not* necessarily continuous with respect to the grey level maps ϕ_{ki} . We focus on particularly important subsets of $Con(\mathcal{F}^*(X))$, namely, those sections of \mathcal{T}_N obtained by fixing the IFS maps w_k and varying the grey level maps ϕ_{ki} one at a time.

DEFINITION 3.1. Let (w, Φ) be an IFZS with corresponding operator $T \in \mathcal{T}_N$. For $i^* \in \{1, 2, \dots, N\}$ define $E_{i^*}(T) \subset \mathcal{T}_N$, the “ ϕ_{i^*} -axis” of \mathcal{T}_N at T , to be the section of \mathcal{T}_N obtained by fixing the IFS maps $w_i, i \in \{1, 2, \dots, N\}$, and the grey-level maps $\phi_i, i \in \{1, 2, \dots, N\} \setminus \{i^*\}$, of T and varying $\phi_{i^*} \in \mathcal{M}^+([0, 1])$.

THEOREM 3.1. Let $T_1 \in \mathcal{T}_N$, with corresponding IFZS (w, Φ_1) , where the IFS maps satisfy the following condition: for some $i^* \in \{1, 2, \dots, N\}$,

$$d_{i^*} \equiv \sup_{x \in w_{i^*}(X)} \inf_{y \in w_{i^*}(X) \cup \bigcup_{j \neq i^*} (w_j(X) \cap w_j(X))} d(x, y) > 0. \tag{3.12}$$

(In other words, $w_{i^*}(X)$ differs from $\bigcup_{j \neq i^*} w_j(X)$ by at least one point.) Also let $T_2 \in E_{i^*}(T_1)$, with corresponding IFZS (w, Φ_2) such that $\phi_{1i^*} \neq \phi_{2i^*}$ and $\phi_{1j} = \phi_{2j}, j \neq i^*$. Then

$$d_{Con(\mathcal{F}^*(X))}(T_1, T_2) \geq d_{i^*}, \tag{3.13}$$

independently of the ϕ_{ki} maps.

PROOF. Without loss of generality, let $i^* = 1$. Using the notation of [3], define

$$\beta_1(\alpha) \equiv \inf \{t \in [0, 1] : \phi_{1,1} \geq \alpha\}, \quad \beta_2(\alpha) \equiv \inf \{t \in [0, 1] : \phi_{2,1} \geq \alpha\}. \quad (3.14)$$

Assume, again without loss of generality, that $\phi_{1,1}(t^*) < \phi_{2,1}(t^*)$ for a certain $t^* \in (0, 1]$. Since the functions ϕ_{ki} are nondecreasing and right continuous, there exists an $\alpha^* \in [0, 1]$ such that $\beta_2(\alpha^*) < \beta_1(\alpha^*)$; in fact, any $\alpha^* \in (\phi_{1,1}(t^*), \phi_{2,1}(t^*))$ will work since $\beta_2(\alpha^*) < t^* \leq \beta_1(\alpha^*)$.

Let $x_0 \in X$ be fixed so that $y_0 = w_1(x_0)$ satisfies

$$\sup_{y \in w_1(X) \setminus \bigcup_{2 \leq j \leq N} (w_1(X) \cap w_j(X))} d(y_0, y) = d_1, \quad (3.15)$$

and let $u_0 \in \mathcal{F}^*(X)$ be defined as

$$u_0(x) = \begin{cases} 1, & x = x_0, \\ \tilde{\beta}, & x \neq x_0, \end{cases} \quad (3.16)$$

where $\beta_2(\alpha^*) < \tilde{\beta} < \beta_1(\alpha^*)$. It follows that

$$\begin{aligned} [\phi_{1,1} \circ u_0]^{\alpha^*} &= [u_0]^{\beta_1(\alpha^*)} = \{x_0\}, \\ [\phi_{2,1} \circ u_0]^{\alpha^*} &= [u_0]^{\beta_2(\alpha^*)} = X, \\ [\phi_{1i} \circ u_0]^{\alpha^*} &= [\phi_{2i} \circ u_0]^{\alpha^*}, \quad i = 2, 3, \dots, N. \end{aligned} \quad (3.17)$$

Then

$$\begin{aligned} & d_{Con(\mathcal{F}^*(X))}(T_1, T_2) \\ &= \sup_{u \in \mathcal{F}^*(X)} d_\infty(T_1 u, T_2 u) \\ &\geq d_\infty(T_1 u_0, T_2 u_0) \\ &= \sup_{0 \leq \alpha \leq 1} h([T_1 u_0]^\alpha, [T_2 u_0]^\alpha) \\ &\geq h([T_1 u_0]^{\alpha^*}, [T_2 u_0]^{\alpha^*}) \\ &= h\left(\bigcup_{1 \leq i \leq N} w_i([\phi_{1i} \circ u_0]^{\alpha^*}), \bigcup_{1 \leq j \leq N} w_j([\phi_{2j} \circ u_0]^{\alpha^*})\right) \\ &= h\left(w_1(\{x_0\}) \cup \left(\bigcup_{2 \leq i \leq N} w_i([\phi_{1i} \circ u_0]^{\alpha^*})\right), w_1(X) \cup \left(\bigcup_{2 \leq j \leq N} w_j([\phi_{1j} \circ u_0]^{\alpha^*})\right)\right) \\ &= \sup_{x \in w_1(X) \cup \bigcup_{2 \leq i \leq N} w_i([\phi_{1i} \circ u_0]^{\alpha^*})} \inf_{y \in w_1(\{x_0\}) \cup \bigcup_{2 \leq j \leq N} w_j([\phi_{1j} \circ u_0]^{\alpha^*})} d(x, y) \\ &= \sup_{x \in w_1(X)} \inf_{y \in w_1(\{x_0\}) \cup \bigcup_{2 \leq j \leq N} w_j([\phi_{1j} \circ u_0]^{\alpha^*})} d(x, y) \\ &\geq \sup_{x \in w_1(X) \setminus \bigcup_{2 \leq j \leq N} (w_1(X) \cap w_j(X))} \inf_{y \in w_1(\{x_0\}) \cup \bigcup_{2 \leq j \leq N} w_j(X)} d(x, y) \\ &= d_1, \end{aligned}$$

and the theorem is proved.

This theorem shows that, in general, the T operator is not continuous with respect to the grey level maps ϕ_i . (The condition in (3.12) is not too restrictive. In fact, it omits only the rather special cases where each set $w_i(X)$ is contained in the union of all the other $w_j(X)$.) Unlike Proposition 3.4, however, we *cannot* guarantee, using the $d_{Con(\mathcal{F}^*(X))}$ metric, that the d_∞ distance between two attractors u_1 and u_2 will be arbitrarily small by making the differences between the associated grey level vectors Φ_1 and Φ_2 sufficiently small.

As a final comment in this section, note that the lack of continuity does not contradict the property that the set of IFZS attractors is dense in the space $(\mathcal{F}^*(X), d_\infty)$ [5]. In this paper we are concerned only with IFZS having a fixed number, N , of IFS and grey level maps.

4. Continuity of u with respect to w and Φ

In the previous section, the inequality in (3.3) shows that when $d_{Con(Y)}$ is the “natural” metric in (3.10), then continuity of IFZS attractors u on the space $Y = \mathcal{F}^*(X)$ is uniform on the set of all contraction maps in $(Con(Y), d_{Con(Y)})$. However, this uniform continuity is too restrictive: Theorem 3.1 shows that given an $\epsilon > 0$ sufficiently small, the condition $d_{Con(\mathcal{F}^*(X))}(T_1, T_2) < \epsilon$ implies that $d_\infty(u_1, u_2) = 0$. In this section, we show that a more practical continuity *at each specific attractor* $u \in \mathcal{F}^*(X)$ can be established, permitting a continuous variation of u as both the w and the Φ components are perturbed. However, the perturbation of the Φ must be performed in a controlled manner: along the ϕ_i -axes of \mathcal{T}_N at the T_k (Definition 3.1) together with restrictions as given in Theorem 4.2 below. Distance functions for the IFZS components will be defined as follows:

- (a) For the IFS map vectors, we use the metric $d_w^N(w_1, w_2)$ defined in (3.5).
- (b) For the grey level maps, first define the following distance function between two maps $\phi, \bar{\phi} \in \mathcal{M}^+([0, 1])$ with respect to an element $u \in \mathcal{F}^*(X)$,

$$d_\phi(\phi, \bar{\phi}; u) \equiv \sup_{0 \leq \alpha \leq 1} h([\phi \circ u]^\alpha, [\bar{\phi} \circ u]^\alpha), \tag{4.1}$$

and the corresponding pseudometric for two grey level map vectors with respect to $u \in \mathcal{F}^*(X)$,

$$d_\phi^N(\Phi_1, \Phi_2; u) \equiv \max_{1 \leq i \leq N} d_\phi(\phi_{1i}, \phi_{2i}; u). \tag{4.2}$$

THEOREM 4.1. *Let (w_k, Φ_k) , $k = 1, 2$ be two IFZS with corresponding contractive operators $T_k \in \mathcal{T}_N$ and attractors u_k , i.e., $T_k u_k = u_k$, $k = 1, 2$. Furthermore assume that $\phi_{1i}(1) = \phi_{2i}(1)$, $i = 1, 2, \dots, N$. Let s_1 be the contractivity factor of w_1 . Then*

$$d_\infty(u_1, u_2) \leq \frac{s_1}{1 - s_1} d_\phi^N(\Phi_1, \Phi_2; u_1) + \frac{1}{1 - s_1} d_w^N(w_1, w_2). \tag{4.3}$$

PROOF. Let $T_3 \in \mathcal{T}_N$ be the contractive operator corresponding to the “mixed” IFZS (\mathbf{w}_1, Φ_2) . Then

$$\begin{aligned} d_\infty(u_1, u_2) &= d_\infty(T_1u_1, T_2u_2) \\ &\leq d_\infty(T_1u_1, T_3u_1) + d_\infty(T_3u_1, T_2u_2) \\ &\leq d_\infty(T_1u_1, T_3u_1) + d_\infty(T_3u_1, T_3u_2) + d_\infty(T_3u_2, T_2u_2) \\ &\leq d_\infty(T_1u_1, T_3u_1) + s_1d_\infty(u_1, u_2) + d_\infty(T_3u_2, T_2u_2). \end{aligned} \tag{4.4}$$

Hence

$$(1 - s_1)d_\infty(u_1, u_2) \leq d_\infty(T_1u_1, T_3u_1) + d_\infty(T_3u_2, T_2u_2). \tag{4.5}$$

We now examine the two distances on the right side of (4.5). Since

$$\begin{aligned} h\left(\bigcup_{i=1}^N w_{1i}([\phi_{1i} \circ u_1]^\alpha), \bigcup_{j=1}^N w_{1j}([\phi_{2j} \circ u_1]^\alpha)\right) \\ \leq \max_{1 \leq i \leq N} h(w_{1i}([\phi_{1i} \circ u_1]^\alpha), w_{1i}([\phi_{2i} \circ u_1]^\alpha)) \\ \leq s_1 \max_{1 \leq i \leq N} h([\phi_{1i} \circ u_1]^\alpha, [\phi_{2i} \circ u_1]^\alpha), \end{aligned} \tag{4.6}$$

it follows that

$$d_\infty(T_1u_1, T_3u_1) \leq s_1d_\phi^N(\Phi_1, \Phi_2; u_1). \tag{4.7}$$

(Note that the assumption $\phi_{1i}(1) = \phi_{2i}(1), i = 1, 2, \dots, N$ is necessary to guarantee that the level sets $[\phi_{1i} \circ u_1]^\alpha$ and $[\phi_{2i} \circ u_1]^\alpha$ in (4.6) exist for the same range of α -values.)

Moreover,

$$\begin{aligned} h\left(\bigcup_{i=1}^N w_{1i}([\phi_{2i} \circ u_2]^\alpha), \bigcup_{j=1}^N w_{2j}([\phi_{2j} \circ u_2]^\alpha)\right) \\ \leq \max_{1 \leq i \leq N} h(w_{1i}([\phi_{2i} \circ u_2]^\alpha), w_{2i}([\phi_{2i} \circ u_2]^\alpha)) \\ \leq d_w^N(\mathbf{w}_1, \mathbf{w}_2). \end{aligned} \tag{4.8}$$

Thus, (4.5), (4.7) and (4.8) imply that

$$(1 - s_1)d_\infty(u_1, u_2) \leq s_1d_\phi^N(\Phi_1, \Phi_2; u_1) + d_w^N(\mathbf{w}_1, \mathbf{w}_2) \tag{4.9}$$

and the desired result follows.

The following lemmas will be useful for the proof of Theorem 4.2, the major result of this section.

LEMMA 4.1. *Given $A_i, B_i, C_i \in \mathcal{H}(X), i = 1, 2$, such that*

$$A_1 \subseteq B_1 \subseteq C_1, \quad A_2 \subseteq B_2 \subseteq C_2, \tag{4.10}$$

then

$$h(B_1, B_2) \leq \max\{h(A_2, C_1), h(A_1, C_2)\}. \tag{4.11}$$

PROOF.

$$\begin{aligned}
 D(B_1, B_2) &= \sup_{x \in B_1} \inf_{y \in B_2} d(x, y) \\
 &\leq \sup_{x \in C_1} \inf_{y \in A_2} d(x, y) \\
 &= D(C_1, A_2) \\
 &\leq h(C_1, A_2).
 \end{aligned}$$

Similarly, $D(B_2, B_1) \leq D(C_2, A_1) \leq h(C_2, A_1)$. The result in (4.12) then follows.

LEMMA 4.2. *If $u \in \mathcal{F}^*(X)$, then*

- (i) $[u]^\alpha$ is closed for all $\alpha \in [0, 1]$,
- (ii) If $\beta \uparrow \beta^*$, then

$$[u]^\beta \rightarrow \bigcap_{\beta < \beta^*} [u]^\beta = [u]^{\beta^*} \tag{4.12}$$

in Hausdorff metric.

Property (i) follows from the upper semicontinuity of $u \in \mathcal{F}^*(X)$. Property (ii) was proved in [4].

The following lemma establishes the existence of ϕ maps in a restricted neighbourhood of a given $\bar{\phi} \in \mathcal{M}^+([0, 1])$ in terms of the distance function $d_\phi(\phi, \bar{\phi}; u)$ introduced in (4.2) and defined with respect to a given $u \in \mathcal{F}^*(X)$. These ϕ maps are obtained by suitably perturbing $\bar{\phi}$, which is nonconstant on $[0, 1]$. (Recall the definition of $\mathcal{M}^+([0, 1])$.)

LEMMA 4.3. *Let $u \in \mathcal{F}^*(X)$ and $\bar{\phi} \in \mathcal{M}^+([0, 1])$. Assume that for some $\alpha^* > 0$ and some $t^* \in (0, 1)$, $\bar{\phi}(t^*) = \alpha^*$. Let*

$$\bar{\beta}^* := \bar{\beta}(\alpha^*) \equiv \inf \{t \in [0, 1] : \bar{\phi}(t) \geq \alpha^*\}.$$

Then for every $\epsilon > 0$, there exists an $\eta^, 0 < \eta^* < \bar{\beta}^*$, such that for all $\phi \in \mathcal{M}^+([0, 1])$ satisfying the following properties:*

- (a) $\phi(t) = \bar{\phi}(t)$ for $0 \leq t < \bar{\beta}^* - \eta^*$ and $\bar{\beta}^* \leq t \leq 1$,
- (b) $\phi(t) \geq \bar{\phi}(t)$ for $\bar{\beta}^* - \eta^* \leq t \leq \bar{\beta}^*$,

we have

$$d_\phi(\bar{\phi}, \phi; u) < \epsilon. \tag{4.13}$$

PROOF. Since $\bar{\phi}$ is right continuous and $\bar{\phi}(0) = 0$, it follows that $\bar{\beta}^* > 0$. Property (ii) of Lemma 4.2 implies that for every $\epsilon > 0$ there exists an $\eta^* > 0$, $\bar{\beta}^* - \eta^* > 0$ such that

$$h([u]^\beta, [u]^{\bar{\beta}^*}) < \epsilon, \quad \forall \beta \in [\bar{\beta}^* - \eta^*, \bar{\beta}^*].$$

Now let $\phi \in \mathcal{M}^+([0, 1]) \setminus \{\bar{\phi}\}$ be a function that satisfies (a) and (b) above. If $0 \leq \alpha_1 \leq \alpha \leq \alpha_2 \leq 1$, then for any $\phi, \bar{\phi} \in \mathcal{M}^+([0, 1])$, with

$$\beta(\alpha) \equiv \inf \{t \in [0, 1] : \phi(t) > \alpha\}$$

and

$$\bar{\beta}(\alpha) \equiv \inf \{t \in [0, 1] : \bar{\phi}(t) > \alpha\},$$

we have the following relations:

$$\begin{aligned} [\bar{\phi} \circ u]^{\alpha_2} \subseteq [\bar{\phi} \circ u]^\alpha \subseteq [\bar{\phi} \circ u]^{\alpha_1}, \quad \text{i.e.,} \quad [u]^{\bar{\beta}(\alpha_2)} \subseteq [u]^{\bar{\beta}(\alpha)} \subseteq [u]^{\bar{\beta}(\alpha_1)}, \\ [\phi \circ u]^{\alpha_2} \subseteq [\phi \circ u]^\alpha \subseteq [\phi \circ u]^{\alpha_1}, \quad \text{i.e.,} \quad [u]^{\beta(\alpha_2)} \subseteq [u]^{\beta(\alpha)} \subseteq [u]^{\beta(\alpha_1)}. \end{aligned}$$

If, in particular,

$$\alpha_2 = \alpha^*, \quad \alpha_1 = \phi(\bar{\beta}^* - \eta^*),$$

and ϕ satisfies the conditions (a)–(c) above, then

$$\begin{aligned} A_1 &\equiv [\bar{\phi} \circ u]^{\alpha_2} = [\phi \circ u]^{\alpha_2} \equiv A_2, \\ C_1 &\equiv [\bar{\phi} \circ u]^{\alpha_1} \subseteq [\phi \circ u]^{\alpha_1} \equiv C_2. \end{aligned}$$

Letting $B_1 = [\bar{\phi} \circ u]^\alpha$ and $B_2 = [\phi \circ u]^\alpha$, we have, from Lemma 4.1 and the fact that $C_1 \subseteq C_2$,

$$h([\bar{\phi} \circ u]^\alpha, [\phi \circ u]^\alpha) \leq h([\bar{\phi} \circ u]^{\alpha_2}, [\phi \circ u]^{\alpha_1}) = h([u]^{\bar{\beta}^*}, [u]^{\bar{\beta}^* - \eta^*}) < \epsilon. \tag{4.14}$$

On the other hand, by construction, $h([\bar{\phi} \circ u]^\alpha, [\phi \circ u]^\alpha) = 0$ for $\alpha \in [0, \bar{\phi}(\bar{\beta}^* - \eta^*) \cup (\bar{\phi}(\bar{\beta}^*), 1]$. (Note that $\bar{\phi}(\bar{\beta}^* - \eta^*) < \bar{\phi}(\bar{\beta}^*)$, from the definition of $\bar{\beta}^*$ and the right continuity of $\bar{\phi}$.) Hence

$$d_\phi(\bar{\phi}, \phi; u) = \sup_{0 \leq \alpha \leq 1} h([\bar{\phi} \circ u]^\alpha, [\phi \circ u]^\alpha) < \epsilon$$

and the proof is complete.

We now come to the major result regarding continuity of IFZS attractor functions.

THEOREM 4.2. *Given an IFZS (w_1, Φ_1) with corresponding contractive operator $T_1 \in \mathcal{T}_N$ and fixed point $u_1 \in \mathcal{F}^*(X)$, then for every $\epsilon > 0$ and every $\epsilon_1, \epsilon_2 > 0$ such that (see Theorem 4.1)*

$$\frac{s_1}{1 - s_1} \epsilon_2 + \frac{1}{1 - s_1} \epsilon_1 < \epsilon, \tag{4.15}$$

there exist the following:

(1) an IFS $w_2 \neq w_1$ for which

$$d_w^N(w_1, w_2) < \epsilon_1, \tag{4.16}$$

(2) a set of grey level maps $\Phi_2 \neq \Phi_1$ such that

$$d_\phi^N(\Phi_1, \Phi_2; u_1) < \epsilon_2. \tag{4.17}$$

As such, the operator $T_2 \in \mathcal{T}_N$ corresponding to (w_2, Φ_2) has a fixed point $u_2 \in \mathcal{F}^*(X)$, such that

$$d_\infty(u_1, u_2) < \epsilon. \tag{4.18}$$

Moreover, for every $\epsilon > 0$, there exist $\delta_1 > 0$ ($\delta_1 = \epsilon_1(1 - s_1)^{-1}$) and $\delta_2 > 0$ ($\delta_2 = s_1\epsilon_2(1 - s_1)^{-1}$) such that $d_w^N(w_1, w_2) < \delta_1$ and $d_\phi^N(\Phi_1, \Phi_2; u_1) < \delta_2$ imply (4.18).

PROOF. Regarding (1), the existence of an IFS w_2 satisfying the inequality in (1) is straightforward, see [1]. Regarding (2), recall that for each $i^* \in \{1, 2, \dots, N\}$ there exists at least one value $t^* \in [0, 1]$, such that $\phi_{1,i^*}(t^*) > 0$. Then, from Lemma 4.3, the following set Φ_2 of grey level maps satisfies condition (2):

$$\begin{aligned} \phi_{2,i} &= \phi_{1,i}, & i \in \{1, 2, \dots, N\} \setminus \{i^*\}, \\ \phi_{2,i^*}(t) &= \phi_{1,i^*}(t), & t \in [0, 1] \setminus [t^* - \eta^*, t^*), \\ \phi_{2,i^*}(t) &\geq \phi_{1,i^*}(t), & t \in [t^* - \eta^*, t^*), \end{aligned}$$

where η^* is such that (see Lemma 4.3)

$$d_\phi(\phi_{1,i^*}, \phi_{2,i^*}; u_1) < \epsilon_2. \tag{4.19}$$

Since

$$d_\phi^N(\Phi_1, \Phi_2; u_1) = \sup_{1 \leq i \leq N} d(\phi_{2,i}, \phi_{1,i}; u_1) = d(\phi_{2,i^*}, \phi_{1,i^*}; u_1), \tag{4.20}$$

(4.18) follows from Theorem 4.1 and the proof is complete.

Note that the conditions $\phi_{1i}(1) = \phi_{2i}(1)$, $i = 1, 2, \dots, N$, assumed to hold in Theorem 4.1, are essential for the existence of a Φ_2 such that (4.17) holds for a given $\epsilon_2 > 0$ and u_1 . The last part of Theorem 4.2 (continuity of u at u_1) is a direct consequence of Theorem 4.1.

5. Discussion

For practical purposes, e.g., animation or the inverse problem of fractal construction, it is desirable that an IFZS attractor $u_k \in \mathcal{F}^*(X)$ may be varied in a continuous

fashion by varying the IFS maps w_{ki} and the grey level maps ϕ_{ki} . The IFZS attractors u_k are trivially continuous with respect to the operators T_k in the $d_{Con}_T^N(\mathcal{F}^*(X))$ metric. However, we have shown that the T_k may be discontinuous along the sections $E_i(T_k)$ obtained by varying the ϕ_{ki} one at a time. These sections are naturally important in the “fine-tuning” of grey level distributions on u_k . (We should mention that our use of the $E_i(T_k)$ sections in the above analysis is in no way restrictive: a standard triangular inequality argument clearly shows that a continuous variation in the grey level vector $\Phi = (\phi_1, \phi_2, \dots, \phi_N)$ with respect to the d_Φ^N metric can be achieved by moving along the $E_i(T_k)$ sections, i.e., varying the ϕ_{ki} one map at a time.)

In order to remedy this situation, a “local” metric for the grey level maps with respect to an element $u \in \mathcal{F}^*(X)$ was constructed. The continuity of attractors u_k with respect to the ϕ_{ki} maps was then established, subject to some restrictions on their variations (Section 4). These restrictions are not too serious: the perturbation method outlined in Theorem 4.2 still allows freedom in the amount of variation in the ϕ_{ki} . In fact, any additional regularity in the ϕ_{ki} maps, e.g., bijectivity, will allow even more freedom in the variation. However, the restriction that the values $\phi_{ki}(1)$ be fixed is essential. It can easily be shown (along the lines of Theorem 3.1) that any δ -variation in this value produces an attractor \bar{u}_k whose d_∞ distance from u_k is bounded away from zero independently of δ .

An even greater flexibility in controlling the u_k attractors could be accomplished if the grey level maps ϕ_{ki} were functions of the base space variable $x \in X$, i.e., $\phi : X \times [0, 1] \rightarrow [0, 1]$. As well, it may be advantageous to consider operators other than the T operator of (2.7) and, consequently, other metrics on $\mathcal{F}^*(X)$ (in order to preserve the contractivity of T or to possibly guarantee uniform continuity of the attractors). Recently, in fact, we have succeeded in modifying the original IFZS method [8]. One of the modifications involves a new distance function which, in special cases, becomes the $L^1(X)$ distance with respect to a measure μ on X . The attractors of this new IFS-type method on $L^1(X)$ are then trivially continuous with respect to the grey-level maps ϕ_i . This new approach has been very effective in solving the inverse problem of function approximation and image representation.

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Appendix A: IFS with probabilities and invariant measures

An N -map contractive IFS on X with probabilities (IFSP), (\mathbf{w}, \mathbf{p}) , is a set of N contraction maps $\mathbf{w} = (w_1, w_2, \dots, w_N)$, $w_i \in \text{Con}(X)$ with associated probabilities $\mathbf{p} = (p_1, p_2, \dots, p_N)$, $p_i \geq 0$, $\sum_{i=1}^N p_i = 1$. The uniqueness of the attractor A defined by the set \mathbf{w} was discussed in Section 2.

Now let $\mathcal{M}(X)$ denote the set of probability measures on $\mathcal{B}(X)$, the σ -algebra of Borel subsets of X . Consider the following metric on $\mathcal{M}(X)$,

$$d_H(\mu, \nu) = \sup_{f \in \text{Lip}(X)} \left[\int f d\mu - \int f d\nu \right] \quad \mu, \nu \in \mathcal{M}(X), \tag{A.1}$$

where

$$\text{Lip}(X) = \{f : X \rightarrow \mathbf{R}, |f(x) - f(y)| \leq d(x, y), x, y \in X\}.$$

The d_H metric has often been called the Hutchinson metric due to its use in [9]. Note that $(\mathcal{M}(X), d_H)$ is a complete metric space.

Given an IFSP (\mathbf{w}, \mathbf{p}) , define a ‘‘Markov operator’’ $M : \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ as follows: for $\nu \in \mathcal{M}(X)$, let

$$M\nu = \sum_{i=1}^N p_i \nu \circ w_i^{-1}.$$

Then there exists a unique measure $\mu \in \mathcal{M}(X)$, the *invariant measure* of the IFSP (\mathbf{w}, \mathbf{p}) , for which $M\mu = \mu$. M is a contraction mapping on $(\mathcal{M}(X), d_H)$, i.e., for $\mu, \nu \in \mathcal{M}(X)$ [9],

$$d_H(M\mu, M\nu) < s d_H(\mu, \nu),$$

where s is the contractivity factor of \mathbf{w} .

EXAMPLES. Return to the 2-map IFS on $[0, 1]$ given earlier in Section 2. (i) When $s = 1/2$ and $p_1 = p_2 = 1/2$, then μ is Lebesgue measure on $[0, 1]$. (ii) When $s = 1/3$ and $p_1 = p_2 = 1/2$, then μ is the uniform Cantor-Lebesgue measure on the ternary Cantor set on $[0, 1]$.

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