SYMMETRY RESULT FOR SOME OVERDETERMINED VALUE PROBLEMS

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Abstract

The aim of this article is to prove a symmetry result for several overdetermined boundary value problems. For the two first problems, our method combines the maximum principle with the monotonicity of the mean curvature. For the others, we use essentially the compatibility condition of the Neumann problem.


Keywords and phrases: compatibility condition, mean curvature, Neumann problem, overdetermined problem, Serrin problem, shape optimization, symmetry.

1. Introduction

We assume throughout that \( D \subset \mathbb{R}^N \) \((N \geq 2)\) is a bounded ball which contains all the domains we use. If \( \omega \) is an open subset of \( D \), let \( \nu \) be the outward normal to \( \partial \omega \) and let \( |\partial \omega| \) (respectively \( |\omega| \)) be the perimeter (respectively the volume) of \( \omega \).

Consider the following overdetermined boundary value problem:

\[
S(k) \begin{cases} 
-\Delta u_\Omega = 1 & \text{in } \Omega, \\
u_\Omega = 0 & \text{on } \partial \Omega, \\
\frac{-\partial u_\Omega}{\partial \nu} = k & \text{on } \partial \Omega.
\end{cases}
\]

Notice that since \( u_\Omega \) vanishes on \( \partial \Omega \) then \((\partial u_\Omega / \partial \nu) = |\nabla u_\Omega|\).

In 1971, Serrin [19] proved that if Problem \( S(k) \) has a solution \( u_\Omega \in C^2(\overline{\Omega}) \) then \( \Omega \) must be a ball and \( u_\Omega \) is radially symmetric. The method used by Serrin combines the maximum principle together with the device of moving planes [11] to a critical position and then showing that the solution is symmetric about the limiting plane. In the same year, Weinberger [21] gave a simplified proof for this problem. His

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strategy of proof consists first of showing that $|\nabla u|^2 + (2/N)u = k^2$ in $\Omega$ and then deriving a radial symmetry from this. Concerning other methods, we refer the reader to the paper by Payne [15] and the references therein. The last problem studied in Section 5 comes from this paper. For more details about the symmetry results see [9, Introduction] and the references therein. Fragalà et al. [9], obtained their symmetry result by combining a maximum principle for a suitable $P$-function with some geometric arguments involving the mean curvature of $\partial \Omega$. However, they assumed the solution $\Omega$ to be star-shaped with respect to the origin. This assumption seems to be crucial for the proof of their main results and they cannot remove it.

The method we present here needs the use of the maximum principle together with the monotonicity of the mean curvature. Therefore, it can be extended to more general divergence operators such as the $p$-Laplacian for which one can use Hopf’s comparison principle or the operator $-\text{div}(A(|\nabla u|)\nabla u)$ for which a boundary point principle is considered [9]. The novelty of our method is the following. First, to prove the main results of Section 3, we do not ask to be star-shaped with respect to the origin. Second, this method can be extended to other problems such as $P(c)$, see Section 4:

$$P(c) \begin{cases} -\Delta u_\Omega = 1 & \text{in } \Omega, \ u_\Omega = 0 \text{ on } \partial \Omega \text{ denoted } P(\Omega), \\ -\Delta v_\Omega = u_\Omega & \text{in } \Omega, \ v_\Omega = 0 \text{ on } \partial \Omega \text{ denoted } Q(\Omega), \\ |\nabla u_\Omega||\nabla v_\Omega| = c & \text{on } \partial \Omega. \end{cases}$$

The problem $P(c)$ arises from the variational problem in probability [10, 13]. Fromm and McDonald [10] related this problem to the fundamental result of Serrin. Then, using the moving plane method combined with Serrin’s boundary point lemma, they showed that if this problem admits a solution $\Omega$ then it must be a ball. Huang and Miller [12] established the variational formulas for maximizing the functionals (which they considered) over $C^k$ domains with a volume constraint and obtained the same symmetry result for their maximizers.

Section 2 contains some preliminary results which are useful for solving the shape optimization problems presented in Sections 3 and 4. Section 3 is devoted to the problem $S(k)$ whereas Section 4 concerns the problem $P(c)$. In Section 5, by using the compatibility condition of the Neumann problem [14], we obtain the same symmetry result for other boundary value problems for which the overdetermined condition is not constant.

2. Preliminaries

**Definition 2.1.** Let $K_1$ and $K_2$ be two compact subsets of $D$. We call a Hausdorff distance of $K_1$ and $K_2$ (or briefly $d_H(K_1, K_2)$) the following positive number:

$$d_H(K_1, K_2) = \max[\rho(K_1, K_2), \rho(K_2, K_1)],$$

where $\rho(K_i, K_j) = \max_{x \in K_i} d(x, K_j)$, $i, j = 1, 2$, and $d(x, K_j) = \min_{y \in K_j}|x - y|$. 

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DEFINITION 2.2. Let $\omega_n$ be a sequence of open subsets of $D$ and let $\omega$ be an open subset of $D$. Let $K_n$ and $K$ be their complements in $\overline{D}$. We say that the sequence $\omega_n$ converges in the Hausdorff sense, to $\omega$ (or briefly $\omega_n \overset{H}{\rightarrow} \omega$) if
\[
\lim_{n \to +\infty} d_H(K_n, K) = 0.
\]

DEFINITION 2.3. Let $\{\omega_n, \omega\}$ be a sequence of open subsets of $D$. We say that the sequence $\omega_n$ converges in the compact sense to $\omega$ (or briefly $\omega_n \overset{K}{\rightarrow} \omega$) if:
- every compact subset of $\omega$ is included in $\omega_n$ for $n$ large enough; and
- every compact subset of $\overline{\omega}^c$ is included in $\overline{\omega}_n^c$ for $n$ large enough.

DEFINITION 2.4. Let $\{\omega_n, \omega\}$ be a sequence of open subsets of $D$. We say that the sequence $\omega_n$ converges in the sense of characteristic functions to $\omega$ (or briefly $\omega_n \overset{L}{\rightarrow} \omega$) if $\chi_{\omega_n}$ converges to $\chi_{\omega}$ in $L^p_{\text{loc}}(\mathbb{R}^N)$, $p \neq \infty$ ($\chi_{\omega}$ is the characteristic function of $\omega$).

DEFINITION 2.5 ([3]). Let $C$ be a compact convex set. The bounded domain $\omega$ satisfies C-GNP if:
1. $\omega \supset \text{int}(C)$;
2. $\partial \omega \setminus C$ is locally Lipschitz;
3. for any $c \in \partial C$ there is an outward normal ray $\Delta_c$ such that $\Delta_c \cap \omega$ is connected; and
4. for every $x \in \partial \omega \setminus C$ the inward normal ray to $\omega$ (if exists) meets $C$.

REMARK 2.6. If $\Omega$ satisfies the C-GNP and $C$ has a nonempty interior, then $\Omega$ is connected.

Put
\[ O_C = \{ \omega \subset D \mid \omega \text{satisfies C-GNP} \}. \]

THEOREM 2.7. If $\omega_n \in O_C$, then there exist an open subset $\omega \subset D$ and a subsequence (again denoted by $\omega_n$) such that (i) $\omega_n \overset{H}{\rightarrow} \omega$, (ii) $\omega_n \overset{K}{\rightarrow} \omega$, (iii) $\chi_{\omega_n}$ converges to $\chi_\omega$ in $L^1(D)$ and (iv) $\omega \in O_C$. Furthermore, the assertions (i), (ii) and (iii) are equivalent.

Barkatou proved this theorem [3, Theorem 3.1] and the equivalence between (i), (ii) and (iii) [3, Propositions 3.4, 3.5, 3.6, 3.7 and 3.8].

PROPOSITION 2.8. Let $\{\omega_n, \omega\} \subset O_C$ such that $\omega_n \overset{H}{\rightarrow} \omega$. Let $u_n$ and $u_\omega$ be respectively the solutions of $P(\omega_n)$ and $P(\omega)$. Then $u_n$ converges strongly in $H^1_0(D)$ to $u_\omega$ ($u_n$ and $u_\omega$ are extended by zero in $D$).

This proposition was proven for $N = 2$ or 3 [3, Theorem 4.3].
DEFINITION 2.9. Let $C$ be a convex set. We say that an open subset $\omega$ has the $C$-SP if:

1. $\omega \supset \text{int}(C)$;
2. $\partial \omega \setminus C$ is locally Lipschitz;
3. for any $c \in \partial C$ there is an outward normal ray $\Delta_c$ such that $\Delta_c \cap \omega$ is connected; and
4. for all $x \in \partial \omega \setminus C$ $K_x \cap \omega = \emptyset$, where $K_x$ is the closed cone defined by

$$\{y \in \mathbb{R}^N \mid (y - x) \cdot (z - x) \leq 0, \text{ for all } z \in C\}.$$ 

REMARK 2.10. $K_x$ is the normal cone to the convex hull of $C$ and $\{x\}$.

PROPOSITION 2.11 ([3, Proposition 2.3]). $\omega$ has the $C$-GNP if and only if $\omega$ satisfies the $C$-SP.

DEFINITION 2.12 ([8]). We say that a domain $\omega$ satisfies the $\varepsilon$-cone property if for all $x \in \partial \omega$ there exists a direction vector $\xi \in \mathbb{R}^N$ such that the cone $C(y, \xi, \varepsilon) \subset \omega$ for all $y \in B(x, \varepsilon) \cap \omega$. $\varepsilon$ denotes both the angle and the height of the cone.

Denoting by $\mathcal{O}_\varepsilon$ the class of domains which have the $\varepsilon$-cone property, we have the following lemma.

LEMMA 2.13 ([8]). If $\omega_n \in \mathcal{O}_\varepsilon$, then there exist an open subset $\omega \subset D$ and a subsequence (again denoted by $\omega_n$) such that (i) $\omega_n \xrightarrow{H} \omega$, (ii) $\omega_n \xrightarrow{H} \omega$, (iii) $\partial \omega_n \xrightarrow{H} \partial \omega$, (iv) $\chi_{\omega_n}$ converges to $\chi_\omega$ in $L^1(D)$, (v) $\omega \in \mathcal{O}_\varepsilon$ and (vi) $u_{\omega_n}$ converges strongly in $H^1_0(D)$ to $u_\omega$ ($u_\omega$ is the solution of $P(\omega)$).

PROPOSITION 2.14 ([5, Theorem 3.5]). Let $\omega_n$ and $u_\omega$ be respectively the solutions of the Dirichlet problems $P(\omega_n, g_n)$ and $P(\omega, g)$. If $g_n$ converges strongly in $H^{-1}(D)$ to $g$ then $\omega_n$ converges strongly in $H^1_0(D)$ to $\omega$ ($\omega_n$ and $\omega$ are extended by zero in $D$).

LEMMA 2.15 ([6, 17]). Let $\omega_n$ be a sequence of open and bounded subsets of $D$. There exist a subsequence (again denoted by $\omega_n$) and some open subset $\omega$ of $D$ such that:

1. $\omega_n$ converges to $\omega$ in the Hausdorff sense; and
2. $|\partial \omega| \leq \liminf_{n \to \infty} |\partial \omega_n|.$

2.1. Shape derivative In this subsection, we use the standard tool of the domain derivative to write down the optimality conditions. Before doing this, recall the definition of the domain derivative [20]. Suppose that $\omega$ is of class $C^2$. Consider a deformation field $V \in C^2(\mathbb{R}^N; \mathbb{R}^N)$ and set $\omega_t = \{x + tV(x) \mid x \in \omega\}$, $t > 0$. The application $\text{Id} + tV$ (a perturbation of the identity) is a Lipschitz diffeomorphism for $t$ small enough and, by definition, the derivative of $J$ at $\omega$ in the direction $V$ is

$$dJ(\omega, V) = \lim_{t \to 0} \frac{J(\omega_t) - J(\omega)}{t}.$$
As the functional $J$ depends on the domain $\omega$ through the solution of some Dirichlet problem, we need to also define the domain derivative $u'_\omega$ of $u_\omega$:

$$u'_\omega = \lim_{t \to 0} \frac{u_{\omega t} - u_\omega}{t}.$$  

Furthermore, $u'_\omega$ is the solution of the following problem [20]:

$$\begin{cases}
-\Delta u'_\omega = 0 & \text{in } \omega, \\
u'_\omega = -\frac{\partial u_\omega}{\partial \nu} V \cdot \nu & \text{on } \partial \omega.
\end{cases}$$  

(2.1)

Now to compute the derivative of the functionals we consider below, recall the following [20].

(1) The shape derivative of the volume is

$$\int_{\partial \omega} V \cdot \nu \, d\sigma.$$  

(2.2)

(2) The shape derivative of the perimeter is

$$\int_{\partial \omega} (N - 1) H_{\partial \omega} V \cdot \nu \, d\sigma.$$  

(2.3)

(3) Suppose that $u_\omega$ is in $H^1_0(D)$ and $\omega$ is of class $C^2$.

(a) If $F(\omega) = \int_\omega u^2_\omega \, dx$, then the Hadamard formula gives

$$dF(\omega, V) = 2 \int_\omega u_\omega u'_\omega \, dx.$$  

But $v_\omega$ is in $H^1_0(D)$ and $-\Delta v_\omega = u_\omega$ in $\omega$, so by Green’s formula we obtain

$$dF(\omega, V) = 2 \int_{\partial \omega} |\nabla u_\omega| |\nabla v_\omega| V \cdot \nu \, d\sigma.$$  

(b) If $G(\omega) = \int_\omega |\nabla u_\omega|^2 \, dx$, by the Hadamard formula we get

$$dG(\omega, V) = \int_{\partial \omega} |\nabla u_\omega|^2 V \cdot \nu \, d\sigma.$$  

Since the set $\omega$ satisfies some geometric property (the $\varepsilon$-cone property or the C-GNP), we ask the deformation set $\omega_t$ to satisfy the same property (for $t$ sufficiently small). To keep the $\varepsilon$-cone property any direction is admissible. The aim in what follows is to prove the same thing for the C-GNP. With $\omega$ having the C-GNP, by Proposition 2.11, it satisfies the C-SP. Then

for all $x \in \partial \omega \setminus C : K_x \cap \omega = \emptyset$. 

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For $t$ sufficiently small, let $\omega_t = \omega + tV(\omega)$ be the deformation of $\omega$ in the direction $V$. Let $x_t \in \partial \omega_t$. There exists $x \in \partial \omega$ such that $x_t = x + tV(x)$. Using the definition of $K_{x_t}$ and the equality above, we get (for $t$ small enough and for every displacement $V$)

$$\text{for all } x_t \in \partial \omega_t \setminus C : K_{x_t} \cap \omega_t = \emptyset,$$

which means that $\omega_t$ satisfies the $C$-SP (and so the $C$-GNP) for every direction $V$ when $t$ is sufficiently small.

3. Problem $S(k)$

3.1. Auxiliary lemmas

**Lemma 3.1.** Let $B_\rho$ be a solution of $S(k)$ so then $\rho = Nk$.

**Proof.** Let $u_\rho$ be the solution of $P(B_\rho)$. Using polar coordinates, $u_\rho$ verifies

$$-u''_\rho - \frac{N-1}{r}u'_\rho = 1 \quad \text{for } r \in ]0, \rho[,$$

$$u_\rho(\rho) = 0.$$

By the first equation, $(r^{N-1}u'_\rho)' = -r^{N-1}$. Since $u_\rho(\rho) = 0$, we get

$$r^{N-1}u'_\rho(r) = \rho^{N-1}u'_\rho(\rho) + \int_{\rho}^r s^{N-1} \, ds.$$

As $r \to 0$, $r^{N-1}u'_\rho(r) \to 0$ (otherwise we get a distributional contribution to $\Delta u_\rho$ at the origin). Thus

$$-u'_\rho(\rho) = \frac{1}{\rho^{N-1}} \int_{0}^{\rho} s^{N-1} \, ds = \frac{\rho}{N}.$$

Now if $B_\rho$ is a solution of $S(k)$ then $-u'_\rho(\rho) = k$. Thus $\rho = Nk$. \hfill \Box

**Lemma 3.2.** Let $\Omega$ be a solution of $S(k)$. Let $\omega \supset \Omega$ and let $u_\omega$ be the solution of $P(\omega)$. If $\partial \omega \cap \partial \Omega \neq \emptyset$ and if $|\nabla u_\omega| \leq k$ on $\partial \omega \cap \partial \Omega$ then $\omega = \Omega$.

**Proof.** Suppose by contradiction that $\Omega$ is different to $\omega$. As $\omega \supset \Omega$, $\partial \omega \neq \partial \Omega$. But $\partial \omega \cap \partial \Omega \neq \emptyset$, so applying the maximum principle to $u_\omega$ and $u_\Omega$ and using the fact that $\Omega$ is a solution of $S(k)$, we obtain

$$k = |\nabla u_\Omega| < |\nabla u_\omega| \leq k \quad \text{on } \partial \omega \cap \partial \Omega,$$

which gives the contradiction. \hfill \Box

As a consequence, we have the following corollary.

**Corollary 3.3.** Let $\omega$ and $\Omega$ be two solutions of $S(k)$. Suppose that $\Omega \subset \omega$ and $\partial \omega \cap \partial \Omega \neq \emptyset$, then $\omega = \Omega$. 

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3.2. Shape optimization problems  Let $\Omega$ be a solution of $S(k)$. Let $B_\rho$ be the ball centred at the origin of radius $\rho$. Denote by $B$ the greatest ball contained in $\Omega$. Denote by $O_B$ (respectively $O_{B_\rho}$) the class of all domains which satisfy the $B$-GNP (respectively the $B_\rho$-GNP). Set

$$
O_\Omega = \{ D \ni \omega \ni \Omega \mid \omega \in O_B \},
$$

$$
O^\Omega = \{ \omega \subset \Omega \subset D \mid \omega \in O_e \},
$$

and

$$
O_\rho = \{ D \ni \omega \ni B_\rho \mid \omega \in O_{B_\rho} \}.
$$

Consider the following functionals:

$$
\begin{align*}
  j_1(\omega) &= \frac{N}{N-1} k^2 |\partial \omega| - \frac{1}{2} \int_\omega |\nabla u_\omega|^2, \\
  j_2(\omega) &= \frac{N}{N-1} k |\partial \omega| - |\omega|,
\end{align*}
$$

and

$$
\begin{align*}
  j_3(\omega) &= \frac{k^2}{2} |\omega| - \frac{1}{2} \int_\omega |\nabla u_\omega|^2.
\end{align*}
$$

Here $u_\omega$ is the solution of $P(\omega)$.

We then have the following propositions.

**Proposition 3.4.** Suppose $N \in \{2, 3\}$. There exists $\Omega_1 \in O_\Omega$ which is of class $C^2$ such that:

1. $j_1(\Omega_1) = \min\{ j_1(\omega) \mid \omega \in O_\Omega \}$ and $u_{\Omega_1}$ is the solution of $P(\Omega_1)$;
2. 

$$
\begin{align*}
  |\nabla u_{\Omega_1}| &\leq N k^2 H_{\partial \Omega_1} & \text{on } \partial \Omega_1 \cap \partial \Omega, \\
  |\nabla u_{\Omega_1}| &\leq N k H_{\partial \Omega_1} & \text{on } \partial \Omega_1 \backslash \partial \Omega. 
\end{align*}
$$

**Proof.** To get item (1), we use Theorem 2.7, Proposition 2.8 and item (2) of Lemma 2.15. For item (2), using the same notation as in Section 2.1, to get $\Omega$ in $(\Omega_1)_t$ (for $t$ small enough) the admissible directions $V$ must satisfy

$$
V \cdot \nu \geq 0 \quad \text{on } \partial \Omega \cap \partial \Omega_1.
$$

Notice that, for $\partial \Omega_1 \backslash \partial \Omega$, each $V$ is admissible. Now since $u_{\Omega_1}$ vanishes on $\partial \Omega_1$, (2.3) and 3(b) above imply

$$
d j_1(\Omega_1, V) = \int_{\partial \Omega_1} (NH_{\partial \Omega_1} k^2 - |\nabla u_{\Omega_1}|^2) V \cdot \nu \, d\sigma.
$$

Since $d j_1(\Omega_1, V) \geq 0$ for each admissible direction $V$, according to the preceeding calculations we obtain (3.1). \qed
**Proposition 3.5.** Suppose that \( N \geq 2 \). There exists \( \Omega_2 \in \mathcal{O}^\Omega \) such that:

1. \( j_2(\Omega_2) = \min\{j_2(\omega) \mid \omega \in \mathcal{O}^\Omega\} \);
2. if \( \Omega_2 \) is of class \( C^2 \) then
   \[
   \begin{cases}
   NkH_{\partial\Omega_2} \leq 1 & \text{on } \partial\Omega_2 \cap \partial\Omega, \\
   NkH_{\partial\Omega_2} = 1 & \text{on } \partial\Omega_2 \setminus \partial\Omega.
   \end{cases}
   \tag{3.2}
   \]

**Proof.** The first item is obtained by using (iv) and (v) of Lemma 2.13 together with item (2) of Lemma 2.15. The continuity with respect to the domains for the Dirichlet problem \( P(\Omega_2) \) is obtained by (vi) of Lemma 2.13. For the second item, on \( \partial\Omega_2 \setminus \partial \), any direction \( V \) is admissible whereas \( V \) must satisfy

\[ V \cdot \nu \leq 0 \quad \text{on } \partial\Omega \cap \partial\Omega_2. \]

Then, arguing as above, (2.2) and (2.3) imply (3.2). \( \square \)

**Proposition 3.6.** Suppose that \( N \in \{2, 3\} \). There exists \( \Omega_3 \in \mathcal{O}_\rho \) which is of class \( C^2 \) such that:

1. \( j_3(\Omega_3) = \min\{j_3(\omega) \mid \omega \in \mathcal{O}_\rho\} \) and \( u_{\Omega_3} \) is the solution of \( P(\Omega_3) \);
2. \[
   \begin{cases}
   |\nabla u_{\Omega_3}| \leq k & \text{on } \partial\Omega_3 \cap \partial B_\rho, \\
   |\nabla u_{\Omega_3}| = k & \text{on } \partial\Omega_3 \setminus \partial B_\rho.
   \end{cases}
   \tag{3.3}
   \]

**Proof.** The first item is due to Theorem 2.7 and Proposition 2.8. For the second item, the admissible directions \( V \) must satisfy \( V \cdot \nu \geq 0 \) on \( \partial\Omega \cap \partial\Omega_3 \). Then (2.2) and 3(b) imply (3.3). \( \square \)

**Remark 3.7.** The \( C^2 \) regularity obtained for \( \Omega_1 \) and \( \Omega_3 \) is due to [4, Theorem 1.4].

**Remark 3.8.** The continuity-compactness result obtained by Bucur and Trebeschi [5] allows us to extend the previous propositions to other divergence operators such as \( \text{div}(a(x, Du)) \), especially for the \( p \)-Laplacian case.

### 3.3. Main results

**Theorem 3.9.** Suppose that \( N = 2 \). Let \( \Omega_2 \) be as in Proposition 3.5, then \( \Omega = \Omega_2 = B_{2k} \).

The proof of this theorem uses the following lemma.

**Lemma 3.10 ([7, Section 30.4.1]).** Suppose that \( N = 2 \) and let \( \Omega \) be a simply connected domain which is of class \( C^{2,\alpha} \). If \( H_{\partial\Omega} \leq 1/\varrho \) then \( \Omega \) contains a ball of radius \( \varrho \).

**Remark 3.11 ([7, Section 30.4.2]).** The result of the previous lemma cannot be extended to \( N \geq 3 \).
PROOF OF THEOREM 3.9. Item (2) of Proposition 3.5 gives $H_{\partial \Omega_2} \leq 1/Nk$ on $\partial \Omega_2$. Since $\Omega_2 \subset \Omega$ and $N = 2$, Lemma 3.10 implies that $B_{2k} \subset \Omega_2 \subset \Omega$. Without loss of generality we may assume that $B_{2k}$ touches $\partial \Omega$ tangentially at a point $x_0$, so that they have the same outward normal vector $v_0$ (otherwise we shift $B_{2k}$). So $|\nabla u_{B_{2k}}(x_0)| = k = |\nabla u_{\Omega}(x_0)|$. Suppose now that $B_{2k} \neq \Omega$. As $B_{2k} \subset \Omega$, $\partial B_{2k} \neq \partial \Omega$. Now as $-\Delta u_{B_{2k}} = 1 = -\Delta u_{\Omega}$ in $B_{2k}$ and $u_{B_{2k}} \leq u_{\Omega}$ on $\partial B_{2k}$, the maximum principle gives $|\nabla u_{B_{2k}}(x_0)| < |\nabla u_{\Omega}(x_0)|$ which is absurd. It then follows that $\Omega = \Omega_2 = B_{2k}$. □

THEOREM 3.12. Suppose that $N \in \{2, 3\}$. Let $\Omega_3$ be as in Proposition 3.6, then $\Omega_3 = \Omega = B_{Nk}$.

PROOF. $B_{Nk} \subset \Omega_3 \subset \Omega$ by the definition of $\mathcal{O}_\rho$. Then using the same arguments as in the proof of Theorem 3.9, we obtain the same result for $\Omega$, $\Omega_3$ and $B_{Nk}$. □

THEOREM 3.13. Suppose that $N \in \{2, 3\}$. Let $\Omega_1$ and $\Omega_2$ be as in Propositions 3.4 and 3.5. If $\partial \Omega_1 \cap \partial \Omega_2 \neq \emptyset$ then $\Omega_1 = \Omega = \Omega_2 = B_{Nk}$.

PROOF. Since $\Omega_2 \subset \Omega \subset \Omega_1$, $\partial \Omega_1 \cap \partial \Omega_2 \neq \emptyset$ implies $\partial \Omega_1 \cap \partial \Omega_2 \cap \partial \Omega \neq \emptyset$. Suppose by contradiction that $\Omega_1 \neq \Omega_2$, then $\partial \Omega_1 \neq \partial \Omega$. According to (3.1) and (3.2), the monotonicity of the mean curvature together with the maximum principle implies

$$k = |\nabla u_\Omega| < |\nabla u_{\Omega_1}| \leq Nk^2 H_{\partial \Omega_1} \leq Nk^2 H_{\partial \Omega_2} \leq k$$

on $\partial \Omega_1 \cap \partial \Omega_2 \cap \partial \Omega$, which is absurd. So $\Omega_1 = \Omega$. Therefore (3.1) gives $k = |\nabla u_\Omega| \leq Nk^2 H_{\partial \Omega}$ on $\partial \Omega$. Thus $1 \leq Nk H_{\partial \Omega}$ which can be combined with (3.2) and the monotonicity of the mean curvature to get $H_{\partial \Omega_2} = 1/Nk$ on $\partial \Omega_2$, that is $\Omega_2$ is a ball with radius $Nk$ [1]. Now Lemma 3.1 implies that $\Omega_2$ is a solution of $S(k)$ and, since $\partial \Omega \cap \partial \Omega_2 \neq \emptyset$, Corollary 3.3 gives $\Omega = \Omega_2$. □

4. Problem $P(c)$

4.1. Auxiliary lemmas

LEMMA 4.1. Let $B_R$ be a solution of $P(c)$, so then $R = \sqrt[4]{N^3(N+2)c}$.

PROOF. Let $u_R$ (respectively $v_R$) be the solution of $P(B_R)$ (respectively $Q(B_R)$). On the one hand, replacing $\rho$ by $R$ in the proof of Lemma 3.1, one obtains $-u_R'(R) = R/N$. Then a simple calculation shows that

$$u_R(r) = \frac{1}{2N} (R^2 - r^2) \quad \text{for } r \in [0, R[.$$ 

On the other hand, there exists a radial function $v_R$ satisfying

$$\begin{cases}
-v''_R + \frac{N - 1}{r} v'_R = u_R & \text{for } r \in [0, R[,

v_R(R) = 0',

\end{cases}$$

$$-v'_R(R) = \frac{1}{R^{N-1}} \int_0^R s^{N-1} u_R(s) \, ds = \frac{N}{N+2} \left( \frac{R}{N} \right)^3.$$ 

Therefore, since $B_R$ is a solution of $P(c)$, $R = \sqrt[4]{N^3(N+2)c}$.
**Lemma 4.2.** Let $\Omega^*$ be a solution of $P(c)$. Let $\omega \supset \Omega^*$ and let $u_\omega$ (respectively $v_\omega$) be the solution of $P(\omega)$ (respectively $Q(\omega)$). If $\partial \omega \cap \partial \Omega^* \neq \emptyset$ and if $|\nabla u_\omega||\nabla v_\omega| \leq c$ on $\partial \omega \cap \partial \Omega^*$ then $\omega = \Omega^*$.

**Proof.** Suppose by contradiction that $\omega \neq \Omega^*$. Since $\omega \supset \Omega^*$, $\partial \omega \neq \partial \Omega^*$. On the one hand, $\Omega^*$ is a solution of $P(c)$ so the maximum principle implies $0 \leq u_{\Omega^*} \leq u_\omega$ in $\Omega^*$ and $|\nabla u_{\Omega^*}| < |\nabla u_\omega|$ on $\partial \omega \cap \partial \Omega^*$ which is nonempty. On the other hand, one can apply the maximum principle to $v_\omega$ and $v_{\Omega^*}$ and obtain $|\nabla v_{\Omega^*}| < |\nabla v_\omega|$ on $\partial \omega \cap \partial \Omega^*$. Therefore,

$$c = |\nabla u_{\Omega^*}||\nabla v_{\Omega^*}| < |\nabla u_\omega||\nabla v_\omega| \leq c \quad \text{on} \quad \partial \omega \cap \partial \Omega^*,$$

which gives the contradiction. \hfill \square

**Corollary 4.3.** Let $\omega^*$ and $\Omega^*$ be two solutions of $P(c)$. Suppose that $\Omega^* \subset \omega^*$ and $\partial \omega^* \cap \partial \Omega^* \neq \emptyset$, then $\omega^* = \Omega^*$.

**4.2. Shape optimization problems** Let $\Omega^*$ be a solution of $P(c)$. Denote by $B$ the greatest ball contained in $\Omega^*$. Replacing $\Omega$ by $\Omega^*$ in $O_\Omega$ (respectively in $O^{\Omega^*}$) we obtain the definition of $O_{\Omega^*}$ (respectively $O^{\Omega^*}$). Set

$$O_R = \{ D \supset \omega \supset B_R \mid \omega \in O_{B_R} \}.$$

Consider the following functionals:

$$J_1(\omega) = c \frac{R}{N-1} |\partial \omega| - \frac{1}{2} \int_{\omega} u_\omega^2, \quad J_2(\omega) = \frac{R}{N-1} |\partial \omega| - |\omega|,$$

and

$$J_3(\omega) = c |\omega| - \frac{1}{2} \int_{\omega} u_\omega^2.$$

By the Green formula,

$$J_1(\omega) = c \frac{R}{N-1} |\partial \omega| - \frac{1}{2} \int_{\omega} v_\omega$$

and

$$J_3(\omega) = c |\omega| - \frac{1}{2} \int_{\omega} v_\omega.$$

$u_\omega$ and $v_\omega$ are respectively the solutions of $P(\omega)$ and $Q(\omega)$.
Suppose that $N \in \{2, 3\}$. There exists $\Omega_1^* \in \mathcal{O}_{\Omega^*}$ which is of class $C^2$ such that:

1. $J_1(\Omega_1^*) = \min\{J_1(\omega) \mid \omega \in \mathcal{O}_{\Omega^*}\}$ and $u_{\Omega_1^*}$ (respectively $v_{\Omega_1^*}$) is the solution of $P(\Omega_1^*)$ (respectively $Q(\Omega_1^*)$);

2. 
\[
\begin{cases}
|\nabla u_{\Omega_1^*}| |\nabla v_{\Omega_1^*}| \leq c R H_{d_{\Omega_1^*}} & \text{on } \partial \Omega_1^* \cap \partial \Omega^*, \\
|\nabla u_{\Omega_1^*}| |\nabla v_{\Omega_1^*}| = c R H_{d_{\Omega_1^*}} & \text{on } \partial \Omega_1^* \setminus \partial \Omega^*.
\end{cases}
\]

**Proof.** (1) Let $u_D$ be the solution of the Dirichlet problem $P(D)$. By the maximum principle, $0 \leq u_\omega \leq u_D$ so $J_1(\omega) \geq -(1/2) \int_D u_D^2$ and $\inf J_1$ exists. Let $\Omega_n$ be a minimizing sequence in $\mathcal{O}_{\Omega^*}$. Since $\Omega_n \subset D$, then there exist an open set $\Omega_1^*$ and a subsequence of $\Omega_n$ (still denoted by $\Omega_n$) such that $\Omega_n \to H \Omega_1^*$. Now according to (iii) of Theorem 2.7 and Proposition 2.8, $\int_D u_n^2 \chi_{\Omega_n}$ converges to $\int_D u_D^2 \chi_{\Omega_1^*}$ and by item (2) of Lemma 2.15 we get $J_1(\Omega_1^*) \leq \liminf_{n \to +\infty} J_1(\Omega_n)$. Then, according to (iv) of Theorem 2.7, $\Omega_1^* \in \mathcal{O}_{\Omega^*}$; therefore $J_1(\Omega_1^*) = \min_{\omega \in \mathcal{O}_{\Omega^*}} J_1(\omega)$. Now, on the one hand, Proposition 2.8 implies that $u_{\Omega_1^*}$ is the solution of $P(\Omega_1^*)$. On the other hand, Proposition 2.8 together with Proposition 2.14 implies that $v_{\Omega_1^*}$ is the solution of $Q(\Omega_1^*)$.

(2) Since $u_{\Omega^*} = 0$ on $\partial \Omega^*$, (2.3) and 3(a) above imply that
\[
d J_1(\Omega^*, V) = \int_{\partial \Omega_1^*} (c R H_{d_{\Omega_1^*}} - |\nabla u_{\Omega_1^*}| |\nabla v_{\Omega_1^*}|) V \cdot \nu \ d\sigma
\]
for all admissible directions $V$. Thus we obtain (4.1).

**Proposition 4.5.** Suppose that $N \geq 2$. There exists $\Omega_2^* \in \mathcal{O}_{\Omega^*}$ such that:

1. $J_2(\Omega_2^*) = \min\{J_2(\omega) \mid \omega \in \mathcal{O}_{\Omega^*}\}$, $u_{\Omega_2^*}$ (respectively $v_{\Omega_2^*}$) is the solution of $P(\Omega_2^*)$ (respectively $Q(\Omega_2^*)$);

2. if $\Omega_2^* \in \mathcal{O}_{\Omega^*}$ is of class $C^2$ then
\[
\begin{cases}
R H_{d_{\Omega_2^*}} \leq 1 & \text{on } \partial \Omega_2^* \cap \partial \Omega^*, \\
R H_{d_{\Omega_2^*}} = 1 & \text{on } \partial \Omega_2^* \setminus \partial \Omega^*.
\end{cases}
\]

**Proof.** (1) The first assertion is due to Lemma 2.13 and item (2) of Lemma 2.15. Then (vi) of Lemma 2.13 together with Proposition 2.14 gives the continuity with respect to the domains for the Dirichlet problems $P(\Omega_2^*)$ and $Q(\Omega_2^*)$.

(2) Arguing as in the proof of Proposition 3.5 and using (2.2) and (2.3), we obtain (4.2).
Suppose that \( N \in \{2, 3\} \). There exists \( \Omega^{*}_2 \in \mathcal{O}_R \) which is of class \( C^2 \) such that:

1. \( J_3(\Omega^{*}_3) = \min \{ J_3(\omega) \mid \omega \in \mathcal{O}_R \} \) and \( u_{\Omega^*_3} \) (respectively \( v_{\Omega^*_3} \)) is the solution of \( P(\Omega^*_3) \) (respectively \( Q(\Omega^*_3) \));
2. \[
\begin{cases}
|\nabla u_{\Omega^*_3}|/|\nabla v_{\Omega^*_3}| \leq c & \text{on } \partial \Omega^*_3 \cap \partial B_R, \\
|\nabla u_{\Omega^*_3}|/|\nabla v_{\Omega^*_3}| = c & \text{on } \partial \Omega^*_3 \setminus \partial B_R.
\end{cases}
\]

**Proof.** (1) Theorem 2.7 and Proposition 2.8 imply the existence of the minimum \( \Omega^*_3 \). Propositions 2.8 and 2.14 give the continuity with respect to the Dirichlet problems \( P(\Omega^*_3) \) and \( Q(\Omega^*_3) \).

(2) Arguing as in the proof of Proposition 4.4, (2.3) and 3(a) above imply (4.3). \( \square \)

**Remark 4.7.** The \( C^2 \) regularity obtained for \( \Omega^*_2 \) and \( \Omega^*_3 \) is due to [4, Theorem 1.4].

## 4.3. Main results

Let \( \Omega^* \) be a solution of \( P(c) \). By applying the maximum principle to \( (u_{\Omega^*}; v_{\Omega^*}) \) and \( (u_{B_R}; v_{B_R}) \), the proofs of the two first theorems are similar to those of Theorems 3.9 and 3.12.

**Theorem 4.8.** Suppose that \( N = 2 \). Let \( \Omega^*_2 \) be as in Proposition 4.5, so \( \Omega^* = \Omega^*_2 = B_2 \). \( \sqrt{\gamma} \).

**Theorem 4.9.** Suppose that \( N \in \{2, 3\} \). Let \( \Omega^*_3 \) be as in Proposition 4.6, so \( \Omega^*_3 = B_R \).

**Theorem 4.10.** Suppose that \( N \in \{2, 3\} \) and \( R = J\sqrt{N^3(N + 2)c} \). Let \( \Omega^*_1 \) and \( \Omega^*_2 \) be as in Propositions 4.4 and 4.5. If \( \partial \Omega^*_1 \cap \partial \Omega^*_2 \neq \emptyset \) then \( \Omega^*_1 = \Omega^* = \Omega^*_2 = B_R \).

**Proof.** Since \( \Omega^*_2 \subset \Omega^* \subset \Omega^*_1 \), \( \partial \Omega^*_1 \cap \partial \Omega^*_2 \neq \emptyset \) implies that \( \partial \Omega^*_1 \cap \partial \Omega^*_2 \cap \partial \Omega^* \neq \emptyset \). Suppose by contradiction that \( \Omega^*_1 \neq \Omega^* \), then \( \partial \Omega^*_1 \neq \partial \Omega^* \). Using (4.1) and (4.2), the monotonicity of the mean curvature together with the maximum principle imply that

\[
\begin{align*}
c &= |\nabla u_{\Omega^*}|/|\nabla v_{\Omega^*}| < |\nabla u_{\Omega^*_1}|/|\nabla v_{\Omega^*_1}| \leq c R H_{\partial \Omega^*_1} \\
&\leq c R H_{\partial \Omega^*_2} \leq c & \text{on } \partial \Omega^*_1 \cap \partial \Omega^*_2 \cap \partial \Omega^*,
\end{align*}
\]

which is absurd. So \( \Omega^*_1 = \Omega^* \). Therefore (4.3) gives \( c = |\nabla u_{\Omega^*}|/|\nabla v_{\Omega^*}| \leq c R H_{\partial \Omega^*} \) on \( \partial \Omega^* \). Thus \( 1 \leq R H_{\partial \Omega^*} \) which can be combined with (4.2) and the monotonicity of the mean curvature to get \( H_{\partial \Omega^*_2} = 1/R \) on \( \partial \Omega^*_2 \), that is \( \Omega^*_2 \) is a ball with radius \( R \). Since \( R = J\sqrt{N^3(N + 2)c} \), Lemma 4.1 implies that \( \Omega^*_2 \) is a solution of \( P(c) \) and since \( \partial \Omega^* \cap \partial \Omega^*_2 \neq \emptyset \), Corollary 4.3 gives \( \Omega^* = \Omega^*_2 \). \( \square \)

## 5. Other problems

This section is concerned with several overdetermined boundary value problems for which the overdetermined condition is not constant. The aim here is to prove for them the same symmetry result obtained for \( S(k) \) and \( P(c) \).
Theorem 5.1. Let $u_{\Omega}$ be a solution of $P(\Omega)$ such that $(x \cdot \nu)|\nabla u_{\Omega}|^3 = c(N+2)$ on $\partial \Omega$. Let $v_{\Omega}$ be the solution of $Q(\Omega)$. If (i) $|\nabla u_{\Omega}| |\nabla v_{\Omega}| \leq c$ on $\partial \Omega$ or if (ii) $|\nabla u_{\Omega}| |\nabla v_{\Omega}| \geq c$ on $\partial \Omega$ then $\Omega$ is a solution of $P(c)$. As a consequence $\Omega$ is a ball of radius $\sqrt[4]{N^3(N+2)c}$.

Proof. The proof needs the well-known Rellich formula [18], valid for any $v \in C^1(\overline{\Omega}) \cap H^2(\Omega)$,

$$2 \int_{\partial \Omega} (x \cdot \nu) \frac{\partial v}{\partial \nu} d\sigma - \int_{\partial \Omega} (x \cdot \nu)|\nabla v|^2 d\sigma = 2 \int_{\Omega} (x \cdot \nabla v) \Delta v \, dx + (2-N) \int_{\Omega} |\nabla v|^2 \, dx. \tag{5.1}$$

Replacing in (5.1) $v$ by $u_{\Omega}$ and using $\nabla u_{\Omega} = -|\nabla u_{\Omega}|\nu$ on the boundary, we find that

$$\int_{\partial \Omega} (x \cdot \nu)|\nabla u_{\Omega}|^2 d\sigma = -2 \int_{\Omega} (x \cdot \nabla u_{\Omega}) dx + (2-N) \int_{\Omega} |\nabla u_{\Omega}|^2 dx. \tag{5.2}$$

But the Green formula gives

$$\int_{\Omega} (x \cdot \nabla u_{\Omega}) \, dx = -N \int_{\Omega} u_{\Omega} = -N \int_{\Omega} |\nabla u_{\Omega}|^2 \, dx.$$

We then obtain the identity

$$\int_{\partial \Omega} (x \cdot \nu)|\nabla u_{\Omega}|^2 d\sigma = (2+N) \int_{\Omega} u_{\Omega} dx. \tag{5.3}$$

By the Compatibility Condition of the Neumann Problem (CCNP), there exists a $w$ solution of

$$-\Delta w = u_{\Omega} \text{ in } \Omega \quad \text{and} \quad -\frac{\partial w}{\partial \nu} = \frac{1}{N+2} (x \cdot \nu)|\nabla u_{\Omega}|^2 \text{ on } \partial \Omega.$$ 

Put $h = v_{\Omega} - w$. Then $\Delta h = 0$ in $\Omega$ and

$$\frac{\partial h}{\partial \nu} = \frac{\partial v_{\Omega}}{\partial \nu} + \frac{1}{N+2} (x \cdot \nu)|\nabla u_{\Omega}|^2 \text{ on } \partial \Omega. \tag{5.4}$$

Multiplying (5.4) by $\partial u_{\Omega}/\partial \nu$ and using $-\partial u_{\Omega}/\partial \nu = |\nabla u_{\Omega}|$, we obtain

$$\frac{\partial h}{\partial \nu} \frac{\partial u_{\Omega}}{\partial \nu} = \frac{\partial v_{\Omega}}{\partial \nu} \frac{\partial u_{\Omega}}{\partial \nu} - \frac{1}{N+2} (x \cdot \nu)|\nabla u_{\Omega}|^3 \text{ on } \partial \Omega. \tag{5.5}$$

Now $(x \cdot \nu)|\nabla u_{\Omega}|^3 = c(N+2)$ on $\partial \Omega$, so (5.5) becomes

$$\frac{\partial h}{\partial \nu} \frac{\partial u_{\Omega}}{\partial \nu} = \frac{\partial v_{\Omega}}{\partial \nu} \frac{\partial u_{\Omega}}{\partial \nu} - c \text{ on } \partial \Omega. \tag{5.6}$$

Since $\partial u_{\Omega}/\partial \nu = -|\nabla u_{\Omega}| < 0$, (i) or (ii) implies that $\partial h/\partial \nu$ has a constant sign on $\partial \Omega$. But $h$ is harmonic in $\Omega$ so by the Green formula $\int_{\partial \Omega} (\partial h/\partial \nu) = 0$. It then follows that $\partial h/\partial \nu = 0$ and so $(\partial v_{\Omega}/\partial \nu)(\partial u_{\Omega}/\partial \nu) = c$ on $\partial \Omega$, that is $\Omega$ is a solution of $P(c)$ and so it is a ball with radius $\sqrt[4]{N^3(N+2)c}$. \qed
For the following theorems we recall that \( u_\Omega \) (respectively \( v_\Omega \)) is the solution of \( P(\Omega) \) (respectively \( Q(\Omega) \)).

**Theorem 5.2.** Suppose that \( |\nabla u_\Omega||\nabla v_\Omega| = c(x \cdot v) \) on \( \partial \Omega \). If \( (a) \ |\nabla u_\Omega| \leq \sqrt[3]{c(N+2)} \) on \( \partial \Omega \) or if \( (b) \ |\nabla u_\Omega| \geq \sqrt[3]{c(N+2)} \) on \( \partial \Omega \) then \( \Omega \) is a solution of \( S(\sqrt[3]{c(N+2)}) \). As a consequence \( \Omega \) is a ball of radius \( N \sqrt[3]{c(N+2)} \).

**Proof.** If \( |\nabla u_\Omega||\nabla v_\Omega| = c(x \cdot v) \) on \( \partial \Omega \) then (5.5) becomes

\[
\frac{\partial h}{\partial v} \frac{\partial u_\Omega}{\partial v} = (x \cdot v) \left[ c - \frac{1}{N+2} |\nabla u_\Omega|^3 \right] \quad \text{on} \ \partial \Omega. \tag{5.7}
\]

Since \( x \cdot v > 0 \), (5.7) with (a) or with (b) implies that \( \frac{\partial h}{\partial v} = 0 \) on \( \partial \Omega \) and so

\[ |\nabla u_\Omega| = \sqrt[3]{c(N+2)} \quad \text{on} \ \partial \Omega. \]

Therefore \( \Omega \) is a solution of \( S(\sqrt[3]{c(N+2)}) \), that is \( \Omega \) is a ball with radius \( N \sqrt[3]{c(N+2)} \). \( \square \)

**Theorem 5.3.** Suppose that \( (H1) \ |\nabla u_\Omega||\nabla v_\Omega| = (c^3/(N+2))r^4(\partial r/\partial v) \) on \( \partial \Omega \). If \( (1) \ |\nabla u_\Omega| \leq cr \) on \( \partial \Omega \) or if \( (2) \ |\nabla u_\Omega| \geq cr \) on \( \partial \Omega \) then \( |\nabla u_\Omega| = cr \) on \( \partial \Omega \). As a consequence \( \Omega \) is a ball.

**Proof.** Let \( r = |x| \), then \( \Delta r^2 = 2N \). An integration by parts gives

\[
-2N \int_\Omega u_\Omega(x) \, dx = \int_\Omega \nabla (r^2) \cdot \nabla u_\Omega = 2 \int_\Omega r \frac{\partial u_\Omega}{\partial r}. \tag{5.8}
\]

But \( \Delta (r(\partial u_\Omega/\partial r)) = -2 \), so by the Green formula we obtain

\[
\int_\Omega 2u_\Omega - r \frac{\partial u_\Omega}{\partial r} = \int_\Omega -u_\Omega \Delta \left( r \frac{\partial u_\Omega}{\partial r} \right) + r \frac{\partial u_\Omega}{\partial v} \frac{\partial u_\Omega}{\partial v} = \int_{\partial \Omega} r \frac{\partial u_\Omega}{\partial r} \frac{\partial u_\Omega}{\partial v} \, d\sigma.
\]

By (5.8) we get the identity

\[
(N+2) \int_\Omega u_\Omega = \int_{\partial \Omega} r \frac{\partial r}{\partial v} |\nabla u_\Omega|^2 \, d\sigma. \tag{5.9}
\]

Now, as in the proof of Theorem 5.1, the CCNP implies the existence of a function \( w \) solution of the following Neumann problem:

\[
-\Delta w = u_\Omega \quad \text{in} \ \Omega \quad \text{and} \quad -\frac{\partial w}{\partial v} = \frac{1}{N+2} r \frac{\partial r}{\partial v} |\nabla u_\Omega|^2 \quad \text{on} \ \partial \Omega.
\]

As above, if \( h = v_\Omega - w \) then \( h \) is harmonic in \( \Omega \) and

\[
\frac{\partial h}{\partial v} \frac{\partial u_\Omega}{\partial v} = \frac{\partial v_\Omega}{\partial v} \frac{\partial u_\Omega}{\partial v} - \frac{1}{N+2} r \frac{\partial r}{\partial v} |\nabla u_\Omega|^3 \quad \text{on} \ \partial \Omega. \tag{5.10}
\]
or again using (H1),
\[
\frac{\partial h}{\partial v} \frac{\partial u_\Omega}{\partial v} = \frac{1}{N+2} r \frac{\partial r}{\partial v} \left[(cr)^3 - |\nabla u_\Omega|^3\right] \quad \text{on } \partial \Omega.
\] (5.11)

Now arguing as above, (1) (or (2)) allows us to get \( \frac{\partial h}{\partial v} = 0 \) on \( \partial \Omega \). It then follows that \( |\nabla u_\Omega| = cr \) on \( \partial \Omega \) and so \( \Omega \) is some ball \([2]\).

**Theorem 5.4.** Suppose (H2) \((N+2)(x \cdot v)^2|\nabla u_\Omega||\nabla v_\Omega| = (C_0r^2 + C_1)^3 \) on \( \partial \Omega \). If (3) \((x \cdot v)|\nabla u_\Omega| \leq C_0r^2 + C_1\) or if (4) \((x \cdot v)|\nabla u_\Omega| \geq C_0r^2 + C_1\) on \( \partial \Omega \) then \((x \cdot v)|\nabla u_\Omega| = C_0r^2 + C_1\) on \( \partial \Omega \). As a consequence \( \Omega \) is a ball if \(2(NC_0 - 1)\) is not a negative integer while it is an ellipsoid if \(C_0 = 0\).

**Proof.** Applying (H2) to (5.5), we obtain
\[
(N+2)(x \cdot v)^2 \frac{\partial h}{\partial v} \frac{\partial u_\Omega}{\partial v} = (C_0r^2 + C_1)^3 - (x \cdot v)^3 |\nabla u_\Omega|^3 \quad \text{on } \partial \Omega.
\] (5.12)

Arguing as above, (3) (or (4)) implies that \( \frac{\partial h}{\partial v} = 0 \) on \( \partial \Omega \). Then
\[
(x \cdot v)|\nabla u_\Omega| = C_0r^2 + C_1 \quad \text{on } \partial \Omega,
\]
which gives the conclusion \([15]\). \(\square\)

**Remark 5.5.** Suppose that \(u_\Omega\) is the solution of \(P(\Omega)\). If \(|\nabla u_\Omega| = cr\) on \(\partial \Omega\) then one can prove that \(\Omega\) is a ball. In fact, replacing \(|\nabla u_\Omega|\) by \(cr\) in (5.9), one can obtain
\[
c^2(N+2) \int_\Omega r^2 = \frac{c^2}{4} \int_\Omega \Delta(r^4) = \frac{c^2}{4} \int_{\partial \Omega} \frac{\partial (r^4)}{\partial v} = c^2 \int_{\partial \Omega} r^3 \frac{\partial r}{\partial v} = (N+2) \int_\Omega u_\Omega.
\]

So
\[
\int_\Omega u_\Omega = c^2 \int_\Omega r^2.
\]

Put \(u = u_\Omega\) and \(\phi = u_i u_i - c^2 r^2\). A simple calculation \([2]\) shows that \(\Delta \phi \geq 0\) in \(\Omega\) and \(\int_\Omega \phi = \int_\Omega u_\Omega - c^2 \int_\Omega r^2 = 0\). Then the maximum principle gives \(\phi \equiv 0\) in \(\Omega\). One can derive that \(u\) is radially symmetric and \(\Omega\) is a ball.

**Remark 5.6.** Because of the use of the compatibility condition of the Neumann problem, Theorems 5.1, 5.2, 5.3 and 5.4 can be extended to the divergence operator \(\text{div}(a(x)Du(x))\) \([14]\).

**References**


