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The Algebraic de Rham Cohomology of Representation Varieties

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Abstract. The $\mathrm{SL}(2,\mathbb{C})$-representation varieties of punctured surfaces form natural families parameterized by monodromies at the punctures. In this paper, we compute the loci where these varieties are singular for the cases of one-holed and two-holed tori and the four-holed sphere. We then compute the de Rham cohomologies of these varieties of the one-holed torus and the four-holed sphere when the varieties are smooth via the Grothendieck theorem. Furthermore, we produce the explicit Gauß-Manin connection on the natural family of the smooth $\mathrm{SL}(2,\mathbb{C})$-representation varieties of the one-holed torus.

1 Prelude

Let $\Sigma = \Sigma_{g,m}$ be a compact oriented surface of genus g with m punctures $\mathcal{C} = \{\mathcal{C}_1,\dots,\mathcal{C}_m\}$. Denote by $\pi = \pi_1(\Sigma)$ its fundamental group. Let G be a reductive complex algebraic group and $\mathrm{Hom}(\pi,G)$ the space of homomorphisms (representations) from π to G. $\mathrm{Hom}(\pi,G)$ inherits a variety structure from G, and G acts on $\mathrm{Hom}(\pi,G)$ by equivalence of representations (conjugation)

$$G \times \mathrm{Hom}(\pi,G) \longrightarrow \mathrm{Hom}(\pi,G), \quad (g,\rho) \longmapsto g\rho g^{-1}.$$

Denote by

$$\mathcal{M}(G) = \mathrm{Hom}(\pi,G)/G$$

the categorical quotient of equivalent representations. Fix a conjugacy class $C_i \subset G$ for each puncture $\mathcal{C}_i$ and let $C = \{C_1,\dots,C_m\}$. Let

$$\mathrm{Hom}_C(\pi,G) = \left\{\rho \in \mathrm{Hom}(\pi,G) : \rho(\mathcal{C}_i) \in C_i,\ \text{for } 1 \le i \le m\right\}.$$

The G-action preserves $\mathrm{Hom}_C(\pi,G)$, and the representation variety is the categorical quotient

$$\mathcal{M}_C(G) = \mathrm{Hom}_C(\pi,G)/G.$$

The representation variety $\mathcal{M}_C(G)$ is of great interest because it is the (coarse) moduli space of integrable G-connections on $\Sigma_{g,m}$; see [4, 5]. Fix $G = \mathrm{SL}(2,\mathbb{C})$ and let

$$\mathcal{M} = \mathcal{M}(\mathrm{SL}(2,\mathbb{C})), \quad \mathcal{M}_C = \mathcal{M}_C(\mathrm{SL}(2,\mathbb{C})).$$

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As the conjugacy classes in C vary in G, the moduli spaces $\mathcal{M}_C$ vary; that is, the varieties $\mathcal{M}_C$ form a family parameterized by C. In this paper, we first identify the singular loci for the cases of $\Sigma_{1,1}$, $\Sigma_{1,2}$, and $\Sigma_{0,4}$.

In the cases of $\Sigma_{1,1}$ and $\Sigma_{0,4}$, the $\mathcal{M}_C$'s are 2-dimensional. In these two cases, their homologies have been calculated via topological methods [9]. A remarkable theorem of Grothendieck [11] states that the hypercohomology of the algebraic de Rham complex of a smooth variety computes its smooth de Rham cohomology. This provides an algebraic method for computing $\mathrm{H}^*_{dR}(\mathcal{M}_C)$. We then carry out the computations for the representation varieties for these two cases. There are pure algorithmic approaches to these problems; however, these general methods tend to overwhelm computers; see [18, 19, 22]. We compute our results directly while taking advantage of the computer resources available, especially *Macaulay2* [10].

These families have natural integrable connections, namely, the Gauß-Manin connections [15]. We compute this connection explicitly for the family of representation varieties of $\Sigma_{1,1}$.

2 Generalities

2.1 Smooth Varieties and their Cohomologies

Let X be a smooth algebraic variety over $\mathbb{C}$ with structure sheaf $\mathcal{O}_X$. Denote by $(\Omega^\bullet_X, d)$ the algebraic de Rham complex of X:

$$(\Omega^\bullet_X, d)\colon \Omega^0_X \xrightarrow{d_0} \Omega^1_X \xrightarrow{d_1} \cdots$$

We drop the subscript on d when the context is clear. A remarkable theorem of Grothendieck [11] states the following.

Theorem 2.1 (Grothendieck) *The (hyper-)cohomologies of $(\Omega^\bullet_X, d)$ coincide with the smooth de Rham cohomologies of X.*

2.2 Relative de Rham Complex and Cohomologies

We begin by briefly introducing the algebraic de Rham (hyper-)cohomology. The standard references for homological algebra and hypercohomology in particular are [12], [8, §III.7.14], and [23, §5.7].

Let $Y \to \mathbf{Spec}(\mathbb{C})$ be a smooth $\mathbb{C}$-variety and $f\colon X \to Y$ a smooth Y-variety. Denote by $f^*\colon \mathcal{O}_Y \to \mathcal{O}_X$ the corresponding morphism between the structure sheaves. From these come the three de Rham complexes [13, §2.8]]:

$$(\Omega^\bullet_X, d), \quad (\Omega^\bullet_Y, d), \quad (\Omega^\bullet_{X/Y}, d).$$

Each complex is associated with their respective (hyper-)cohomologies. The relative de Rham cohomologies associated with $(\Omega^\bullet_{X/Y}, d)$ are cohomologies of $\mathcal{O}_Y$-sheaves

$$\mathcal{H}^i := \mathcal{H}^i(X) := R^i f_*(\Omega^\bullet_{X/Y}).$$

Let $\Phi\colon S \to Y$ be a flat morphism. Then by base extension, we obtain the S-scheme $\Phi^*(X)$ and its associated de Rham complex.

Proposition 2.2 $\Phi^*(\mathcal{H}^i) \xrightarrow{\cong} R^i(f \circ \Phi)_*(\Omega_{\Phi^*(X)/S})$.

Proof See [13, Proposition 5.2]. ∎

In particular, this is true for localization at a closed point $P \in Y$, *i.e.*, for $\Phi\colon Y_P \to Y$. Let $\mathbb{C} = k(P)$ be the residue field at the closed point P and

$$\phi\colon \mathbf{Spec}(k(P)) \longrightarrow Y.$$

Then we obtain the S-scheme $U = \phi^*(X)$ by base extension and the associated de Rham complex $(\Omega_U^\bullet, d)$. Denote by $\mathrm{H}^\bullet$ the de Rham cohomologies of U.

2.3 The Gauß–Manin Connection

Assume f to be smooth. Then there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X \otimes_{f^*} \Omega_Y^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X/Y}^1 \longrightarrow 0.$$

This gives rise to a filtration F on $\Omega_X^\bullet$:

$$F^i = \operatorname{im}(\Omega_X^{\bullet - i} \otimes_{f^*} \Omega_Y^i \xrightarrow{\wedge} \Omega_X^\bullet).$$

The (E_1, d_1) pair of the resulting spectral sequence satisfies $E_1^{p,q} \cong \Omega_Y^p \otimes_{f^*} \mathcal{H}^q$; see [15]. The Gauß–Manin connection [15] on $\mathcal{H}^q$ is the differential $\nabla = d_1^{0,q}$ in the following complex

$$0 \longrightarrow \mathcal{H}^q \xrightarrow{d_1^{0,q}} \Omega_Y^1 \otimes_{f^*} \mathcal{H}^q \xrightarrow{d_1^{1,q}} \Omega_Y^2 \otimes_{f^*} \mathcal{H}^q \longrightarrow \cdots.$$

3 Singular and Smooth Varieties

For the rest of the paper, unless otherwise specified, we assume all varieties are affine over $\mathbb{C}$.

3.1 Rings, Modules, and Affine Varieties

Denote by $\mathbf{x}$ the set $\{x_1, x_2, \dots, x_n\}$ and $\mathcal{O} \cong \mathbb{C}[\mathbf{x}]$, the coordinate ring of $\mathbb{C}^n$ and by $(\Omega^\bullet, d)$ the algebraic de Rham complex over $\mathbf{Spec}(\mathbb{C}[\mathbf{x}])$. Let $a \in \mathbb{Z}_{\geq 0}^n$. We will use the standard notation

$$|a| := \sum_{i=1}^n a_i, \qquad \mathbf{x}^a := \prod_{i=1}^n x_i^{a_i}.$$

Let $N = \{j : 1 \leq j \leq n\}$ be the ordered index set of n elements. For an ordered subset $K \subseteq N$, write dx_K for $\wedge_{j \in K} dx_j$. Then Ω^i is generated as an $\mathcal{O}$-module by $\{dx_K : |K| = i\}$.

Definition 3.1 Let $w \in \Omega^i$.

(i) Denote by ∂_j the differential operator $\frac{\partial}{\partial x_j}$.
(ii) w is a monomial form of degree $|a|$ if $w = \mathbf{x}^a dx_K$.
(iii) $\deg(w)$ denotes the maximum degree of the monomial forms in w.

Let $I = (\phi_i : 1 \leq i \leq k) \subseteq \mathcal{O}$ be the (finitely generated) ideal of definition of U, which is to say that $U = \mathbf{Spec}(\mathcal{O}_U)$ with

$$0 \longrightarrow I \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_U \longrightarrow 0. \tag{3.1}$$

This induces an inclusion $\iota: U \to \mathbb{C}^n$.

Remark 3.2 For any module M, we will always use elements of M to denote elements of quotients of M when contexts are clear.

Remark 3.3 The map

$$P: \Omega^0 \longrightarrow \Omega^n, \quad f \longmapsto f dx_N$$

is an $\mathcal{O}$-module isomorphism.

3.2 Gröbner Basis and Singularity

Let F be a free $\mathcal{O}$-module and let W be a complete order on the monomials of F [§15, [6]]. For any $v, w \in F$, we write $W(v) > W(w)$ if the leading monomial of v is greater than the leading monomial of w according to the order W. A monomial order corresponds to a weight matrix, and we do not distinguish the two [2, §2.4].

Definition 3.4 ([6, §16]) Let $\mathcal{J} \subseteq \mathcal{O}$ be the ideal generated by the $c \times c$ minors of the Jacobian $[\partial_j \phi_i]$, where $1 \leq j \leq n$ and c is the codimension of $I(U)$. The Jacobian ideal of I is $J(I) := I + \mathcal{J} \subseteq \mathcal{O}$.

Proposition 3.5 ([6, §16]) *U is smooth if and only if $J(I) = \mathcal{O}$.*

Hence, one can determine whether U is smooth by computing a Gröbner basis J_G for $J(I)$.

Remark 3.6 Let w be a monomial form. Then either

$$dw = 0 \quad \text{or} \quad \deg(w) = \deg(dw) + 1.$$

Definition 3.7 A monomial order W on Ω^i is degree-modified if $\deg(\eta_2) > \deg(\eta_1)$ implies $W(\eta_2) > W(\eta_1)$.

If W is a monomial order on Ω^0, then W induces a monomial order on Ω^n and vice versa, via P: $W(\mathbf{x}^a) \leftrightarrow W(P(\mathbf{x}^a))$. With respect to P, our order will always satisfy $W(w) > W(v)$ if and only if $W(P(w)) > W(P(v))$ (see Remark 3.3).

4 Computing Algebraic de Rham Cohomology

Designing effective algorithms to compute algebraic de Rham cohomologies for smooth Noetherian varieties is an interesting problem. There is a general algorithm for smooth projective varieties [22]. For the affine case, there is a general algorithm to compute the upper bounds of the Betti numbers [19]. As is typical with these methods, they depend on the non-commutative Gröbner basis computation and the computational complexity is often large. This section describes how to explicitly compute the top algebraic de Rham cohomology ($\mathrm{H}^{\dim(U)}$) of a smooth affine variety corresponding to a principal ideal domain.

We begin by recalling the inclusion morphism $\iota: U \to \mathbb{C}^n$. The first thing to notice is that coherent sheaves on affine varieties are acyclic. This implies that hypercohomology reduces to cohomology of complexes.

Corollary 4.1 *Suppose U is affine. Then the hypercohomologies of the algebraic de Rham complex $(\Omega_U^\bullet, d)$ are*

$$\mathrm{H}^i := \mathrm{H}^i(U) := \frac{\ker(d_i)}{\mathrm{im}(d_{i-1})}.$$

Define $h^i := \dim(\mathrm{H}^i)$.

4.1 Algebraic de Rham Cohomology

Again, let $I \subseteq \mathcal{O}$ be the ideal of definition of U and let $l = \dim(U)$. From sequence (3.1), we obtain an exact sequence of $\mathcal{O}$-modules where Q is the quotient

$$dI \wedge \Omega^{\bullet-1} \longrightarrow \Omega^\bullet \xrightarrow{\text{proj}} Q^\bullet \longrightarrow 0.$$

Pulling back this sequence by ι, we have

$$\Omega_U^\bullet = \iota^*(Q^\bullet) = \mathcal{O}_U \otimes_{\iota^*} Q^\bullet$$

which is both an $\mathcal{O}$- and an $\mathcal{O}_U$-module. We obtain the exact sequence of $\mathcal{O}$-modules

$$dI \wedge \Omega^{\bullet-1} + I\Omega^\bullet \longrightarrow \Omega^\bullet \longrightarrow \Omega_U^\bullet \longrightarrow 0.$$

Definition 4.2 For two i-forms w_1, w_2, write $w_1 \sim w_2$ (cohomologous) if $w_1 = w_2 + du$ for some $(i-1)$-form u.

Remark 4.3 $\mathbb{C}^n$ is (de Rham) acyclic.

Remark 4.4 d is not $\mathcal{O}$-linear, so it is important to distinguish $\mathbb{C}$-linear morphisms and $\mathcal{O}$-morphisms.

4.2 The Top Cohomology

Assume U to be smooth of dimension l for the rest of this section. Then $\Omega_U^{l+1} = 0$ and every form in Ω_U^l is closed. Hence, we have the $\mathbb{C}$-linear commutative diagram in Figure 1 with exact rows. The up arrows are projections. This means that we obtain a rather simple set of generators (compare [19, 20]).

Proposition 4.5 *The cohomology H^l is generated by the monomials*

$$\left\{ p(\mathbf{x}^a dx_K) : a \in \mathbb{Z}_{\geq 0}^n, \quad K \subseteq N, \quad |K| = l \right\}.$$

Lemma 4.6 *For any j,*

$$d(dI \wedge \Omega^{j-1}) = dI \wedge d\Omega^{j-1} \subseteq d(I\Omega^j).$$

$$\begin{tikzcd}
d\Omega_U^{l-1} \arrow[r, hook] & \Omega_U^l \arrow[r] & \mathrm{H}^l \arrow[r] & 0 \\
d\Omega^{l-1} + I\Omega^l + dI \wedge \Omega^{l-1} \arrow[r, hook] \arrow[u] & \Omega^l \arrow[r, "p"] \arrow[u] & \mathrm{H}^l \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Figure 1

Proof The first equality is trivial. Let $du \wedge dw \in dI \wedge d\Omega^{j-1}$. Then $udw \in I\Omega^j$ and

$$d(u \wedge dw) = dh \wedge dw + ud^2(w) = du \wedge dw. \qquad \blacksquare$$

4.3 Principal Ideals and Codimension-one Subvarieties

The smooth representation varieties of $\Sigma_{1,1}$ and $\Sigma_{0,4}$ are two-dimensional, defined by principal ideals $I \subset \mathbb{C}[\mathbf{x}]$ (with $n = 3$) and their top de Rham cohomology is H^{n-1}. For this subsection, we consider the case of $l = n-1$. Since $\mathbb{C}^n$ is acyclic and $\Omega^{n+1} = 0$,

$$d\colon \Omega^{n-1} \longrightarrow \Omega^n$$

is onto, and a form in Ω^{n-1} is closed if and only if it is exact. This implies that a form $w \in \Omega_U^{n-1}$ is exact if and only if $dw = 0$ in Ω^n. This, together with Remark 3.3 and Lemma 4.6, extend Figure 1 to the following commutative diagram with exact rows:

$$\begin{tikzcd}
d\Omega_U^{n-2} \arrow[r, hook] & \Omega_U^{n-1} \arrow[r] & \mathrm{H}^{n-1} \arrow[r] & 0 \\
d\Omega^{n-2} + I\Omega^{n-1} + dI \wedge \Omega^{n-2} \arrow[r, hook] \arrow[u] \arrow[d, "d"] & \Omega^{n-1} \arrow[r, "p"] \arrow[u] \arrow[d, "d"] & \mathrm{H}^{n-1} \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
d(I\Omega^{n-1}) \arrow[r, hook] & \Omega^n \arrow[r, "q"] & \mathrm{H}^{n-1} \arrow[r] & 0.
\end{tikzcd}$$

Figure 2

5 Free Groups and their $\mathrm{SL}(2,\mathbb{C})$-representation Varieties

The fundamental group of the one-holed torus is a free group of two generators, while those of the four-holed sphere and the two-holed torus are free groups of three generators. The traces of elements in $\mathrm{SL}(2,\mathbb{C})$ are $\mathrm{SL}(2,\mathbb{C})$-conjugate invariant. Therefore, the moduli spaces $\mathcal{M}$ and $\mathcal{M}_C$ have trace coordinates. Moreover conjugacy classes of $\mathrm{SL}(2,\mathbb{C})$ are characterized by traces if we remove the identity class $\{\mathbb{I}\}$. In this section, we introduce the trace coordinates for the free groups of two and three generators. For a detailed and excellent exposition, see [7].

5.1 The Free Group on two Generators

Let $\mathbb{F}_2 = \langle F_1, F_2 \rangle$ be the free group on two generators. For $\rho \in \mathrm{Hom}(\mathbb{F}_2, \mathrm{SL}(2,\mathbb{C}))$, let

$$z_1 = \mathrm{tr}\big(\rho(F_1)\big), \quad z_2 = \mathrm{tr}\big(\rho(F_2)\big), \quad z_{12} = \mathrm{tr}\big(\rho(F_1F_2)\big).$$

Then the representation variety is $\mathcal{M} = \mathbb{C}^3$ with $\mathcal{O} = \mathbb{C}[\mathbf{z}]$, where $\mathbf{z} = \{z_1, z_2, z_{12}\}$.

5.2 The Free Group on Three Generators

Let $\mathbb{F}_3 = \langle F_1, F_2, F_3 \rangle$ be the free group of three generators. Let $\rho \in \mathrm{Hom}(\mathbb{F}_3, \mathrm{SL}(2,\mathbb{C}))$. For $1 \le i < j < k \le 3$, let

$$z_i = \mathrm{tr}\big(\rho(F_i)\big), \quad z_{ij} = \mathrm{tr}\big(\rho(F_iF_j)\big), \quad z_{ijk} = \mathrm{tr}\big(\rho(F_iF_jF_k)\big).$$

Then $\mathcal{M}$ is defined by the quotient $\mathcal{O} = \mathbb{C}[\mathbf{z}]/(u)$ (see [7, §5.1]), where

$$\begin{aligned}\mathbf{z} &= \{z_i, z_{ij}, z_{ijk} : 1 \le i < j < k \le 3\},\\ u &= 4 - z_1^2 - z_2^2 - z_3^2 - z_1z_2z_3z_{123} - z_{123}^2 + z_1z_2z_{12}\\ &\quad + z_3z_{123}z_{12} - z_{12}^2 + z_1z_3z_{13} + z_2z_{123}z_{13} - z_{13}^2\\ &\quad + z_2z_3z_{23} + z_1z_{123}z_{23} - z_{12}z_{13}z_{23} - z_{23}^2.\end{aligned}$$

6 The Representation Varieties of the One-holed Torus

This section studies the representation varieties of the one-holed torus with structure group $\mathrm{SL}(2,\mathbb{C})$ and describes the Gauß–Manin connection on a natural family. Let $g = m = 1$. Then the fundamental group π is isomorphic to $\mathbb{F}_2$, the free group on two generators [7]. We begin by renaming the variables in Section 5.1. Let $\mathbf{x} = \{x_1, x_2, x_3\}$ such that

$$x_1 = z_1, \quad x_2 = z_2, \quad x_3 = z_{12}.$$

With respect to the two generators, the boundary element is

$$T = F_1F_2F_1^{-1}F_2^{-1},$$

see [7]. Let $\rho \in \mathrm{Hom}(\pi, \mathrm{SL}(2,\mathbb{C}))$ and $t = \mathrm{tr}(\rho(T))$. Then

$$t = -2 + x_1^2 + x_2^2 - x_1x_2x_3 + x_3^2 \in \mathbb{C}[\mathbf{x}].$$

Following the notation of Section 2, denote by $\mathcal{O} \cong \mathbb{C}[\mathbf{x}]$ the coordinate ring of $\mathcal{M} \cong \mathbb{C}^3$ and by $(\Omega^\bullet, d) = (\Omega^\bullet_{\mathcal{M}}, d)$ its algebraic de Rham complex. We have a morphism

$$f_1 : \mathcal{M} \longrightarrow \mathbf{Spec}(\mathbb{C}[y]) \cong \mathbb{C}$$

induced by the ring homomorphism

$$f_1^* : \mathbb{C}[y] \longrightarrow \mathbb{C}[\mathbf{x}], \quad f_1^*(y) = t.$$

The representation varieties $\mathcal{M}_C$ are the fibres of f_1. For a fixed $b \in \mathbb{C}$, the representation variety $\mathcal{M}_C$ is defined by $I_b = (t - b)$, *i.e.*, $\mathcal{M}_C = \mathbf{Spec}(\mathcal{O}/I_b)$. We rename $\mathcal{M}_C$ as $\mathcal{M}_b$. Let

$$\psi_1(y) = y^2 - 4 \in \mathbb{C}[y].$$

Proposition 6.1 *For a fixed $b \in \mathbb{C}$, $\mathcal{M}_b$ is singular if and only if $\psi_1(b) = 0$.*

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Proof Let $J(I_b)$ be the Jacobian ideal of I_b. For the Gröbner basis computation for $J(I_b)$, we treat b as a variable and use the elimination degree-lexicographic order on $\mathbf{x}$. More specifically, we use the monomial order matrix

$$W = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ on } \{x_3, x_2, x_1, b\}.$$

Denote by J_G the resulting Gröbner basis of $J(I_b)$. The (constant) term in J_G containing only b is $\psi_1(b)$. In other words, $\psi_1(b) \neq 0$ if and only if $J(I_b) = \mathcal{O}$ if and only if $\mathcal{M}_b$ is smooth by Proposition 3.5. ■

6.1 Computing de Rham H^2

We assume here that $\psi_1(b) \neq 0$ unless otherwise specified and use the last row of Figure 2 to compute H^2.

Theorem 6.2 *H^2 has dimension $h^2 = 5$ and a $\mathbb{C}$-basis*

$$B = \{1, x_1, x_2, x_3, x_1^2\} \otimes (x_1 dx_{23}).$$

These are parallel results to [9].

Proof We will use the last row of Figure 2 to show that $q(dB)$ is a basis for H^2. Notice that W is degree-modified on $\mathbf{x}$. Denote also by W the induced weight on Ω^3, according to Definition 3.7.

We first observe that $t \in \mathcal{O}$ is symmetric and

$$dt = (2x_1 - x_2x_3)dx_1 + (2x_2 - x_3x_1)dx_2 + (2x_3 - x_1x_2)dx_3.$$

Let $a = (a_1, a_2, a_3) \in \mathbb{Z}^3_{\geq 0}$.

Lemma 6.3 *Fix i and set $a_i \geq 0$ and $a_j > 0$ for $j \neq i$. Then $\mathbf{x}^a dx_{123} \sim v dx_{123}$ for some $v \in \mathcal{O}$ with* $\deg(v) < |a|$.

Proof Since t is a symmetric polynomial, without the loss of generality, we can assume that $a_1 \geq 0$ and $a_j > 0$ for $j > 1$. Let $s = a - (0, 1, 1)$ and

$$w = d\big((t-b)\mathbf{x}^s dx_{23}\big) = -\big((a_1+1)\mathbf{x}^a + v\big)dx_{123}.$$

Then $w \in d(I_b\Omega^2)$ and $\deg(v) < |a|$. Hence, $\mathbf{x}^a dx_{123} \sim -\frac{v}{a_1+1} dx_{123}$. This also means that $W(v dx_{123}) < W(\mathbf{x}^a dx_{123})$, since W is degree modified. ■

Lemma 6.4 *Fix i and set $a_i > 2$ and $a_j = 0$ for $j \neq i$. Then $\mathbf{x}^a dx_{123} \sim v dx_{123}$ for some $v \in \mathcal{O}$ with $W(v dx_{123}) < W(\mathbf{x}^a dx_{123})$.*

Proof Again since t is a symmetric polynomial, we can assume that $i = 1$. Let $a = (a_1, 0, 0)$ with $a_1 > 2$ and

$$w_1 = d\big((t-b)(2x_1^{a_1-1}dx_{23} + a_1x_1^{a_1-2}x_2 dx_{13})\big) = (2\mathbf{x}^a + v)dx_{123}.$$

Then $w_1 \in d(I_b\Omega^2)$. Moreover, v has the following properties: if $\mathbf{x}^s$ is a monomial in v, then either $|s| < a_1$ or $\mathbf{x}^s$ satisfies the hypothesis of Lemma 6.3 with $|s| = a_1$. In the latter case, $\mathbf{x}^s dx_{123}$ is cohomologous to a 3-form of degree less than a_1 by Lemma 6.3. Hence in both cases, $\mathbf{x}^a dx_{123}$ is cohomologous to a 3-form with a strictly lower weight.

Since t is symmetric, similar arguments take care of the cases of $a = (0, a_2, 0)$ and $a = (0, 0, a_3)$ for $a_2, a_3 > 2$, respectively, by permuting the indices of the items in w_1. More specifically, for $a = (0, a_2, 0)$, permute by $1 \leftrightarrow 2$ in the expression of w_1 to obtain

$$w_2 = d(-(t-b)(2x_2^{a_2-1}dx_{13} + a_2x_2^{a_2-2}x_1dx_{23})).$$

For for $a = (0, 0, a_3)$, permute by $1 \leftrightarrow 3$ in the expression of w_1 to obtain

$$w_3 = d\big((t-b)(2x_3^{a_3-1}dx_{12} + a_3x_3^{a_3-2}x_2dx_{13})\big). \qquad \blacksquare$$

From the above two lemmas, we can assume $a_i \leq 2$ and $a_j = 0$ for $j \neq i$. Suppose $a = (0, 2, 0)$. Set $a_2 = 2$ for the expression of w_2 above; we get $w_2 = 4(x_2^2 - x_1^2)dx_{123} \in d(I_b\Omega^3)$. Hence, $x_2^2dx_{123} \sim x_1^2dx_{123}$.

Suppose $a = (0, 0, 2)$. Set $a_3 = 2$ for the expression of w_3 above and obtain $w_3 = 4(x_3^2 - x_2^2)dx_{123}$. Hence, $x_3^2dx_{123} \sim x_2^2dx_{123}$.

Hence, we conclude that $q(dB)$ generates H^2. By Remark 3.6, one needs only to check a finite (very few) number of at most cubic polynomial 2-forms to verify the linear independence of $q(dB)$. $\blacksquare$

In this relatively simple situation, one can also compute the algebraic de Rham cohomologies for the two singular cases.

Suppose $b = -2$. Then $\mathcal{M}_b$ has one singular point at origin and

$$dI_b \wedge \Omega^2 + I_b\Omega^3 = \{x_1, x_2, x_3\} \otimes dx_{123}.$$

This implies $\Omega^3_{\mathcal{M}_b} = \mathbb{C} \otimes dx_{123}$. Hence,

$$x_1dx_{23} \notin \ker(d_2) \subseteq \Omega^3_{\mathcal{M}_b}.$$

Therefore, $\{x_1, x_2, x_3, x_1^2\} \otimes x_1dx_{23}$ is a basis for H^2 and $h_2 = 4$.

Suppose $b = 2$. Then a Gröbner basis (with monomial order W) for $dI_b \wedge \Omega^2 + I_b\Omega^3$ is

$$\{-4 + x_1^2, x_1x_2 - 2x_3, -4 + x_2^2, -2x_2 + x_1x_3, -2x_1 + x_2x_3, -4 + x_3^2\} \otimes dx_{123}.$$

This implies

$$\Omega^3_{\mathcal{M}_b} = \mathbb{C} \otimes \{1, x_1, x_2, x_3\} \otimes dx_{123}.$$

Hence, no 2-form of degree less than 3 is in $\ker(d_2)$. Hence, $\{x_1^2\} \otimes x_1dx_{23}$ is a basis for H^2 and $h_2 = 1$.

Theorem 2.1 only guarantees that the algebraic de Rham cohomologies agree with the smooth de Rham cohomologies. These two results show that this is also true for these two particular singular spaces (compare [9]).

6.2 What the Computer Says

Modern computer algebra has come of age, and one can obtain much information directly from packages such as *Macaulay2* [10]. Denote by H^i_c the compact support cohomology of $\mathcal{M}_b$ and $h^i_c = \dim(\mathrm{H}^i_c)$.

Theorem 6.5 *If $\mathcal{M}_b$ is smooth, then*

$$h^0_c = 0, \quad h^1_c = 0, \quad h^2_c = 5, \quad h^3_c = 0, \quad h^4_c = 1.$$

Proof We have $\mathcal{M}_b \subseteq \mathbb{C}^3$ as a subvariety. Since $\mathbb{C}^3$ is acyclic, Alexander duality [22] gives

$$\mathrm{H}^i_c(\mathcal{M}_b)^* \cong \mathrm{H}^{6-i-1}_{dR}(\mathbb{C}^3 \smallsetminus \mathcal{M}_b) \text{ for } i \leq 4, \quad \mathrm{H}^5_c(\mathcal{M}_b) = 0.$$

One then uses *Macaulay2* to compute $\mathrm{H}^*_{dR}(\mathbb{C}^3 \smallsetminus \mathcal{M}_b)$ via the Oaku–Takayama algorithm to obtain the above numbers [10,18]. ■

The singularity of $\mathcal{M}_{-2}$ at $(0,0,0)$ is isolated. Let $\mathcal{B}$ be a small ϵ-ball of $(0,0,0)$. Then for some $b \in f(\mathcal{B})$ near but not equal to -2, $f_1^{-1}(b) \cap \mathcal{B}$ is homotopic to a bouquet of 2-spheres [17]. One can apply Schultze's algorithm using the Brieskorn lattice method to compute the monodromy of $f_1|_{\mathcal{B}}$ around -2; see [1,21].

Theorem 6.6 *$f_1^{-1}(b) \cap \mathcal{B}$ consists of one 2-sphere and the monodromy action is the -1 map.*

This means that if one goes around a small loop around $-2 \in \mathbb{C}$, the monodromy action on $f_1^{-1}(b) \cap \mathcal{B}$ is the antipodal map on the small sphere in $f_1^{-1}(b) \cap \mathcal{B}$ around $(0,0,0)$. Schultz implemented his algorithm in *Singular*; see [3]. Notice also that this monodromy action does not arise from a Dehn twist action on $\Sigma_{1,1}$ because any Dehn twist induced monodromy action is the identity on the above 2-sphere [9].

There is a natural compactification via the projectivization of $\mathcal{M}_b$ and much more information can be obtained from this projectivized object. Consider the projective surface defined by the homogeneous polynomial

$$\Psi(X_1, X_2, X_3, V) = V(X_1^2 + X_2^2 + X_3^2) - X_1 X_2 X_3 - V^3(2 + b).$$

Then Ψ defines the (projective) compactification $\bar{\mathcal{M}}_b \subseteq \mathbb{P}^3$. A direct calculation shows that Ψ is irreducible and that $\overline{\mathcal{M}}_b$ is smooth if and only if $\psi_1(b) \neq 0$. We assume this is the case for the rest of this section. Geometrically, $\bar{\mathcal{M}}_b \smallsetminus \mathcal{M}_b$ consists of three copies of $\mathbb{P}^1$ defined by the equation $V = 0$, pairwise intersecting at a point (with a total of three points of intersections). *Macaulay2* gives us the following theorem.

Theorem 6.7 *The non-zero Hodge numbers of $\bar{\mathcal{M}}_b$ are*

$$h^{0,0}(\bar{\mathcal{M}}_b) = h^{2,2}(\bar{\mathcal{M}}_b) = 1, \quad h^{1,1}(\bar{\mathcal{M}}_b) = 7.$$

Corollary 6.8 *The Betti numbers of $\bar{\mathcal{M}}_b$ are*

$$h^0(\bar{\mathcal{M}}_b) = h^4(\bar{\mathcal{M}}_b) = 1, \quad h^2(\bar{\mathcal{M}}_b) = 7.$$

Notice that one may obtain the Betti numbers using the algebraic de Rham complex [22].

6.3 The Groups H^0 and H^1

The algebraic de Rham cohomology satisfies many of the usual cohomological axioms. There is an excision sequence [13, Theorem 3.3].

Proposition 6.9 (Excision) *Suppose $\mathcal{U}$ is smooth and $\mathcal{V} \subseteq \mathcal{U}$ a smooth subvariety of codimension r. Let $\mathcal{W} = \mathcal{U} \setminus \mathcal{V}$. Then there is an exact sequence*

$$\cdots \longrightarrow \mathrm{H}^{i-2r}(\mathcal{V}) \longrightarrow \mathrm{H}^{i}(\mathcal{U}) \longrightarrow \mathrm{H}^{i}(\mathcal{W}) \longrightarrow \mathrm{H}^{i-2r+1}(\mathcal{V}) \longrightarrow \cdots$$

Corollary 6.10 *The Euler characteristics are additive: $\chi(\mathcal{U}) = \chi(\mathcal{V}) + \chi(\mathcal{W})$.*

Corollary 6.11 *If $\psi_1(b) \neq 0$, then $h^0 = 1$ and $h^1 = 0$.*

Proof The locus at infinity defined by $V = 0$ consists of three $\mathbb{P}^1$'s pairwise intersecting at one point. Let $\mathcal{V}$ be the disjoint union:

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3, \text{ where } \mathcal{V}_1 = \mathbb{P}^1, \mathcal{V}_2 = \mathbb{C}, \mathcal{V}_3 = \mathbb{C} \setminus \{0\},$$

each of which has codimension 1 in $\bar{\mathcal{M}}_b$. Let $\mathcal{U}_0 = \bar{\mathcal{M}}_b$ and $\mathcal{U}_{i+1} = \mathcal{U}_i \setminus \mathcal{V}_{i+1}$ for $0 \leq i \leq 2$. Notice that $\mathcal{M}_b = \mathcal{U}_3$. Proposition 6.9 gives the exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathcal{U}_i) \longrightarrow \mathrm{H}^0(\mathcal{U}_{i+1}) \longrightarrow 0.$$

Hence, $h^0(\mathcal{U}_i) = 1$ for $0 \leq i \leq 3$.

By Corollary 6.10,

$$\chi(\bar{\mathcal{M}}_b) = \chi(\mathcal{M}_b) + \sum_{i=1}^{3} \chi(\mathcal{V}_i).$$

By Corollary 6.8, $\chi(\bar{\mathcal{M}}_b) = 9$. Hence,

$$\chi(\mathcal{M}_b) = \chi(\bar{\mathcal{M}}_b) - \sum_{i=1}^{3} \chi(\mathcal{V}_i) = 9 - (2+1+0) = 6.$$

By Theorem 6.2, $h^2(\mathcal{M}_b) = 5$. Since $h^0(\mathcal{M}_b) = 1$,

$$h^1(\mathcal{M}_b) = (5+1) - \chi(\mathcal{M}_b) = 0. \qquad \blacksquare$$

Remark 6.12 All of these calculations are done in the algebraic category, and these results parallel those in [9].

6.4 The Gauß–Manin Connection

Recall the map $f_1 \colon \mathcal{M} \to \mathbb{C}$ corresponding to the ring morphism

$$f_1^* \colon \mathbb{C}[y] \longrightarrow \mathbb{C}[\mathbf{x}], \quad y \longmapsto t.$$

Note that f_1 is not smooth; however, it becomes smooth when the fibres over $\mathbf{Spec}(\mathbb{C}[y]/(\psi_1(y)))$ are removed. Let X and Y be the respective localizations defined by

$$\mathcal{O}_X = \mathbb{C}[\mathbf{x}]_{f_1^*(\psi_1(y))}, \quad \mathcal{O}_Y = \mathbb{C}[y]_{\psi_1(y)}.$$

Then $f_1 \colon X \to Y$ is smooth. The Gauß–Manin connection on $\mathcal{H}^2$ is

$$\nabla \colon \mathcal{H}^2 \longrightarrow \Omega^1_Y \otimes_{f_1^*} \mathcal{H}^2.$$

Recall our choice of a basis for H^2:

$$B = \{1, x_1, x_2, x_3, x_1^2\} \otimes x_1 dx_{23},$$
$$dB = \{1, 2x_1, x_2, x_3, 3x_1^2\} \otimes dx_{123}.$$

For each $b \in \mathbb{C}$ with $\psi_1(b) \neq 0$, $P = (y - b)$ is a maximal prime of $\mathcal{O}_Y$. Let $\Phi_1 \colon Y_P \to Y$ be the localization map. By Proposition 2.2,

$$\Phi_1^*(\mathcal{H}^2) \xrightarrow{\cong} R^2(f_1 \circ \Phi_1)_*(\Omega_{\Phi_1^*(X)/Y_P}).$$

Let $\phi_1 \colon \mathbf{Spec}(k(P)) \to Y_P$. Then the following diagram

$$\begin{tikzcd} \Omega^i_{Y_P} \arrow[r, "d"] \arrow[d, "\phi_1^*"] & \Omega^{i+1}_{Y_P} \arrow[d, "\phi_1^*"] \\ \Omega^i_{\mathcal{M}_b} \arrow[r, "d"] & \Omega^{i+1}_{\mathcal{M}_b} \end{tikzcd}$$

commutes. By Nakayama's lemma, B generates $\Phi_1^*(\mathcal{H}^2)$. Since this is true for every maximal P, by the local to global principle [6, Corollary 2.9], B generates $\mathcal{H}^2$. Hence B serves as a basis for $\mathcal{H}^2$.

Let $u \in \mathcal{H}^2$. Then du is of the form $w \wedge dt$ for some $w \in \Omega^2_X$ and $\nabla(u) = w \otimes_{f_1^*} dy$. With the global basis B, we can write $\nabla = d + E(t) \otimes_{f_1^*} dy$, where d is the exterior differential operator of $(\Omega^\bullet_Y, d)$.

We now factor each element in dB as a product of $w \wedge dt$ and write w as a linear combination of basis elements in B. This is carried out with the help of *Macaulay2* [10]. Let

$$\eta = \frac{(t-2)x_1 dx_{23} + x_3(x_3^2 - 4)dx_{12} + (2x_1x_3 + 2x_2 - tx_2 - x_2x_3^2)dx_{13}}{2(t^2 - 4)}.$$

Then $dx_{123} = \eta \wedge dt$. Hence,

$$d(x_1 dx_{23}) = \eta \wedge dt, \quad d(x_1^2 dx_{23}) = 2x_1 \eta \wedge dt, \quad d(x_1^3 dx_{23}) = 3x_1^2 \eta \wedge dt,$$
$$d(x_1x_2 dx_{23}) = x_2 \eta \wedge dt, \quad d(x_1x_3 dx_{23}) = x_3 \eta \wedge dt.$$

We need to write the elements in the set

$$D = \{1, 2x_1, 3x_1^2, x_2, x_3\} \otimes \eta$$

as linear combinations of elements in B. Let $b \in \mathbb{C} \setminus \{\pm 2\}$ and consider

$$D \subseteq \mathcal{H}^2 \otimes_{f_1^*} \big(\mathcal{O}_Y/(y-b)\big) \cong \mathrm{H}^2.$$

Now we apply the algorithm in the proof of Theorem 6.2 to D. This results in

$$\eta \sim \Big(\frac{9x_1 - x_1^3}{6(b+2)} + \frac{-3x_1 + x_1^3}{6(b-2)}\Big)dx_{23},$$
$$2x_1\eta \sim \frac{3x_1^2}{2(b-2)}dx_{23}, \quad 3x_1^2\eta \sim \frac{-6x_1 + 2x_1^3}{b-2}dx_{23},$$
$$x_2\eta \sim \frac{3x_1x_2}{2(b-2)}dx_{23}, \quad x_3\eta \sim \frac{3x_1x_3}{2(b-2)}dx_{23}.$$

Since the Jacobson radical of $\mathcal{O}_Y$ is $\{0\}$, $E(t)$ with respect to the basis B is

$$E(t) = \begin{pmatrix} \frac{3}{2(t+2)} + \frac{-1}{2(t-2)} & 0 & 0 & 0 & \frac{-1}{6(t+2)} + \frac{1}{6(t-2)} \\ 0 & \frac{3}{2(t-2)} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2(t-2)} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{2(t-2)} & 0 \\ \frac{-6}{t-2} & 0 & 0 & 0 & \frac{2}{t-2} \end{pmatrix}.$$

Notice the $t+2$ and $t-2$ terms in the denominators. These are the singular values around which the monodromy of ∇ is not trivial. From $E(t)$, we see that ∇ is a direct sum of three rank-1 systems and one rank-2 system. Denote by $\mathcal{D}$ the Gauß–Manin connection for the rank-2 subsystem. Then $Y = \mathbb{C} \setminus \{-2, 2\}$ is the three-holed sphere and

$$\mathcal{D} = d + \Big(\frac{A_2}{y-2} + \frac{A_{-2}}{y+2}\Big)dy,$$

where

$$A_2 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{6} \\ -6 & 2 \end{pmatrix}, \quad A_{-2} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{6} \\ 0 & 0 \end{pmatrix}.$$

The exponential matrix at infinity is then

$$A_\infty = -(A_2 + A_{-2}) = \begin{pmatrix} -1 & 0 \\ 6 & -2 \end{pmatrix}.$$

The eigenvalues of A_2 and A_{-2} are 0 and $\frac{3}{2}$. The eigenvalues of A_∞ are -1 and -2. Since the difference of the two eigenvalues of A_∞ is a non-zero integer, one must take special care to compute the monodromy at ∞. We make a change of variable $y \to \frac{1}{z}$. Then

$$\mathcal{D} = d + \frac{\mathcal{A}(z)}{z}dz, \quad \mathcal{A}(z) = -\Big(\frac{A_2}{(1-2z)} + \frac{A_{-2}}{(1+2z)}\Big).$$

Now we follow [16, §6] to compute the monodromy at $z = 0$. First we compute the Taylor series of $\mathcal{A}$ at $z = 0$. This gives us $\mathcal{A}(0) = A_\infty$ and

$$\frac{d\mathcal{A}}{dz}(0) = 2A_{-2} - 2A_2 = \begin{pmatrix} 4 & -\frac{2}{3} \\ 12 & -4 \end{pmatrix}.$$

Following [16, §6], we set

$$\varphi = \begin{pmatrix} -2 & -\frac{2}{3} \\ 0 & -2 \end{pmatrix}.$$

Then the monodromy at $z = 0$ is

$$\mathcal{N}_\infty = e^{-2\pi i\varphi} = \begin{pmatrix} 1 & \frac{4\pi i}{3} \\ 0 & 1 \end{pmatrix}.$$

The classical result of Riemann says that the global monodromy is determined by the local ones at the three punctures [14]. A direct computation then shows that the monodromy group of this rank-2 subsystem is generated by the following elements:

$$\mathcal{N}_{-2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{N}_2 = \begin{pmatrix} 1 & -\frac{4\pi i}{3} \\ 0 & -1 \end{pmatrix}, \quad \mathcal{N}_\infty = \begin{pmatrix} 1 & \frac{4\pi i}{3} \\ 0 & 1 \end{pmatrix}.$$

Notice that $\mathcal{M}_b$ is not projective. Moreover, the locus at infinity of $\bar{\mathcal{M}}_b$ consists of three copies of $\mathbb{P}^1$, pairwise intersecting at one point. The long exact sequence in Proposition 6.9 then shows that the map $\iota^*\colon \mathrm{H}^2(\bar{\mathcal{M}}_b) \to \mathrm{H}^2(\mathcal{M}_b)$ has rank equal to 4, where $\iota\colon \mathcal{M}_b \to \bar{\mathcal{M}}_b$ is the inclusion.

7 The Representation Varieties of the Four-holed Sphere

This section computes the cohomologies $\mathrm{H}^\bullet$ of the smooth $\mathrm{SL}(2,\mathbb{C})$-representation varieties of a four-holed sphere. Let $g = 0, m = 4$. Then the fundamental group π is isomorphic to $\mathbb{F}_3$, the free group on three generators [7]. Again we rename the variables in Section 5.2. Let $\mathbf{x} = \{x_1, x_2, x_3\}$ such that

$$x_1 = z_{12}, \quad x_2 = z_{13}, \quad x_3 = z_{23}.$$

The four punctures correspond to

$$F_1, F_2, F_3, F_4 := (F_1F_2F_3)^{-1}.$$

For $\rho \in \mathrm{Hom}(\mathbb{F}_3, G)$, let $\mathbf{t} = \{t_1, t_2, t_3, t_4\}$ with

$$t_i = \mathrm{tr}(\rho(F_i)), \quad 1 \le i \le 4.$$

With this new notation, let

$$\begin{aligned} u_4 = u_4(\mathbf{t}) = 4 - t_1^2 - t_2^2 - t_3^2 - t_1t_2t_3t_4 - t_4^2 + t_1t_2x_1 \\ + t_3t_4x_1 - x_1^2 + t_1t_3x_2 + t_2t_4x_2 - x_2^2 \\ + t_2t_3x_3 + t_1t_4x_3 - x_1x_2x_3 - x_3^2. \end{aligned}$$

Then $\mathcal{M} = \mathbf{Spec}(\mathbb{C}[\mathbf{x},\mathbf{t}]/(u_4))$. Let $\mathbf{y} = \{y_1, y_2, y_3, y_4\}$. Then we have a morphism

$$f_4\colon \mathcal{M} \longrightarrow \mathbf{Spec}(\mathbb{C}[\mathbf{y}])$$

induced by the ring homomorphism

$$f_4^*\colon \mathbb{C}[\mathbf{y}] \longrightarrow \mathbb{C}[\mathbf{x},\mathbf{t}]/(u_4), \quad f_4^*(y_i) = t_i.$$

For a fixed element $\mathrm{b} = (b_1, b_2, b_3, b_4) \in \mathbb{C}^4$, representing the (fixed) monodromies at the punctures, we rename $\mathcal{M}_C$ as $\mathcal{M}_{\mathrm{b}}$. Then $\mathcal{M}_{\mathrm{b}}$ is defined by the ideal

$$I_{\mathrm{b}} = (t_1 - b_1, t_2 - b_2, t_3 - b_3, t_4 - b_4, u_4) \subseteq \mathbb{C}[\mathbf{x},\mathbf{t}].$$

Remark 7.1 $\mathcal{M}_{\mathsf{b}}$ is a subvariety of $\mathbb{C}^3$ defined by the principal ideal

$$L_{\mathsf{b}} := (u_4(\mathsf{b})) = I_{\mathsf{b}} \cap \mathbb{C}[\mathbf{x}] \subseteq \mathbb{C}[\mathbf{x}]$$

for a fixed $\mathsf{b} \in \mathbb{C}^4$. Let $\mathcal{O} = \mathbb{C}[\mathbf{x}]$.

Introducing the symmetric coordinates, let $\mathbf{s} = \{s_1, s_2, s_3, s_4\}$ be the elementary symmetric polynomials in $\mathbb{C}[\mathbf{y}]$, *i.e.*,

$$s_i = \sum_{|a|=i, a_j \le 1} \mathbf{y}^a, \quad 1 \le i \le 4.$$

Let

$$\begin{aligned}\Delta(\mathbf{y}) = &(y_1^4 + y_2^4 + y_3^4 + y_4^4) - 2(y_1^2y_2^2 + y_1^2y_3^2 + y_1^2y_4^2 + y_2^2y_3^2 + y_2^2y_4^2) + 8y_1y_2y_3y_4\\ &+ (y_1^2y_2^2y_3^2 + y_1^2y_3^2y_4^2 + y_2^2y_3^2y_4^2)\\ &- (y_1^3y_2y_3y_4 + y_1y_2^3y_3y_4 + y_1y_2y_3^3y_4 + y_1y_2y_3y_4^3)\\ = &\, s_1^4 - (4s_1^2s_2 + s_1^2s_4 + s_4s_1^2) + (8s_1s_3 + s_3^2).\end{aligned}$$

Let $\psi_4(\mathbf{y}) \in \mathbb{C}[\mathbf{s}] \subseteq \mathbb{C}[\mathbf{y}]$ be the symmetric polynomial

$$\psi_4(\mathbf{y}) = \Delta(\mathbf{y})^2 \prod_{i=1}^{4} (y_i^2 - 4).$$

Theorem 7.2 *The singularity locus is defined by the symmetric polynomial ψ_4. This is to say that $\mathcal{M}_{\mathsf{b}}$ is singular if and only if $\psi_4(\mathsf{b}) = 0$.*

Proof Let $J(L_{\mathsf{b}})$ be the Jacobian ideal of L_{b}. For the Gröbner basis computation, we treat b as variables and use monomial order

$$W = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

on variables $\{x_3, x_2, x_1, b_4, b_3, b_2, b_1\}$. Denote by J_G the resulting Gröbner basis of $J(L_{\mathsf{b}})$. The (constant) term in J_G that contains only b is $\psi_4(\mathsf{b})$. In other words, $\psi_4(\mathsf{b}) \ne 0$ if and only if $J(L_{\mathsf{b}}) = \mathcal{O}$ if and only if $\mathcal{M}_{\mathsf{b}}$ is smooth by Proposition 3.5. ∎

The $\mathrm{SL}(2, \mathbb{C})$-representation variety of the 4-holed sphere is of importance. As far as the author is aware, this is the first explicit computation of the singularity locus. The factor $\prod_{i=1}^4 (b_i^2 - 4)$ corresponds to the representation varieties of the three-holed sphere. The three-fold defined by Δ is worthy of further analysis.

7.1 Computing $\mathrm{H}^{\bullet}_{dR}$

Theorem 7.3 *If $\mathcal{M}_{\mathsf{b}}$ is smooth, then H^2 has dimension $h^2 = 5$ and a $\mathbb{C}$-basis*

$$B = \{1, x_1, x_2, x_3, x_1^2\} \otimes (x_1 dx_{23}).$$

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Proof All (smooth) $\mathcal{M}_{\mathsf{b}}$ have isomorphic $\mathrm{H}^\bullet$ for $\psi_4(\mathsf{b}) \neq 0$. Let $\mathsf{b} = (1,0,0,0)$. Then $\psi_4(\mathsf{b}) = 1 \neq 0$. Hence $f_4^{-1}(\mathsf{b})$ is a smooth fibre.

Recall the function f_1 and the objects b and $t(\mathbf{x})$ from Section 6. Let $b = 1$; make a change of coordinates $\mathbf{x} \to -\mathbf{x}$ and consider the ideal $(t(-\mathbf{x}) - b) \subseteq \mathcal{O}$. Then a direct calculation shows

$$I_b \cong (t(-\mathbf{x}) - b) = L_{\mathsf{b}}$$

as ideals of $\mathcal{O}$. Hence, $\mathcal{M}_{\mathsf{b}} \cong \mathcal{M}_b$. The result then follows from Theorem 6.2 (compare [9]). ∎

Remark 7.4 One can similarly prove that $h^0 = 1$ and $h^1 = 0$.

7.2 The Limit of Computer Power

Recall the morphism $f_4\colon \mathcal{M} \to \mathbb{C}^4$ corresponding to the ring morphism

$$f_4^*\colon \mathbb{C}[\mathbf{y}] \longrightarrow \mathbb{C}[\mathbf{x},\mathbf{t}], \quad y_i \longmapsto t_i.$$

Note that f_4 is not smooth; however, it becomes smooth when the fibres over $\mathbf{Spec}(\mathbb{C}[\mathbf{y}]/(\psi_4(\mathbf{y})))$ are removed. Let X and Y be the respective localizations defined by

$$\mathcal{O}_X = \mathbb{C}[\mathbf{x},\mathbf{t}]_{f_4^*(\psi_4(\mathbf{y}))}, \quad \mathcal{O}_Y = \mathbb{C}[\mathbf{y}]_{\psi_4(\mathbf{y})}.$$

Then $f_4\colon X \to Y$ is smooth. The Gauß-Manin connection on $\mathcal{H}^2$ is

$$\nabla\colon \mathcal{H}^2 \longrightarrow \Omega^1_Y \otimes_{f_4^*} \mathcal{H}^2.$$

Again as in the case of 1-holed torus, the fibre over $\mathsf{b} \in Y$ of $\mathcal{H}^2$ is isomorphic to H^2 as a $\mathbb{C}$-vector space and generated by B. In theory, one then follows the method of Section 6 and factors du as

$$du = \sum_{i=1}^{4} w_i \wedge dt_i$$

for $u \in B$. We then have

$$\nabla(u) = du = \sum_{i=1}^{4} w_i \otimes_{f_4^*} dy_i$$

for each $u \in B$ to obtain the connection matrices $E_i(\mathbf{t})$ for $1 \le i \le 4$ so that

$$\nabla = d + \sum_{i=1}^{4} E_i(\mathbf{t}) \otimes_{f_4^*} dy_i.$$

Unfortunately, the computation involved in this factorization overwhelmed the computers in our possession and is likely to overwhelm any currently available computers.

8 The Representation Varieties of the Two-holed Torus

Let $g = 1, m = 2$. Then the fundamental group π is again isomorphic to $\mathbb{F}_3$, the free group on three generators [7]. Change $\mathbf{z}$ to $\mathbf{x}$ as before,

$$x_i = z_i, \quad 1 \le i \le 3; \quad x_{ij} = z_{ij}, \quad 1 \le i < j \le 3,$$
$$\mathbf{x} = \{x_i, 1 \le i \le 3; x_{ij}, 1 \le i < j \le 3\}, \quad \mathbf{y} = \{y_1, y_2\}$$

The two punctures correspond to $F_1F_2F_3$ and $F_1F_3F_2$, respectively. For $\rho \in \mathrm{Hom}(\mathbb{F}_3, \mathrm{SL}(2,\mathbb{C}))$, let $\mathbf{t} = \{t_1, t_2\}$ with

$$t_1 = z_{123} = \mathrm{tr}(\rho(F_1F_2F_3)), \quad t_2 = \mathrm{tr}(\rho(F_1F_3F_2))$$

i.e., $\mathbf{t}$ represents the monodromies at the two punctures. Let

$$\begin{aligned} u_p &= u_p(\mathbf{t}) = t_1t_2 \\ &\quad - (x_1^2 + x_2^2 + x_3^2 + x_{12}^2 + x_{13}^2 + x_{23}^2 - x_1x_{12}x_2 - x_1x_{13}x_3 - x_2x_{23}x_3 + x_{12}x_{13}x_{23} - 4) \\ u_s &= u_s(\mathbf{t}) = t_1 + t_2 - (x_3x_{12} + x_2x_{13} + x_1x_{23} - x_1x_2x_3). \end{aligned}$$

Then $\mathcal{M} = \mathbf{Spec}(\mathbb{C}[\mathbf{x},\mathbf{t}]/(u_p,u_s))$ and we have a morphism

$$f_2 : \mathcal{M} \longrightarrow \mathbf{Spec}(\mathbb{C}[\mathbf{y}])$$

induced by the ring homomorphism

$$f_2^* : \mathbb{C}[\mathbf{y}] \longrightarrow \mathbb{C}[\mathbf{x},\mathbf{t}]/(u_p,u_s), \quad f_2^*(y_1) = t_1, \quad f_2^*(y_2) = t_2.$$

For a fixed element $\mathsf{b} \in \mathbb{C}^2$, representing the (fixed) monodromies at the punctures, we rename $\mathcal{M}_C$ as $\mathcal{M}_{\mathsf{b}}$. Then $\mathcal{M}_{\mathsf{b}}$ is defined by the ideal

$$I_{\mathsf{b}} = (t_1 - b_1, t_2 - b_2, u_p, u_s) \subseteq \mathbb{C}[\mathbf{x},\mathbf{t}].$$

For an elaborate exposition of the above calculations, see [7].

Remark 8.1 $\mathcal{M}_{\mathsf{b}}$ can be considered as a subvariety of $\mathbb{C}^6$ defined by the ideal

$$L_{\mathsf{b}} := (u_p(\mathsf{b}), u_s(\mathsf{b})) = I_{\mathsf{b}} \cap \mathbb{C}[\mathbf{x}] \subseteq \mathbb{C}[\mathbf{x}]$$

for a fixed $\mathsf{b} \in \mathbb{C}^2$. Let $\mathcal{O} = \mathbb{C}[\mathbf{x}]$.

Let $\psi_2(\mathbf{y}) \in \mathbb{C}[\mathbf{y}]$ be the symmetric polynomial

$$\psi_2(\mathbf{y}) = (y_1^2 - 4)(y_2^2 - 4)(y_1 - y_2)^2.$$

Theorem 8.2 *For fixed* $\mathsf{b} \in \mathbb{C}^2$, $\mathcal{M}_{\mathsf{b}}$ *is singular if and only if* $\psi_2(\mathsf{b}) = 0$.

Proof Let $J(L_{\mathsf{b}})$ be the Jacobian ideal of L_{b}. For the Gröbner basis computation, we treat b as variables and use the monomial order

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

on variables $\{x_{23}, x_{13}, x_{12}, x_3, x_2, x_1, b_2, b_1\}$. Denote by J_G the resulting Gröbner basis of $J(L_{\mathsf{b}})$. The (constant) term in J_G that contains only b is $\psi_2(\mathsf{b})$. In other words, $\psi_2(\mathsf{b}) \neq 0$ if and only if $J(L_{\mathsf{b}}) = \mathcal{O}$ if and only if $\mathcal{M}_{\mathsf{b}}$ is smooth by Proposition 3.5. ∎

In this situation, there is no mystery of the singular locus. The cases of $b_i = \pm 2$ mean the monodromy around the i-th puncture is central in $\mathrm{SL}(2,\mathbb{C})$; *i.e.*, $\mathcal{M}_b$ contains the representation varieties of one-holed torus. The case of $b_1 = b_2$ means the moduli space $\mathcal{M}_b$ contains abelian representations.

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