

UTILITY OPTIMIZATION IN CONGESTED QUEUEING NETWORKS

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Abstract

We consider a multiclass single-server queueing network as a model of a packet switching network. The rates packets are sent into this network are controlled by queues which act as congestion windows. By considering a sequence of congestion controls, we analyse a sequence of stationary queueing networks. In this asymptotic regime, the service capacity of the network remains constant and the sequence of congestion controllers act to exploit the network's capacity by increasing the number of packets within the network. We show that the stationary throughput of routes on this sequence of networks converges to an allocation that maximises aggregate utility subject to the network's capacity constraints. To perform this analysis, we require that our utility functions satisfy an exponential concavity condition. This family of utilities includes weighted α -fair utilities for $\alpha > 1$.

Keywords: Utility optimization; queueing network; quasi-reversible; large deviation; congestion window; congestion control

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1. Introduction

We are interested in proving how end-to-end control can provide a mechanism where routes receive a transfer rate that is the solution to a network wide utility optimization problem. Using differential equations to model network dynamics, authors have demonstrated network utility optimization; see [14]. In this paper we consider a sequence of stationary queueing networks and demonstrate network utility optimization. We note here that an accessible heuristic derivation of the papers' main results can be found in Section 2.

As a method of allocating resources and introducing fairness, Kelly [11] considered utility optimization of the form

$$\text{maximise } \sum_{i \in \mathcal{I}} U_i(\Lambda_i) \quad (1)$$

$$\text{subject to } \sum_{\{i: j \in i\}} \Lambda_i \leq C_j, \quad j \in \mathcal{J}, \quad (2)$$

$$\text{over } \Lambda_i \geq 0, \quad i \in \mathcal{I}, \quad (3)$$

where U_i is an increasing strictly concave utility function with derivative satisfying $U_i'(\Lambda_i) \rightarrow \infty$ as $\Lambda_i \rightarrow 0$. We call this optimization problem the *system problem*. In words, it states that one should maximise the aggregate utility of the transfer rate received by users of a network's routes (1) subject to the network's capacity constraints (2). But, the utility preferences of users

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are separate and unknown to the functional operation of a communication network. Similarly, users do not explicitly know the network's topology and exact capacity. So, the network's behaviour and preferences of users must be decomposed.

In [11], Kelly also introduced *proportional fairness* as the unique solution to the optimization problem

$$\text{maximise } \sum_{i \in \mathcal{I}} \bar{m}_i \log \Lambda_i \quad (4)$$

$$\text{subject to } \sum_{\{i: j \in i\}} \Lambda_i \leq C_j, \quad j \in \mathcal{J}, \quad (5)$$

$$\text{over } \Lambda_i \geq 0, \quad i \in \mathcal{I}. \quad (6)$$

We call this optimization problem the *network problem* or the *proportionally fair optimization problem*. Kelly [11] considered the combined solution of the network problem and the following *user problems*: for each $i \in \mathcal{I}$,

$$\text{maximise } U_i \left(\frac{\bar{m}_i}{q_i} \right) - \bar{m}_i \quad (7)$$

over $\bar{m}_i \geq 0$.

This combined solution was considered under the relation

$$\bar{m}_i = \Lambda_i q_i, \quad i \in \mathcal{I}, \quad (8)$$

where $q_i = \sum_{j \in i} q_j$ and $(q_j: j \in \mathcal{J})$ are the Lagrange multipliers associated with the network problem. Theorem 2 of [11] showed that, under (8), the combined solution of the network and user problems yielded the solution to the system problem.

This result was constructed to suggest an end-to-end argument for providing optimization and fairness across a communication network. The result provided a method for decomposing the system problem into a user problem that is independent of the network structure, except through parameter q_i , and a network problem that is independent of users' preferences, except through parameters $\bar{m} = (\bar{m}_i: i \in \mathcal{I})$. Interpreted in the context of a communication network, this separated the preferences of users performing end-to-end communication and the network's preferred optimal behaviour. In [11] the solution was interpreted as setting prices $(q_j: j \in \mathcal{J})$ for sending traffic through the network. With these prices, each user, $i \in \mathcal{I}$, chooses an amount of money, \bar{m}_i , he/she is willing to pay per unit of time for network resources. From this, the user receives an amount of bandwidth $\Lambda_i = \bar{m}_i/q_i$.

By construction, this result considers a static model and the end-to-end argument performed by users is implicit. Subsequent work has successfully used differential equations to add dynamics to this notion of optimization and decomposition [7], [12], [14], [21]. Other work has considered the form of utility optimization achieved by different protocols [15], [18], [23]. Authors have also considered stochastic models of flow across a network [2], [3], [16]. More recently, authors have explicitly used the queue length as a mechanism to provide utility-based fairness [5], [22]. We now describe the approach taken in this paper.

In 1979, Schweitzer [20] studied approximations of closed multiclass queueing networks and considered how asymptotic conditions on such networks might satisfy the Kuhn–Tucker conditions for proportionally fair optimization. In 1989, Kelly [10] studied approximations of closed queueing networks and, by an analogous analysis, considered a similar optimization formulation. In 1999, Massoulié and Roberts [17] studied a fluid-type queueing model and

used these same Kuhn–Tucker conditions to deduce proportional fairness. Using large deviations and heavy traffic analysis, the recent work of Walton [24] and Kelly *et al.* [13] have provided rigorous formalisations of the relationship between closed queueing networks and proportional fairness. The large deviations connection between multiclass queueing networks and proportional fairness gives a much more literal meaning to the network problem (4)–(6).

In light of this work, the first key observation of this paper is that we can express the network problem (4)–(6) in terms of the asymptotic behaviour of a multiclass queueing network. Given this, $(\bar{m}_i : i \in \mathcal{I})$ must represent the number of packets on each route of this network. We let $(\bar{m}_i : i \in \mathcal{I})$ be recorded and controlled by congestion windows. In this paper, for each route, a congestion window sends packets along its route at a rate which is a function of the number of packets on that route. We call this system of queues and congestion windows a *queueing system*.

We wish to associate the congestion windows in a queueing system with the user problem (7). The second key observation of this paper is that the user problem (7) is reminiscent of a Legendre–Fenchel transform. Results like the Gärtner–Ellis theorem [4] relate the large deviations behaviour of sequences of random variables to the Legendre–Fenchel transform of their log moment generating function. So, by controlling the number of packets in transfer in a network, under a large deviations asymptotic, this Legendre–Fenchel transform observation can be used to associate a utility function with a sequence of congestion windows. We interpret each congestion window in this sequence as a congestion controller’s response to the level of congestion within the queueing system. We discuss this point further in Section 2 and, more formally, in Section 3.

A third observation is that in our queueing system statement (8) corresponds to the statement of Little’s law, i.e. the expected number of packets across routes equals the expected sojourn time of packets through the multiclass queueing network multiplied by the rate packets are sent into the multiclass queueing network. Thus, these three observations now place the work of [11] in the context of a network of queues with congestion windowing.

The above three observations prescribe the limiting regime and theoretical approach of this paper. We consider a sequence of stationary queueing systems. The capacity of queues within this network is constant, but the control policy used by each congestion controller is altered in this sequence. Controllers sequentially increase the number of packets in flight within the network. Large deviations results are then employed to show that the corresponding sequence of stationary distributions asymptotically concentrates probability on a point that maximises network utility.

We now locate the main results of this paper. The queues and congestion windows considered in our queueing system are quasi-reversible. Theorem 1 describes standard reversibility results that can be applied to calculate these queueing systems’ stationary distribution. We consider a sequence of stationary queueing systems associated with a particular sequence of congestion windows. We study the large deviations of the stationary distribution of this sequence of queueing systems. In this large deviations limit our above three observations are realised and, thus, these queueing systems are asymptotically able to execute the analysis of Kelly [11, Theorem 2]. We find in our analysis that we require each utility function to satisfy an *exponential concavity condition*, that is, the map $\lambda \mapsto U_i(e^\lambda)$ is concave. Theorem 3 states that this sequence of queueing systems obeys a large deviations principle with rate function given by the system problem

$$\max_{\Lambda \in \mathbb{R}_+^{\mathcal{I}}} \sum_{i \in \mathcal{I}} U_i(\Lambda_i) \quad \text{subject to} \quad \sum_{\{i: j \in i\}} \Lambda_i \leq C_j, \quad j \in \mathcal{J}.$$

From this, in Theorem 5 we prove that the stationary rate of packet transfer converges to the solution of the system problem (1)–(3).

1.1. Organization

In Section 2 we present a heuristic derivation of the main results of this paper. This section should be quick and accessible to most readers.

In Section 3 we define our model of a congestion window. We study its stationary distribution when operating in isolation of a network. We study the large deviations behaviour of a sequence of congestion windows and we consider how to associate a utility function with this sequence.

In Section 4 we consider a well-known model of a multiclass queue. We study these queues’ stationary distribution when operating in isolation of a network and we study their large deviations behaviour.

In Section 5 we connect the congestion windows of Section 3 with the queues of Section 4 to form a *queueing system*. Similarly, we study the stationary distribution of this queueing system and the large deviations of a sequence of queueing systems. In addition, we study the dual relationship between the state and the flow through the queueing system, and we prove that the stationary throughput of packets in the queueing system converges to the solution of the system problem.

1.2. Notation

Let the finite set \mathcal{J} index the set of queues in a network. Let $J = |\mathcal{J}|$. A route through the network is a nonempty subset of queues. Let \mathcal{I} be the set of routes, and let $I = |\mathcal{I}|$. For each route $i = \{j_1^i, \dots, j_{k_i}^i\}$, we associate an order $(j_1^i, \dots, j_{k_i}^i)$. This is the order in which route- i packets traverse their route. Also, we define the set of queue-route incidences, $\mathcal{K} = \{(j, i) \in \mathcal{J} \times \mathcal{I} : j \in i\}$, and let $K = |\mathcal{K}|$.

Our multiclass queueing system will process packets through a network of queues and congestion windows. For each route, there is a congestion window. Let the *window size* \bar{m}_i denote the number of packets in congestion window $i \in \mathcal{I}$. The window size is the number of sent but not yet acknowledged packets on route i . Each queue j processes packets from routes $i \in \mathcal{I}$ with $j \in i$. Let m_{ji} denote the number of route- i packets at queue j . We also define

$$m_j = \sum_{\{i:j \in i\}} m_{ji}, \quad j \in \mathcal{J}, \tag{9}$$

to be the number of packets at queue j . As each congestion window records the number of packets in transfer in the queueing network,

$$\bar{m}_i = \sum_{j \in i} m_{ji}, \quad i \in \mathcal{I}. \tag{10}$$

Although we will often use $\bar{m} = (\bar{m}_i : i \in \mathcal{I}) \in \mathbb{Z}_+^I$ and $m = (m_{ji} : (j, i) \in \mathcal{K}) \in \mathbb{Z}_+^K$ to refer to the numbers of packets at congestion windows and queues, in Sections 3.2, 4.2, and 5.2, when applying large deviations results, we will use $\bar{m} \in \mathbb{R}_+^I$ and $m \in \mathbb{R}_+^K$ to refer to the proportions of packets in the queueing system at different congestion windows and queues.

In addition, for $m \in \mathbb{Z}_+^K$, we define

$$\binom{m_j}{m_{ji} : i \ni j} = \frac{m_j!}{\prod_{\{i : j \in i\}} (m_{ji}!)}$$

For vectors $x \in \mathbb{R}_+^D$ and $\phi \in \mathbb{R}_+^D$, we define $\|x\| = (\sum_{d=1}^D x_d^2)^{1/2}$, the Euclidean norm of x ; $\lfloor x \rfloor = (\lfloor x_d \rfloor : d = 1, \dots, D)$, the low integer part of each component of x ; and $\phi \cdot x = \sum_{d=1}^D \phi_d x_d$, the dot product of ϕ and x .

2. A heuristic derivation of results

In this section we give a heuristic derivation of the results in this paper. The heuristic is an adaption of the arguments of [10], [17], and [20] applied to a modified version of [11]. The formal proof, given in subsequent sections, follows a similar approach to [24].

The Karush–Kuhn–Tucker (KKT) conditions for the system problem (1)–(3) are that there exist positive multipliers ($q_j : j \in \mathcal{J}$) and positive rates ($\Lambda_i : i \in \mathcal{I}$) such that

$$\begin{aligned} U_i'(\Lambda_i) &= \sum_{j \in i} q_j, & i \in \mathcal{I}, \\ q_j \left(C_j - \sum_{\{i: j \in i\}} \Lambda_i \right) &= 0, & j \in \mathcal{J}, \\ \sum_{\{i: j \in i\}} \Lambda_i &\leq C_j, & j \in \mathcal{J}. \end{aligned}$$

Consider a queueing system with routes \mathcal{I} and queues \mathcal{J} , as notated in the previous section. For each route i , packets are injected into the network, traverse the queues in their route in order $j_1^i, \dots, j_{k_i}^i$, and then leave. Let q_j denote the stationary sojourn time of a packet at queue j , let m_{ji} denote the stationary number of route- i packets in transfer at queue j , and let Λ_i denote the stationary sending rate of route- i packets into the network. By Little's law,

$$\Lambda_i q_j = m_{ji} \quad \text{for all } i \in \mathcal{I} \text{ and } j \in i.$$

Summing over the queues on route i and rearranging gives

$$\frac{\bar{m}_i}{\Lambda_i} = \sum_{j \in i} q_j \quad \text{for all } i \in \mathcal{I}, \quad (11)$$

where \bar{m}_i is the stationary number of packets in transfer on route i . Suppose that a congestion window for each route i injects packets into the network at a rate that is a function of \bar{m}_i , i.e. $g_i(\bar{m}_i) = \Lambda_i$. If we chose $g_i(\cdot)$ so that

$$\bar{m}_i = g_i^{-1}(\Lambda_i) = \Lambda_i U_i(\Lambda_i) \quad (12)$$

then (11) implies that

$$U_i'(\Lambda_i) = \sum_{j \in i} q_j, \quad i \in \mathcal{I}. \quad (13)$$

Assuming that all queues are stable, we know that

$$\sum_{\{i: j \in i\}} \Lambda_i \leq C_j \quad \text{for all } j \in \mathcal{J}. \quad (14)$$

If equality (14) is strict then sojourn times will be small, i.e. $q_j \approx 0$. Thus, approximately,

$$q_j \left(C_j - \sum_{\{i: j \in i\}} \Lambda_i \right) = 0 \quad \text{for all } j \in \mathcal{J}. \quad (15)$$

Also,

$$q_j \geq 0 \text{ for all } j \in \mathcal{J} \text{ and } \Lambda_i \geq 0 \text{ for all } i \in \mathcal{I}. \tag{16}$$

Interpreting $(q_j : j \in \mathcal{J})$ as Lagrange multipliers, (13)–(16) are precisely the above KKT conditions for the system problem. So the rates $(\Lambda_i : i \in \mathcal{I})$ and the sojourn times $(q_j : j \in \mathcal{J})$, implicitly defined by the queueing network, solve the system problem (1)–(3).

The remainder of this paper is concerned with making the above arguments rigorous. To make conditions like (15) strict, we will require the network to have a high level of congestion. We achieve this by considering a sequence of congestion controls $g_i^{(c)}$, whilst keeping C_j , each queues’ service capacity, fixed. As c increases, the controllers increase the number of packets in transfer and, thus, more aggressively exploit the network capacity. We think of this increase in congestion in an analogous way to the increase rules employed by the transmission control protocol in Internet communications. We will apply large deviations techniques to a sequence of stationary network models, indexed by c , and prove that probability concentrates on the system optimal operating point.

2.1. Choices of $g_i(\bar{m}_i)$ and $G_i(\bar{m}_i)$

Later, we will apply large deviations results to certain stationary distributions. For this reason, it will be convenient to express g_i through the theory of convex duality. Let G_i be a differentiable function such that $g_i(\bar{m}) = e^{G_i(\bar{m})}$. Then (12) gives

$$G_i'^{-1}(\log \Lambda_i) = \Lambda_i U_i'(\Lambda_i) \text{ or } G_i'^{-1}(\lambda_i) = e^{\lambda_i} U_i'(e^{\lambda_i}). \tag{17}$$

Now if F^* is the Legendre–Fenchel transform of a concave differentiable function F , i.e. $F^*(\bar{m}) = -\max_{\lambda} \{F(\lambda) - \lambda \bar{m}\}$, then, by construction, F^* has a derivative that inverts the derivative of F , i.e. the inverse of $F^{*'}(\cdot)$ equals $F'(\cdot)$. Applying this to (17), with $G_i(\bar{m}) = F^*(\bar{m})$ and $F(\lambda_i) = U_i(e^{\lambda_i})$, we have

$$G_i(\bar{m}) = -\max_{\lambda_i} \{U_i(e^{\lambda_i}) - \bar{m}_i \lambda_i\} = -\max_{\Lambda_i \geq 0} \{U_i(\Lambda_i) - \bar{m}_i \log \Lambda_i\}. \tag{18}$$

It is precisely by this function G_i that we are able to replace $\bar{m}_i \log \Lambda_i$, the network problem summand, with the system user’s utility $U_i(\Lambda_i)$.

So that duality can invert this operation, that is, $U_i(e^{\lambda}) = G_i^*(\lambda)$, we require that $U_i(e^{\lambda})$ is a concave function of λ .

We note that the optimization used to derive G_i , (18), is different to the user problem (7) derived by Kelly [11]. This is, in essence, because we choose the rate packets to be injected based on the number of packets in transfer. This is in contrast to the user problem of Kelly, which could be interpreted as choosing the number of packets in transfer given the current network delay (or round trip time) of packets. A theoretical advantage to our approach is that queueing networks which inject packets based on the number of packets in transfer are more analytically tractable. In particular, quasi-reversibility results can be applied to explicitly give the stationary distribution of such networks.

In the next two sections we construct the components of these queueing networks: congestion windows and multiclass queues. In Section 5 we connect these components to form a sequence of queueing systems that will execute the above heuristic.

3. Congestion windows

Congestion windows keep a record of the number of sent but not yet acknowledged packets in a queueing network. The models of congestion windows considered here are reversible and,

thus, lend well to product-form results when incorporated into a network [1], [10]. We will later connect these congestion windows to the routes of a network of multiclass single-server queues.

When in isolation of a network, we define a congestion window at congestion level c as a continuous-time Markov chain $(\bar{M}_i^{(c)}(t) : t \in \mathbb{R}_+)$ on \mathbb{Z}_+ with transition rates

$$q(\bar{m}_i, \bar{m}'_i) = \begin{cases} g_i^{(c)}(\bar{m}_i) & \text{if } \bar{m}'_i = \bar{m}_i + 1, \\ \Lambda_i & \text{if } \bar{m}'_i = \bar{m}_i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

The transition $\bar{m}_i \mapsto \bar{m}_i + 1$ is thought of as *injecting a packet* into a network, and, for $\bar{m}_i > 0$, a transition $\bar{m}_i \mapsto \bar{m}_i - 1$ is thought of as *acknowledging a packet* that has been transferred. Thus, we think of \bar{m}_i as recording the number of packets currently in transfer. For the purposes of this paper, it will be convenient to define

$$\Lambda_i = e^{\lambda_i} \quad \text{and} \quad g_i^{(c)}(\bar{m}_i) = \exp\{G_i^{(c)}(\bar{m}_i + 1) - G_i^{(c)}(\bar{m}_i)\},$$

where $\lambda_i \in \mathbb{R}$ and $G_i^{(c)} : \mathbb{R}_+ \mapsto [-\infty, \infty)$ is a strictly concave function. Thus, a congestion window is defined by a constant λ_i and a function $(c, \bar{m}_i) \mapsto G_i^{(c)}(\bar{m}_i)$.

3.1. Reversibility and stationary behaviour

We now collect a result about the stationary behaviour of congestion windows.

Proposition 1. *A stationary congestion window is reversible with stationary distribution*

$$\pi_i^{(c)}(\bar{m}_i) = \pi_i^{(c)}(0) \exp\{G_i^{(c)}(\bar{m}_i) - \lambda_i \bar{m}_i\}, \quad \bar{m}_i \in \mathbb{Z}_+. \tag{19}$$

Proof. The result is immediate from the detail balance equations

$$\begin{aligned} \pi_i^{(c)}(\bar{m}_i) &= \exp\{G_i^{(c)}(\bar{m}_i) - G_i^{(c)}(\bar{m}_i - 1) - \lambda_i\} \pi_i^{(c)}(\bar{m}_i - 1) \\ &= \dots \\ &= \exp\{G_i^{(c)}(\bar{m}_i) - \lambda_i \bar{m}_i\} \pi_i^{(c)}(0). \end{aligned}$$

3.2. Large deviations

We think of the pair (c, \bar{m}_i) as giving the state of a congestion controller, where L is a level of congestion and \bar{m}_i is the number of packets in transfer. As noted above, if c is fixed, a congestion controller describes a reversible Markov chain. In this section we study the stationary distribution of a sequence of these Markov chains as $c \rightarrow \infty$. As c increases, the stationary number of packets in transfer will increase. This can be interpreted as the congestion controller more aggressively exploiting the network’s capacity. In this section we use large deviation type arguments to study where the probability concentrates under this limit.

For each $c \in \mathbb{N}$, let $\bar{M}_i^{(c)}$ be a stationary congestion window defined by λ_i and $G_i^{(c)}(\cdot)$, where $G_i^{(c)}(k) = cG_i(k/c + d_i^{(c)}/c)$ for a strictly concave function G_i , differentiable on $(0, \infty)$ with derivatives taking all values in $(-\infty, \infty)$ and with $\{d_i^{(c)}\}_{c \in \mathbb{N}}$ a bounded sequence with values in \mathbb{R} . We define a convex function $G_i^* : \mathbb{R} \rightarrow \mathbb{R}$ from G_i with Legendre–Fenchel transform

$$G_i^*(\lambda_i) = \max_{\bar{m}_i \in \mathbb{R}_+} \{G_i(\bar{m}_i) - \lambda_i \bar{m}_i\}.$$

We also define

$$\bar{m}_{\lambda_i}^* = \operatorname{argmax}_{\bar{m}_i \in \mathbb{R}_+} \{G_i(\bar{m}_i) - \lambda_i \bar{m}_i\}.$$

In the following proposition we use large deviation arguments to identify the most probable state of the sequence of random variables $\bar{M}^{(c)}$, $c \in \mathbb{N}$.

Proposition 2. (a) *We have*

$$\lim_{c \rightarrow \infty} \frac{1}{c} \log \sum_{k=0}^{\infty} \exp\{G_i^{(c)}(k) - \lambda_i k\} = G_i^*(\lambda_i), \quad \lambda_i \in \mathbb{R}_+.$$

(b) *For $\bar{m}_i \in (0, \infty)$ and a bounded sequence $\{\bar{\sigma}_i^{(c)}\}_{c \in \mathbb{N}}$ such that $c\bar{m}_i + \bar{\sigma}_i^{(c)} \in \mathbb{Z}_+$,*

$$\lim_{c \rightarrow \infty} \frac{1}{c} \log \mathbb{P}(\bar{M}^{(c)} = c\bar{m}_i + \bar{\sigma}_i^{(c)}) = G_i(\bar{m}_i) - \lambda_i \bar{m}_i - G_i^*(\lambda_i).$$

Proof. To prove (a), we wish to verify the principle of the largest term for this infinite sum [6, Lemma 2.1]. First we show the upper bound. Let $d = \max_c |d_i^{(c)}|$. Since G_i is strictly concave, for all $\delta > 0$, letting

$$\varepsilon = \frac{1}{\delta} \min\{G_i(\bar{m}_{\lambda_i}^*) - G_i(\bar{m}_{\lambda_i}^* - \delta) + \delta, G_i(\bar{m}_{\lambda_i}^*) - G_i(\bar{m}_{\lambda_i}^* + \delta) - \delta\} > 0,$$

we have, for all $\bar{m} \in \mathbb{R}_+$,

$$G_i(\bar{m}_i) - \lambda_i \bar{m}_i \leq G_i^*(\lambda_i) - \varepsilon(\bar{m}_i - \bar{m}_{\lambda_i}^*) \mathbf{1}[\bar{m}_i \geq \bar{m}_{\lambda_i}^* + \delta] + \varepsilon(\bar{m}_i - \bar{m}_{\lambda_i}^*) \mathbf{1}[\bar{m}_i \leq \bar{m}_{\lambda_i}^* - \delta].$$

Therefore, applying the above inequality and comparing the following with a geometric sum, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \exp\{G_i^{(c)}(k) - \lambda_i k\} &= \sum_{k=0}^{\infty} \exp\left\{c\left(G_i\left(\frac{k}{c} + \frac{d_i^{(c)}}{c}\right) - \lambda_i \frac{k}{c}\right)\right\} \\ &\leq \exp\{cG_i^*(\lambda_i) + d_i^{(c)}\} \left(2(c\delta + d + 1) + 2 \sum_{k \in \mathbb{Z}_+} e^{c\varepsilon(k/c - \bar{m}^*)}\right) \\ &\leq \exp\{cG_i^*(\lambda_i) + d_i^{(c)}\} \left(2(c\delta + d + 1) + 2 \frac{e^{-(c\delta - d - 1)}}{1 - e^{-\varepsilon}}\right). \end{aligned} \tag{20}$$

Hence, as the first term in the brackets dominates,

$$\limsup_{c \rightarrow \infty} \frac{1}{c} \log \sum_{k=0}^{\infty} \exp\{G_i^{(c)}(k) - \lambda k\} \leq G_i^*(\lambda).$$

By proving the lower bound for (a) we can simultaneously verify (b). Using the terminology of (b), for all $\bar{m}_i \in (0, \infty)$,

$$\begin{aligned} G_i(\bar{m}_i) - \lambda_i \bar{m}_i &= \lim_{c \rightarrow \infty} \frac{1}{c} \log \exp\left\{cG_i\left(\bar{m}_i + \frac{\bar{\sigma}_i^{(c)}}{c} + \frac{d_i^{(c)}}{c}\right) - \lambda_i c\bar{m}_i + \bar{\sigma}_i^{(c)}\right\} \\ &\leq \liminf_{c \rightarrow \infty} \frac{1}{c} \log \sum_{k=0}^{\infty} \exp\{G_i^{(c)}(k) - \lambda_i k\}. \end{aligned} \tag{21}$$

Taking $\bar{m}_i = \bar{m}_{\lambda_i}^*$ and $\bar{\sigma}_i^{(c)} = \lfloor \bar{m}_{\lambda_i}^* \rfloor - \bar{m}_{\lambda_i}^*$ verifies (a). Given (a) and (21), we have (b).

We now emphasise the following duality between the state and the flow in congestion windows. Suppose that, as $c \rightarrow \infty$, the number of packets in transfer is approximately $c\bar{m}_i^*$ for some $\bar{m}_i^* \in (0, \infty)$. Thus, the flow out of the congestion window is approximately

$$g_i^{(c)}(c\bar{m}_i^*) \approx \exp\left\{\frac{G_i(\bar{m}_i^* + 1/c) - G_i(\bar{m}_i^*)}{1/c}\right\} \rightarrow e^{G_i'(\bar{m}_i^*)} \quad \text{as } c \rightarrow \infty.$$

When stationary, the average outward flow of packets from the congestion window equals the average inward flow. Thus, we have $G_i'(\bar{m}_i^*) = \lambda_i$, or, in other words,

$$\bar{m}_i^* = \operatorname{argmax}_{\bar{m}_i \in \mathbb{R}_+} \{G_i(\bar{m}_i) - \lambda_i \bar{m}_i\}.$$

By this duality and the correct choice of G_i (as discussed in Section 2.1), we can control the throughput of packets from the congestion window so that it optimises a utility function.

3.3. Utility optimization

The rate packets are acknowledged by a congestion window is $\Lambda_i = e^{\lambda_i}$; thus, the utility associated with λ_i is $U_i(e^{\lambda_i})$. If we wish to maximise the system problem, we must define G_i through the following user problem:

$$-G_i(\bar{m}_i) = \max_{\lambda_i \in \mathbb{R}} \{U_i(e^{\lambda_i}) - \bar{m}_i \lambda_i\}, \quad \bar{m}_i \in \mathbb{R}_+. \tag{22}$$

Similarly, by the duality of Legendre–Fenchel transforms, we may define U_i from G_i by

$$U_i(e^{\lambda_i}) = \min_{\bar{m}_i \in \mathbb{R}_+} \{\lambda_i \bar{m}_i - G_i(\bar{m}_i)\} = -G_i^*(\lambda_i), \quad \lambda_i \in \mathbb{R}. \tag{23}$$

The function $U_i(e^{\lambda_i})$ must be concave as $G_i^*(\lambda_i)$ is convex. Thus, in order to optimise a utility function U_i , we require the following assumption.

Assumption 1. *The utility function U_i is exponentially concave, that is, the map $\lambda_i \mapsto U_i(e^{\lambda_i})$ is strictly concave on \mathbb{R} .*

We also collect the differentiability assumptions that we make on G_i .

Assumption 2. *We assume that G_i defined by (22) is strictly concave and (continuously) differentiable on $(0, \infty)$ with derivatives taking all values in \mathbb{R} .*

Remark 1. (Weighted α -fairness, $\alpha > 1$.) The weighted α -fair family of utility functions considered by Mo and Walrand [18] corresponds to the aggregate utility of users with iso-elastic utility, that is, utilities

$$U_i(\Lambda_i) = \begin{cases} \frac{w_i \Lambda_i^{1-\alpha}}{1-\alpha} & \text{if } w_i \in \mathbb{R}_+, \alpha > 0, \alpha \neq 1, \\ w_i \log \Lambda_i & \text{if } w_i \in \mathbb{R}_+, \alpha = 1. \end{cases}$$

The weighted α -fair class has proved popular as it contains proportional fairness ($\alpha = w_i = 1$) and transmission control protocol fairness ($\alpha = 2, w_i = 1/T_i^2$), and converges to maximum throughput ($\alpha \rightarrow 0, w_i = 1$) and max–min fairness ($\alpha \rightarrow \infty, w_i = 1$).

We can easily verify that $U_i(\Lambda_i)$ is exponentially concave for $\alpha > 1$ and that

$$G_i(\bar{m}_i) = \frac{\bar{m}_i}{1-\alpha} \log \frac{\bar{m}_i}{w_i} - \frac{\bar{m}_i}{1-\alpha} = \frac{1}{1-\alpha} \int_0^{\bar{m}_i} \log \frac{x}{w_i} dx.$$

So U_i and G_i satisfy the two assumptions above for $\alpha > 1$. Thus, our results apply for weighted α -fairness for $\alpha > 1$. The case $\alpha = 1$ is considered in [24]. For weighted α -fairness, $\alpha > 1$, convenient forms for $g_i^{(c)}$ and $G_i^{(c)}$ are

$$g_i^{(c)}(\bar{m}_i) = \left(\frac{cw_i}{\bar{m}_i}\right)^{1/(\alpha-1)} \quad \text{and} \quad e^{G_i^{(c)}(\bar{m}_i)} = \frac{(cw_i)^{\bar{m}_i/(\alpha-1)}}{(\bar{m}_i!)^{1/(\alpha-1)}}.$$

4. Multiclass single-server queues

We define a multiclass single-server queue. These queues are quasi-reversible and are described in [8]. When connected to form a network, these queues process packets over different routes of a queueing system. Congestion windows regulate the number of packets present on each route. After defining these multiclass queues in this section, we define this queueing system in Section 5.

A queue $j \in \mathcal{J}$ is fed packets from classes of the set of routes $\{i \in \mathcal{I} : j \in i\}$. Packets occupy different positions within the queue and have an exponentially distributed mean 1 service requirement. Given there are $m_j \in \mathbb{Z}_+$ packets at queue j , packets occupy positions $1, \dots, m_j$. The total service devoted to these packets is $C_j \in (0, \infty)$. This fixed service is then divided amongst the different positions in the queue. A proportion $\gamma_j(l, m_j)$ of this capacity is devoted to the packet at position $l = 1, \dots, m_j$. Upon completing its service, a packet at position l will depart the queue and the packets at positions $l + 1, \dots, m_j$ will move to positions $l, \dots, m_j - 1$, respectively. In this section we assume that packets of route i arrive at queue j as a Poisson process with rate Λ_i . Upon arrival, a packet will move to position $l = 1, \dots, m_j + 1$ with probability $\delta_j(l, m_j + 1)$, and packets that were in positions l, \dots, m_j will move to positions $l + 1, \dots, m_j + 1$.

Let $s^j = (i_1^j, \dots, i_{m_j}^j) \in \mathcal{I}^{m_j}$ for $m_j > 0$ give the state of queue j . Let $T_{\cdot, (j,l)}^i$ denote the arrival of a class- i packet to position l in queue j , and let $T_{(j,l), \cdot}^i$ denote the departure of a class- i packet from position l . Thus, the state of this queue forms a continuous-time Markov chain with transition rates given by

$$q(s^j, s'^j) = \begin{cases} \Lambda_i \delta_j(l, m_j + 1) & \text{for } s'^j = T_{\cdot, (j,l)}^i s^j, l = 1, \dots, m_j + 1, \\ C_j \gamma_j(l, m_j) & \text{for } s'^j = T_{(j,l), \cdot}^i s^j, i_l^j = i, l = 1, \dots, m_j, \\ 0 & \text{otherwise.} \end{cases}$$

4.1. Quasi-reversibility and stationary behaviour

Multiclass single-server queues are known to be quasi-reversible and their stationary distribution is well understood [1], [8].

Proposition 3. *Provided that the stability condition*

$$\sum_{\{i : j \in i\}} \Lambda_i < C_j, \quad j \in \mathcal{J},$$

holds, a multiclass single-server queue $j \in \mathcal{J}$ is quasi-reversible and $(M_{ji} : i \in \mathcal{I}, j \in i)$, the stationary number of route- i packets at queue j , has distribution

$$P(M_{ji} = m_{ji} \text{ for all } i \ni j) = \frac{C_j - \sum_{\{i : j \in i\}} \Lambda_i}{C_j} \binom{m_j}{m_{ji} : i \ni j} \prod_{\{i : j \in i\}} \left(\frac{\Lambda_i}{C_j}\right)^{m_{ji}} \quad (24)$$

for $m_{ji} \in \mathbb{Z}_+$ and each $i \in \mathcal{I}$ such that $j \in i$, where $m_j = \sum_{\{i : j \in i\}} m_{ji}$.

Proof. Let M_j be the number of packets at queue j . Since the queue does not discriminate between different packet classes, M_j is a reversible Markov chain and, thus, its departures prior to time t form a Poisson process independent of the Poisson process of arrivals after time t . By thinning these Poisson processes with probability $\Lambda_i / \sum_{\{j: r \in j\}} \Lambda_r$, we obtain the arrival and departure processes of route- i packets and, thus, deduce that the queue is quasi-reversible. The stationary distribution of M_j is geometric with parameter $(1/C_j) \sum_{\{r: j \in r\}} \Lambda_i$. Combining this with the same thinning argument and summing over states s^j , giving $(m_{ji} : i \in \mathcal{I}, j \in i)$, we obtain (24). For more details, see [1, Section IV.4] or [8, Theorem 3.1].

4.2. Large deviations

We study the stationary distribution (24) when the number of packets of each class is increased proportionately, but all other queueing parameters are kept constant. We collect some large deviations results on these queues in this regime [19], [24].

Proposition 4. For $j \in \mathcal{J}$, let $(M_{ji} : i \in \mathcal{I}, j \in i)$ have distribution (24).

- (a) Let $m^j = (m_{ji} : i \in \mathcal{I}, j \in i)$ and $\sigma^{j,(c)} = (\sigma_{ji}^{(c)} : i \in \mathcal{I}, j \in i)$, $c \in \mathbb{N}$, be such that $m_{ji} \in \mathbb{R}_+$, $\sup_{c \in \mathbb{N}} \|\sigma^{j,(c)}\| < \infty$, and $cm_{ji} + \sigma_{ji}^{(c)} \in \mathbb{Z}_+$ for all i with $j \in i$. Then

$$\lim_{c \rightarrow \infty} \frac{1}{c} \log P(M_{ji} = cm_{ji} + \sigma_{ji}^{(c)} \text{ for all } i \ni j) = -\beta_j(m^j),$$

where

$$\beta_j(m^j) = \sum_{\substack{\{i: j \in i\} \\ m_j > 0}} m_{ji} \log \frac{m_{ji} C_j}{m_j \Lambda_i}.$$

- (b) The function $\beta_j(m^j)$ is continuous, convex, and such that

$$\inf_{m^j \geq 0} \beta_j(m^j) = \begin{cases} 0 & \text{if } \sum_{\{i: j \in i\}} \Lambda_i \leq C_j, \\ -\infty & \text{otherwise.} \end{cases} \tag{25}$$

Proof. (a) Define $\sigma_j^{(c)} = \sum_{\{i: j \in i\}} \sigma_{ji}^{(c)}$. By Stirling’s formula,

$$\lim_{c \rightarrow \infty} \frac{1}{c} \log(cm_{ji} + \sigma_{ji}^{(c)})! = m_{ji} \log m_{ji} - m_{ji}.$$

Thus,

$$\begin{aligned} & \lim_{c \rightarrow \infty} \frac{1}{c} \log P(M_{ji} = cm_{ji} + \sigma_{ji}^{(c)}, i \ni j) \\ &= \lim_{c \rightarrow \infty} \frac{1}{c} \left(\log(cm_j + \sigma_j^{(c)})! - \sum_{\{i: j \in i\}} \log(cm_{ji} + \sigma_{ji}^{(c)})! + \sum_{\{i: j \in i\}} (cm_{ji} + \sigma_{ji}^{(c)}) \log \frac{\Lambda_i}{C_j} \right) \\ &= - \lim_{c \rightarrow \infty} \sum_{\substack{\{i: j \in i\} \\ m_j > 0}} m_{ji} \log \frac{(m_{ji} + \sigma_{ji}^{(c)}/c) C_j}{(m_j + \sigma_j^{(c)}/c) \Lambda_i} \\ &= -\beta_j(m^j). \end{aligned}$$

(b) Taking $x \log x = 0$ for $x = 0$, $x \log x$ is continuous on \mathbb{R}_+ ; thus, β_j is continuous. We now prove (25). For two probability distributions p and q with the same support on \mathcal{K} , we

define their relative entropy to be $D(p||q) = \sum_s p_s \log(p_s/q_s)$. In particular, we can verify that

$$\min_p D(p||q) = 0, \tag{26}$$

which is minimised when $p = q$. Thus, taking $p = (m_{ji}/m_j : i \in \mathcal{I}, j \in i)$ and $q = (\Lambda_i / \sum_{\{r: j \in r\}} \Lambda_r : i \in \mathcal{I}, j \in i)$,

$$\begin{aligned} \inf_{m^j \geq 0} \beta_j(m^j) &= \inf_{m^j > 0} m_j \left(\sum_{\{i: j \in i\}} \frac{m_{ji}}{m_j} \log \frac{m_{ji} \sum_{\{r: j \in r\}} \Lambda_r}{m_j \Lambda_i} \right) + m_j \log \frac{C_j}{\sum_{\{r: j \in r\}} \Lambda_r} \\ &= \inf_{m^j > 0} m_j \log \frac{C_j}{\sum_{\{r: j \in r\}} \Lambda_r} \\ &= \begin{cases} 0 & \text{if } \sum_{\{r: j \in r\}} \Lambda_r \leq C_j, \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Finally, to show that $\beta_j(\cdot)$ is convex, we can verify that

$$\beta_j(m^j) = \max_{\{\phi_i: j \in i\}} \sum m_{ji} \phi_i \quad \text{subject to} \quad \sum_{\{i: j \in i\}} \Lambda_i e^{\phi_i} \leq C_j \quad \text{over } \phi_i \in \mathbb{R}, i \ni j. \tag{27}$$

Thus, $\beta_j(\cdot)$ is expressible as a Legendre–Fenchel transform and so is convex.

5. A queueing system

We now connect the queues and congestion windows discussed in the last two sections to form a network. The interior of this network consists of a set of multiclass queues with routes defined over these queues. We think of this as a simple model of a packet switching network. The number of packets in transfer on a route are determined by congestion windows, as defined in Section 3. The queueing system, defined by these queues and congestion windows, models the congestion control of a packet-switched communication network with a fixed number of document transfers in progress.

In this section, using quasi-reversibility results and fixed c , we calculate the stationary distribution of this queueing system. For these stationary distributions, we increase c , the congestion level of each congestion window, and, using large deviation techniques, we calculate where the probability concentrates. We show that the most likely state is given an entropy optimization and its dual is the system problem. In particular, we prove that $\Lambda^{(c)}$, the stationary rate packets traverse the network, converges to the solution of the system problem (1)–(3).

We consider a network of queues indexed by the set \mathcal{J} and congestion windows indexed by the set of routes \mathcal{I} . Each queue $j \in \mathcal{J}$ will process packets as described in Section 4, but transitions between queues will be prompted by transitions within the queueing system (rather than by a Poisson process). Similarly, each congestion window will send packets into the network as described in Section 3, but transitions $\bar{m}_i \mapsto \bar{m}_i - 1$ will be prompted by the successful transfer of a packet in the queueing network.

If we associate the route order $(j_1^i, \dots, j_{k_i}^i)$ to route i , a packet injected by congestion window i will prompt an arrival at queue j_1^i . Also, the departure of a route- i packet from queue $j_k^i, k = 1, \dots, k_i - 1$, will prompt an arrival at queue j_{k+1}^i , and, similarly, a departure of a route- i packet from the final queue $j_{k_i}^i$ will prompt a transition $\bar{m}_i \rightarrow \bar{m}_i - 1$ at congestion window i . In this way, packets are sent into the network, transferred along their route, and finally acknowledged.

We more explicitly describe the state of our queueing system as follows. As in Section 4, we let $s^j = (i_1^j, \dots, i_{m_j}^j)$ record the state of queue j and $s = (s^j : j \in \mathcal{J})$ record the state of our queueing system. Also, we let \bar{m}_i record the state of congestion window i and let $\bar{m} = (\bar{m}_i : i \in \mathcal{I})$ record the state of our congestion windows. Finally, let $\underline{s} = (s, \bar{m})$ record the explicit state of our queueing system. We define the transitions in our queueing network as follows. Let $\underline{s} \mapsto T_{(j,l),\underline{s}}^i$ define the transition corresponding to a route- i packet injected by congestion window i and arriving at position l in queue j . Let $\underline{s} \mapsto T_{(j,l),(j',l'),\underline{s}}^i$ denote the departure of a route- i packet from position l of queue j which arrives at position l' in queue j' . Finally, let $\underline{s} \mapsto T_{(j,l),\underline{s}}^i$ denote the departure of a route- i packet from position l of queue j which is then acknowledged at congestion window i .

We define our queueing system corresponding to congestion level c to be a continuous-time Markov chain with the following transition rates:

$$q(\underline{s}, \underline{s}') = \begin{cases} g_i^{(c)}(\bar{m}_i)\delta_j(l, m_j + 1) & \text{for } \underline{s}' = T_{(j,l),\underline{s}}^i, j = j_1^i, l = 1, \dots, m_j + 1, \\ C_j\gamma_j(l, m_j)\delta_{j'}(l', m_{j'} + 1) & \text{for } \underline{s}' = T_{(j,l),(j',l'),\underline{s}}^i, j = j_k^i, \\ & j' = j_{k+1}^i, k = 1, \dots, k_i - 1, \\ & i_l^j = i, l' = 1, \dots, m_{j'} + 1, \\ C_j\gamma_j(l, m_j) & \text{for } \underline{s}' = T_{(j,l),\underline{s}}^i, j = j_{k_i}^i, i_l^j = i, \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

As in Section 4, we let $m = (m_{ji} : (j, i) \in \mathcal{K})$ record the number of route- i packets at queue j . Also, recall the expressions for m_j and \bar{m}_i , (9) and (10), the numbers of packets in each queue and in each congestion window. Finally, recall that we defined $G_i^{(c)}$ from $g_i^{(c)}$ by the relation $g_i^{(c)}(\bar{m}_i) = \exp\{G_i^{(c)}(\bar{m}_i + 1) - G_i^{(c)}(\bar{m}_i)\}$.

5.1. Quasi-reversibility and stationary behaviour

From Proposition 1 and Proposition 3, our queueing system consists of a network of quasi-reversible nodes. Thus, as considered in [9], networks of quasi-reversible nodes have a stationary distribution that is described by multiplying distributions (19) and (24). We prove this in the following theorem.

Theorem 1. *For a stationary queueing system defined by rates (28), let $M = (M_{ji} : (j, i) \in \mathcal{K})$ record the stationary number of packets of each route at each queue. Then M has distribution*

$$P(M = m) = \frac{1}{B_{G^{(c)}}} \prod_{j \in \mathcal{J}} \binom{m_j}{m_{ji} : i \ni j} \frac{1}{C_j^{m_j}} \prod_{i \in \mathcal{I}} e^{G_i^{(c)}(\bar{m}_i)}, \quad m \in \mathbb{Z}_+^K, \tag{29}$$

where

$$B_{G^{(c)}} = \sum_{m \in \mathbb{Z}_+^K} \prod_{j \in \mathcal{J}} \binom{m_j}{m_{ji} : i \ni j} \frac{1}{C_j^{m_j}} \prod_{i \in \mathcal{I}} e^{G_i^{(c)}(\bar{m}_i)}.$$

Note that the distribution $P(M = m)$ is not a product-form stationary distribution because we require the constraint $\bar{m}_i = \sum_{j \in i} m_{ji}$ for all $i \in \mathcal{I}$.

Proof of Theorem 1. A good candidate for the time reversal of this queueing system is defined by rates $\tilde{q}(\cdot, \cdot)$, where packets follow route i in reverse order, i.e. $(j_1^i, \dots, j_{k_i}^i)$, where queues $j \in \mathcal{J}$ operate at capacity C_j with $\tilde{\delta}(l, m_j) = \gamma_j(l, m_j)$ and $\tilde{\gamma}_j(l, m_j) = \delta_j(l, m_j)$,

and where, as before, congestion window $i \in \mathcal{I}$ sends packets into the network at rate $g_i^{(c)}(\bar{m}_i) = \exp\{G_i^{(c)}(\bar{m}_i + 1) - G_i^{(c)}(\bar{m}_i)\}$. We show that

$$\pi^{(c)}(\underline{s}) = \prod_{j \in \mathcal{J}} \frac{1}{C_j^{m_j}} \prod_{i \in \mathcal{I}} e^{G_i^{(c)}(\bar{m}_i)}$$

forms an invariant measure for our explicit Markov chain description. We verify Kelly’s lemma [8, Theorem 1.13] for our three types of transition: packet injections from congestion windows, transitions between queues, and acknowledgements at congestion windows.

For a packet injected by a congestion window,

$$\frac{q(\underline{s}, T_{\cdot, (j,l)}^i \underline{s})}{\tilde{q}(T_{\cdot, (j,l)}^i \underline{s}, \underline{s})} = \frac{g_i^{(c)}(\bar{m}_i) \delta_j(l, m_j + 1)}{C_j \gamma_j(l, m_j + 1)} = \frac{\pi^{(c)}(T_{\cdot, (j,l)}^i \underline{s})}{\pi^{(c)}(\underline{s})}$$

for $j = j_1^i$ and $l = 1, \dots, m_j + 1$. For a transition between queues,

$$\frac{q(\underline{s}, T_{(j,l)(j',l')}^i \underline{s})}{\tilde{q}(T_{(j,l)(j',l')}^i \underline{s}, \underline{s})} = \frac{C_j \gamma_j(l, m_j) \delta_{j'}(l', m'_j + 1)}{C_{j'} \tilde{\gamma}_{j'}(l, m_j + 1) \tilde{\delta}_j(l, m_j)} = \frac{\pi^{(c)}(T_{(j,l)(j',l')}^i \underline{s})}{\pi^{(c)}(\underline{s})}$$

for $j = j_k^i, j' = j_{k+1}^i, k = 1, \dots, k_i - 1, i_1^j = i$, and $l' = 1, \dots, m_{j'} + 1$. For an acknowledgement at a congestion window i ,

$$\frac{q(\underline{s}, T_{(j,l), \cdot}^i \underline{s})}{\tilde{q}(T_{(j,l), \cdot}^i \underline{s}, \underline{s})} = \frac{C_j \gamma_j(l, m_j)}{g_i^{(c)}(\bar{m}_i - 1) \tilde{\delta}_j(l, m_j)} = \frac{\pi^{(c)}(T_{(j,l), \cdot}^i \underline{s})}{\pi^{(c)}(\underline{s})}$$

for $j = j_{k_i}^i$ and $i = i_1^j$. We note that the transition intensity of the reversed process agrees with that of the forward process

$$\tilde{q}(\underline{s}) = - \sum_{\underline{s}'} \tilde{q}(\underline{s}, \underline{s}') = \sum_{i \in \mathcal{I}} g_i^{(c)}(\bar{m}_i) + \sum_{\{j: m_j > 0\}} C_j = q(\underline{s}).$$

This verifies Kelly’s lemma and, thus, $\pi^{(c)}(\underline{s})$ is an invariant measure. Note that $B_{G^{(c)}}$ is expressible as

$$B_{G^{(c)}} = \sum_{m \in \mathbb{Z}_+^K} \prod_{j \in \mathcal{J}} \left[\binom{m_j}{m_{j_i}: i \ni j} \prod_{\{i: j \in i\}} \left(\frac{e^{\lambda_i}}{C_j} \right)^{m_{j_i}} \right] \prod_{i \in \mathcal{I}} \exp\{G_i^{(c)}(\bar{m}_i) - \lambda_i \bar{m}_i\}$$

for all $\lambda \in \mathbb{R}^I$. By Assumption 2, the function $\bar{m}_i \mapsto G_i^{(c)}(\bar{m}_i) - \lambda_i \bar{m}_i$ is bounded from above for all λ_i . Applying this upper bound and choosing $\lambda \in \mathbb{R}^I$ such that $\sum_{j \in i} e^{\lambda_i} < C_j$, the sum for $B_{G^{(c)}}$ is finite. Thus, summing over states of invariant measure, $\pi^{(c)}(\underline{s})$ gives the stationary distribution (29).

5.2. Large deviations

We now study the large deviations behaviour of our stationary queueing systems as $c \rightarrow \infty$. As c increases, the congestion windows increase the number of packets within the queueing system, and we study its large deviations behaviour. Once again, we think of each congestion controller as attempting to exploit the network capacity by increasing c and congesting the

network. We will relate the most likely state in our queueing system to the solution of the system problem.

We use the same notation from Sections 3 and 4. We also suppose that Assumptions 1 and 2 hold. As in Section 3, we consider a sequence of congestion windows defined by

$$G_i^{(c)}(k) = cG_i\left(\frac{k}{c} + \frac{d_i^{(c)}}{c}\right) \quad \text{for } i \in \mathcal{I} \text{ and } c \in \mathbb{N}.$$

Here G_i is expressible in terms of the utility function U_i given in (23). We define the function

$$\beta_{G,\lambda}(m, \tilde{m}) = \sum_{\{(j,i) \in \mathcal{K} : m_{ji} > 0\}} m_{ji} \log \frac{m_{ji} C_j}{m_j e^{\lambda_i}} - \sum_{i \in \mathcal{I}} \{G_i(\tilde{m}_i) - \lambda_i \tilde{m}_i\} \quad (30)$$

for $m \in \mathbb{R}_+^K$, $\tilde{m} \in \mathbb{R}_+^I$, and $\lambda_i \in \mathbb{R}^I$. We use the shorthand $\beta_G = \beta_{G,\lambda}$ for $\lambda = 0$ and the shorthand $\beta_G(m) = \beta_G(m, \tilde{m})$ when $\tilde{m}_i = \sum_{j \in \mathcal{I}} m_{ji}$ for all $i \in \mathcal{I}$. We define

$$\beta_G^* = \min \sum_{\{(j,i) \in \mathcal{K} : m_{ji} > 0\}} m_{ji} \log \frac{m_{ji} C_j}{m_j} - \sum_{i \in \mathcal{I}} G_i(\tilde{m}_i) \quad (31)$$

$$\text{subject to } \sum_j m_{ji} = \tilde{m}_i, \quad i \in \mathcal{I}, \text{ over } m \in \mathbb{R}_+^K, \tilde{m} \in \mathbb{R}_+^I. \quad (32)$$

For each fixed c , consider a stationary queueing system defined by rates (28) with congestion windows defined by $G_i^{(c)}(\cdot)$, $i \in \mathcal{I}$. Let $M^{(c)} = (M_{ji}^{(c)} : (j, i) \in \mathcal{K})$ record the stationary distribution of the number of packets on each route at each queue in our queueing system, (29). In the following theorem we characterise the large deviations of our sequence of queueing systems.

Theorem 2. *The sequence $M^{(c)}/c$, $c \in \mathbb{N}$, obeys a large deviations principle with good rate function $\beta_G(\cdot) - \beta_G^*$. That is, for all $D \subset \mathbb{R}_+^K$,*

$$\begin{aligned} - \inf_{m \in D^o} (\beta_G(m) - \beta_G^*) &\leq \liminf_{c \rightarrow \infty} \mathbf{P}^{(c)}\left(\frac{M^{(c)}}{c} \in D\right) \\ &\leq \limsup_{c \rightarrow \infty} \mathbf{P}^{(c)}\left(\frac{M^{(c)}}{c} \in D\right) \\ &\leq - \inf_{m \in D} (\beta_G(m) - \beta_G^*). \end{aligned}$$

Proof. Assuming that $\lambda \in \mathbb{R}^I$ satisfies

$$\sum_{\{i : j \in \mathcal{I}\}} e^{\lambda_i} < C_j \quad \text{for all } j \in \mathcal{J},$$

we define a product-form stationary distribution on $\mathbb{Z}_+^K \times \mathbb{Z}_+^I$ with

$$\begin{aligned} \tilde{\mathbf{P}}^{(c)}(M^{(c)} = m, \tilde{M}^{(c)} = \tilde{m}) &= \frac{1}{\tilde{B}^{(c)}} \prod_{j \in \mathcal{J}} \binom{m_j}{m_{ji} : i \ni j} \prod_{\{i : j \in \mathcal{I}\}} \frac{e^{\lambda_i m_{ji}}}{C_j^{m_{ji}}} \\ &\times \prod_{i \in \mathcal{I}} \exp\{G_i^{(c)}(\tilde{m}_i) - \lambda_i \tilde{m}_i\} \quad m \in \mathbb{Z}_+^K, \tilde{m} \in \mathbb{Z}_+^I, \quad (33) \end{aligned}$$

where

$$\tilde{B}^{(c)} = \prod_{j \in \mathcal{J}} \frac{C_j}{C_j - \sum_{\{i: j \in i\}} e^{\lambda_i}} \prod_{i \in \mathcal{I}} \left(\sum_{k=0}^{\infty} \exp\{G_i^{(c)}(k) - \lambda_i k\} \right).$$

Note that, by (20), $\tilde{B}^{(c)}$ is finite. Note that $P^{(c)}$ is expressible in terms of $\tilde{P}^{(c)}$ through the conditional probability

$$P^{(c)}(M^{(c)} = m) = \tilde{P}^{(c)}\left(M^{(c)} = m \mid \tilde{M}_i^{(c)} = \sum_{j \in i} M_{ji}^{(c)}, i \in \mathcal{I}\right)$$

for $m \in \mathbb{Z}_+^K$. By proving large deviations results about $\tilde{P}^{(c)}$ we are able to prove a large deviations principle for $P^{(c)}$. First, by Proposition 4, Proposition 2(a), and the definition of U_i , (23), we have

$$\lim_{c \rightarrow \infty} \frac{1}{c} \log \tilde{B}^{(c)} = - \sum_{i \in \mathcal{I}} U_i(e^{\lambda_i}).$$

Thus, by Proposition 2 and Proposition 4, for $m \in \mathbb{R}_+^K$ and $\tilde{m} \in \mathbb{R}_+^I$ with bounded sequences $\sigma^{(c)} \in \mathbb{R}_+^K$ and $\tilde{\sigma}^{(c)} \in \mathbb{R}_+^I$, $c \in \mathbb{N}$, such that $cm + \sigma^{(c)} \in \mathbb{Z}_+^K$ and $c\tilde{m} + \tilde{\sigma}^{(c)} \in \mathbb{Z}_+^I$, we have

$$-\beta_{G,\lambda}(m, \tilde{m}) + \sum_{i \in \mathcal{I}} U_i(e^{\lambda_i}) = \lim_{c \rightarrow \infty} \frac{1}{c} \log \tilde{P}^{(c)}(M^{(c)} = cm + \sigma^{(c)}, \tilde{M}^{(c)} = c\tilde{m} + \tilde{\sigma}^{(c)}).$$

Take $E \subset \mathbb{R}_+^K \times \mathbb{R}_+^I$, either open or, more generally, such that, for all $(m, \tilde{m}) \in E$, there exists a sequence as described above with $cm + \sigma^{(c)} \in \mathbb{Z}_+^K$ and $c\tilde{m} + \tilde{\sigma}^{(c)} \in \mathbb{Z}_+^I$ for which $(m + \sigma^{(c)}/c, \tilde{m} + \tilde{\sigma}^{(c)}/c) \in E$ eventually as $c \rightarrow \infty$. Then

$$\begin{aligned} - \inf_{(m, \tilde{m}) \in E} \beta_{G,\lambda}(m, \tilde{m}) &= - \inf_{(m, \tilde{m}) \in E} \lim_{c \rightarrow \infty} \frac{1}{c} \log \tilde{P}^{(c)}(M^{(c)} = cm + \sigma^{(c)}, \tilde{M}^{(c)} = c\tilde{m} + \tilde{\sigma}^{(c)}) \\ &\leq \liminf_{c \rightarrow \infty} \frac{1}{c} \log \tilde{P}^{(c)}\left(\left(\frac{M^{(c)}}{c}, \frac{\tilde{M}^{(c)}}{c}\right) \in E\right). \end{aligned} \tag{34}$$

This gives us a large deviations lower bound for $\tilde{P}^{(c)}$. We prove the upper bound by using the Gärtner–Ellis theorem [4, p. 44]. We study the moment generating function of $\tilde{P}^{(c)}$ for $\theta \in \mathbb{R}^K$ and $\phi \in \mathbb{R}^I$:

$$\begin{aligned} &E \exp\{\theta \cdot M^{(c)} + \phi \cdot \tilde{M}^{(c)}\} \\ &= \begin{cases} \prod_{j \in \mathcal{J}} \left(\frac{C_j - \sum_{\{i: j \in i\}} e^{\lambda_i}}{C_j - \sum_{\{i: j \in i\}} \exp\{\lambda_i + \theta_{ji}\}} \right) \\ \times \prod_{i \in \mathcal{I}} \left(\frac{\sum_{k=0}^{\infty} \exp\{G_i^{(c)}(k) - (\lambda_i - \phi_i)k\}}{\sum_{k=0}^{\infty} \exp\{G_i^{(c)}(k) - \lambda_i k\}} \right) & \text{if } \sum_{\{i: j \in i\}} \exp\{\lambda_i + \theta_{ji}\} < C_j, j \in \mathcal{J}, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, combining Proposition 2(a) and Proposition 4(b),

$$\begin{aligned} F(\theta, \phi) &= \lim_{c \rightarrow \infty} \frac{1}{c} \log E \exp\left\{\theta \cdot \frac{M^{(c)}}{c} + \phi \cdot \frac{\tilde{M}^{(c)}}{c}\right\} \\ &= \begin{cases} \sum_{i \in \mathcal{I}} U_i(e^{\lambda_i}) - U_i(\exp\{\lambda_i - \phi_i\}) & \text{if } \sum_{\{i: j \in i\}} e^{\lambda_i + \theta_{ji}} < C_j, j \in \mathcal{J}, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, F has Legendre–Fenchel transform

$$\begin{aligned}
 F^*(m, \bar{m}) &= \max_{\theta \in \mathbb{R}^K, \phi \in \mathbb{R}^I} \sum_{ji} m_{ji} \theta_{ji} + \sum_i \tilde{m}_i \phi_i + \sum_i (U_i(\exp\{\lambda_i - \phi_i\}) - U_i(e^{\lambda_i})) \\
 &\quad \text{(subject to } \sum_{i: j \in i} \exp\{\lambda_i + \theta_{ji}\} < C_j, j \in \mathcal{J}) \\
 &= \max_{\theta \in \mathbb{R}^K} \left\{ \sum_{ji} m_{ji} \theta_{ji} : \sum_i \exp\{\lambda_i + \theta_{ji}\} < C_j, j \in \mathcal{J} \right\} \\
 &\quad + \sum_{i \in \mathcal{I}} \max_{\phi_i \in \mathbb{R}} \{U_i(\exp\{\lambda_i + \phi_i'\}) - \tilde{m}_i(\phi_i' + \lambda_i)\} + \sum_i \lambda_i \tilde{m}_i - \sum_i U_i(e^{\lambda_i}) \\
 &= \sum_{ji} m_{ji} \log \frac{m_{ji} C_j}{m_j e^{\lambda_i}} - \sum_i (G_i(\tilde{m}_i) - \lambda_i \tilde{m}_i) - \sum_i U_i(e^{\lambda_i}).
 \end{aligned}$$

In the second equality we collected terms and substituted $\phi_i' = -\phi_i$ for $i \in \mathcal{I}$. In the third equality we applied (27) to the first maximisation and the user problem (22) to the second maximisation. From this, the Gärtner–Ellis theorem [4, p. 44] proves that, for all closed sets $E \subset \mathbb{R}_+^K \times \mathbb{R}_+^I$,

$$\limsup_{c \rightarrow \infty} \frac{1}{c} \log \tilde{P}^{(c)} \left(\left(\frac{M^{(c)}}{c}, \frac{\tilde{M}^{(c)}}{c} \right) \in E \right) \leq - \inf_{(m, \bar{m}) \in E} \left(\beta_{G, \lambda}(m, \bar{m}) - \sum_i U_i(e^{\lambda_i}) \right). \tag{35}$$

In particular, we are interested in the closed set $\bar{E} = \{(m, \tilde{m}) : \sum_{j \in i} m_{ji} = m_i, i \in \mathcal{I}\}$. Note that, if $m \in \mathbb{R}_+^K$ and the sequence $\sigma^{(c)} \in \mathbb{R}_+^I, c \in \mathbb{N}$, is such that $cm + \sigma^{(c)} \in \mathbb{Z}_+^I$, and defining $\bar{\sigma}_i^{(c)} = \sum_{j \in i} \sigma_{ji}^{(c)}$, then $(cm + \sigma^{(c)}, c\bar{m} + \bar{\sigma}^{(c)}) \in \bar{E}$. So we may apply lower bound (34) to \bar{E} and also upper bound (35) to this choice of \bar{E} . Hence, we have

$$\lim_{c \rightarrow \infty} \frac{1}{c} \log \tilde{P}^{(c)} \left(\sum_{j \in i} M_{ji}^{(c)} = \tilde{M}_i^{(c)}, i \in \mathcal{I} \right) = -\beta_G^* - \sum_i U_i(e^{\lambda_i}). \tag{36}$$

Put otherwise, we have, for the normalising constant $B_{G^{(c)}}$,

$$\lim_{c \rightarrow \infty} \frac{1}{c} \log B_{G^{(c)}} = -\beta_G^*. \tag{37}$$

From (33), combining (36) with the large deviations upper bound (35), for all closed sets $D \in \mathbb{R}_+^K$, letting $D' = \{(m, \bar{m}) : m \in D\} \subset \bar{E}$, we have

$$\begin{aligned}
 &\limsup_{c \rightarrow \infty} \frac{1}{c} \log P^{(c)} \left(\frac{M^{(c)}}{c} \in D \right) \\
 &= \limsup_{c \rightarrow \infty} \left(\frac{1}{c} \log \tilde{P}^{(c)} \left(\left(\frac{M^{(c)}}{c}, \frac{\tilde{M}^{(c)}}{c} \right) \in D' \right) - \frac{1}{c} \log \tilde{P} \left(\sum_j M_{ji}^{(c)} = \tilde{M}_i^{(c)}, i \in \mathcal{I} \right) \right) \\
 &\leq - \inf_{m \in D} \beta_G(m) + \beta_G^*.
 \end{aligned}$$

This proves the large deviations upper bound for $P^{(c)}$. The lower bound follows similarly by combining (36) with lower bound (34).

5.3. Duality between state and flow

An important phenomenon we find from our large deviation analysis is that the limiting state of our queueing system and the limiting flow through our queueing system are dual. We demonstrate here that the dual form of the optimization problem (31)–(32) found in our large deviations analysis is exactly the system problem (1)–(3).

Theorem 3. *We have*

$$\beta_G^* = \min_{m \in \mathbb{R}_+^K, \bar{m} \in \mathbb{R}_+^I} \sum_{\{(j,i) \in \mathcal{K} : m_{ji} > 0\}} m_{ji} \log \frac{m_{ji} C_j}{m_j} - \sum_{i \in \mathcal{I}} G_i(\bar{m}_i) \quad (\text{subject to } \sum_{j \in \mathcal{I}} m_{ji} = \bar{m}_i, i \in \mathcal{I}) \tag{38}$$

$$= \max_{\Lambda \in \mathbb{R}_+^I} \sum_{i \in \mathcal{I}} U_i(\Lambda_i) \quad (\text{subject to } \sum_{\{i : j \in \mathcal{I}\}} \Lambda_i \leq C_j, j \in \mathcal{J}). \tag{39}$$

Moreover, the vector $(m^*, \bar{m}^*) \in \mathbb{R}_+^K \times \mathbb{R}_+^I$ optimises (38) and $\Lambda^* \in \mathbb{R}_+^I$ optimises (39) if and only if

$$\sum_{j \in \mathcal{I}} m_{ji}^* = \bar{m}_i^* \quad \text{for all } i \in \mathcal{I}, \quad \sum_{\{i : j \in \mathcal{I}\}} \Lambda_i^* \leq C_j \quad \text{for all } j \in \mathcal{J},$$

$$m_{ji}^* \Lambda_i^* = m_{ji}^* C_j \quad \text{for all } (j, i) \in \mathcal{K}, \tag{40}$$

$$e^{G'_i(\bar{m}_i^*)} = \Lambda_i^* \quad \text{for all } i \in \mathcal{I}. \tag{41}$$

Here G'_i is the derivative of the function G_i .

Proof. Note that the Lagrangian of optimization problem (38) with Lagrange multipliers $\lambda = (\lambda_i : i \in \mathcal{I})$ is exactly $\beta_{G,\lambda}$ defined in (30). Minimising this Lagrangian we have

$$\begin{aligned} \min_{m \in \mathbb{R}_+^K, \bar{m} \in \mathbb{R}_+^I} \beta_{G,\lambda}(m, \bar{m}) &= \min_{m, \bar{m}} \sum_{ji} m_{ji} \log \frac{m_{ji} C_j}{m_j} - \sum_{i \in \mathcal{I}} G_i(\bar{m}) + \sum_i \lambda_i \left(\bar{m}_i - \sum_{j \in \mathcal{I}} m_{ji} \right) \\ &= \min_{m \in \mathbb{R}_+^K} \left\{ \sum_{ji} m_{ji} \log \frac{m_{ji} C_j}{m_j e^{\lambda_i}} \right\} - \sum_i \max_{\bar{m}_i > 0} \{G_i(\bar{m}_i) - \lambda_i \bar{m}_i\} \\ &= \begin{cases} \sum_i U_i(e^{\lambda_i}) & \text{if } \sum_{\{i : j \in \mathcal{I}\}} e^{\lambda_i} \leq C_j, \\ i & \\ -\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{42}$$

In the final equality we used Proposition 4(b) and definition (23). Thus, the dual of optimization problem (38) is

$$\max_{\lambda \in \mathbb{R}^I} \sum_{i \in \mathcal{I}} U_i(e^{\lambda_i}) \quad \text{subject to} \quad \sum_{\{i : j \in \mathcal{I}\}} e^{\lambda_i} \leq C_j, j \in \mathcal{J}.$$

By the strong duality of optimization problem (38) and (39), expressions (38) and (39) are equal.

Now we demonstrate the only if part of the equivalence using (41). Suppose that Λ^* optimises (39). We now consider how our Lagrangian behaves for $\lambda = (\log \Lambda_i^* : i \in \mathcal{I})$. From (42) we

see that (m^*, \bar{m}^*) must solve

$$\min_{\bar{m}_i > 0} \{G_i(\bar{m}_i) - \bar{m}_i \log \Lambda_i^*\} = -U_i(\lambda_i^*), \quad i \in \mathcal{I}, \tag{43}$$

$$\min_{\{m \in \mathbb{R} : m_j > 0\}} \left\{ m_j \sum_{\{i : j \in i\}} \frac{m_{ji}}{m_j} \log \frac{m_{ji} C_j}{m_j \Lambda_i^*} \right\} = 0, \quad j \in \mathcal{J}. \tag{44}$$

From (43) we see that $e^{G_i(\bar{m}_i^*)} = \Lambda_i^*$. We now consider (44). If $m_j^* = 0$ then (44) is satisfied and so is (40). If $m_j^* > 0$ then, given the relative entropy result (26), (44) can only hold if $\sum_{\{i : j \in i\}} \Lambda_i = C_j$ and $m_{ji}^*/m_j^* = \Lambda_i/C_j$ for all $i \in \mathcal{I}$ such that $j \in i$. Thus, both (40) and (41) hold.

Conversely, if (40) and (41) hold, then substituting m^* and \bar{m}^* into the objective function of optimization problem (38) gives

$$\sum_{(j,i) \in \mathcal{K}, m_j^* > 0} m_{ji}^* \log \frac{m_{ji}^* C_j}{m_j^*} - \sum_{i \in \mathcal{I}} G_i(\bar{m}_i^*) = \sum_{i \in \mathcal{I}} \bar{m}_i^* \log \Lambda_i^* - \sum_{i \in \mathcal{I}} G_i(\bar{m}_i^*) = \sum_{i \in \mathcal{I}} U_i(\Lambda_i^*).$$

In the final equality, we used the Legendre–Fenchel transform expression, (23). Thus, \bar{m}_i^* attains the value $\sum_i U_i(\Lambda_i^*)$ for Λ^* feasible and, thus, by the Lagrangian sufficiency theorem, our solutions are optimal for (38) and (39).

5.4. Most likely state and convergence of throughput

We now study the most likely behaviour of our stationary sequence of queueing systems $M^{(c)}/c$, $c \in \mathbb{N}$. Section 5.2 suggests that our sequence of queueing systems implicitly solves the system problem (1)–(3). This section formalizes this assertion. We show that the sequence of stationary queueing systems $M^{(c)}/c$, $c \in \mathbb{N}$, concentrates on the solutions of the primal optimization problem (31)–(32). From this, we show that the stationary rate which packets traverse the network converges to the allocation, solving the system problem. We define the manifold by

$$\mathcal{M} = \{m \in \mathbb{R}_+^K : m_{ji} C_j = m_j \Lambda_i^*, (j, i) \in \mathcal{K}, G'_i(\bar{m}_i) = \log \Lambda_i^*, i \in \mathcal{I}\},$$

where $(\Lambda_i^* : i \in \mathcal{I})$ is the optimal solution to the system problem (1)–(3). From Theorem 3 we know that \mathcal{M} is the set of solutions to the primal optimization problem (31)–(32). The stationary sequence of queues $M^{(c)}/c$, $c \in \mathbb{N}$, considered in Section 5.2 converges in probability to the set of solutions \mathcal{M} .

Theorem 4. *We have*

$$P^{(c)} \left(\inf_{m \in \mathcal{M}} \left\| \frac{M^{(c)}}{c} - m \right\| \geq \varepsilon \right) \rightarrow 0 \quad \text{as } c \rightarrow \infty.$$

Proof. For $\varepsilon > 0$, let $\mathcal{M}_\varepsilon = \{m \in \mathbb{R}_+^K : \inf_{m' \in \mathcal{M}} \|m - m'\| < \varepsilon\}$. As \mathcal{M} is closed and compact,

$$\begin{aligned} \beta_{G,\varepsilon}^* &:= \min_{m \in \mathbb{R}_+^K, m \in \mathbb{R}_+^I} \sum_{\{(j,i) \in \mathcal{K} : m_j > 0\}} m_{ji} \log \frac{m_{ji} C_j}{m_j} \\ &\quad - \sum_i G_i(\bar{m}_i) \quad (\text{subject to } m \notin \mathcal{M}_\varepsilon, \sum_{j \in i} m_{ji} = \bar{m}_i, i \in \mathcal{I}) \\ &> \beta_G^*, \end{aligned}$$

where β_G^* is given by (31)–(32). Thus, by Theorem 2, for all $\varepsilon > 0$,

$$\limsup_{c \rightarrow \infty} \frac{1}{c} \log P^{(c)} \left(\inf_{m \in \mathcal{M}} \left\| \frac{M^{(c)}}{c} - m \right\| \geq \varepsilon \right) \leq -\beta_{G,\varepsilon}^* + \beta_G^*.$$

Thus, for all $\varepsilon' \in (0, \beta_{G,\varepsilon}^* - \beta_G^*)$, eventually as $c \rightarrow \infty$,

$$P^{(c)} \left(\inf_{m \in \mathcal{M}} \left\| \frac{M^{(c)}}{c} - m \right\| \geq \varepsilon \right) \leq \exp\{-c(\beta_{G,\varepsilon}^* - \beta_G^*) + c\varepsilon'\} \rightarrow 0 \quad \text{as } c \rightarrow \infty.$$

The stationary throughput of route- i packets in our queueing system can be expressed as

$$\Lambda_i^{(c)} = E^{(c)} \exp\{G_i^{(c)}(\bar{M}_i^{(c)} + 1) - G_i^{(c)}(\bar{M}_i)\}, \quad i \in \mathcal{I}.$$

That is, the stationary rate packets are sent into the queueing network by the i th congestion window when it is at congestion level c . We now show that this rate converges to the solution to the system problem $(\Lambda_i^* : i \in \mathcal{I})$. In this sense our sequence of queueing systems implicitly solves the system problem.

Theorem 5. *We have, for $i \in \mathcal{I}$,*

$$\Lambda_i^{(c)} \rightarrow \Lambda_i^* \quad \text{as } c \rightarrow \infty.$$

Proof. We first describe a modification of measure $P^{(c)}$ that will be useful. For fixed $i \in \mathcal{I}$, let $\check{P}^{(c)}$ denote the stationary distribution of a queueing system defined by the same rates as $P^{(c)}$, except that the i th congestion window is defined by $\check{G}_i^{(c)}(\bar{m}_i) = G_i^{(c)}(\bar{m}_i + 1)$. Observe that, for all $m \in \mathbb{Z}_+^K$,

$$\begin{aligned} & E^{(c)}[\exp\{G_i^{(c)}(\bar{M}_i^{(c)} + 1) - G_i^{(c)}(\bar{M}_i^{(c)})\} \mathbf{1}[M^{(c)} = m]] \\ &= \frac{1}{B_G} \prod_{j \in \mathcal{I}} \binom{m_j}{m_{jr} : r \ni j} \frac{1}{C_j^{m_j}} \prod_{r \in \mathcal{I}} e^{\check{G}_r(\bar{m}_r)} \\ &= \frac{B_{\check{G}^{(c)}}}{B_{G^{(c)}}} \check{P}^{(c)}(M^{(c)} = m). \end{aligned} \tag{45}$$

Also, by definition, $G_i^{(c)}(\bar{m}_i) = G_i(\bar{m}_i/c + d_i^{(c)}/c)$; thus, $\check{G}_i^{(c)}$ corresponds to taking $\check{d}_i^{(c)} = d_i^{(c)} + 1$. The precise values of the bounded sequence $\{d_i^{(c)}\}_{c \in \mathbb{N}}$ do not determine any of the large deviations behaviour of $P^{(c)}$; thus, both $P^{(c)}$ and $\check{P}^{(c)}$ exhibit exactly the same large deviations behaviour. Hence, given (37) and Theorem 2, we have

$$\lim_{c \rightarrow \infty} \frac{1}{c} \log \frac{B_{\check{G}^{(c)}}}{B_{G^{(c)}}} = 0, \quad \lim_{c \rightarrow \infty} \check{P}^{(c)} \left(\inf_{m \in \mathcal{M}} \left\| \frac{M^{(c)}}{c} - m \right\| \geq \varepsilon \right) = 0. \tag{46}$$

Now let (m^*, \bar{m}^*) be an optimal solution to the primal optimization problem (31)–(32). By assumption, G_i is differentiable with a continuous derivative at \bar{m}_i^* . Thus, by the mean value theorem, for all $\varepsilon' > 0$, there exist a $\varepsilon > 0$ and $c' \in \mathbb{N}$ such that, for all $c > c'$ and $\bar{m}_i > 0$ with $|\bar{m}_i - \bar{m}_i^*| \leq \mathcal{J}\varepsilon$, we have

$$\begin{aligned} & \left| \exp\{G_i^{(c)}(\bar{m}_i + 1) - G_i^{(c)}(\bar{m}_i)\} - e^{G_i'(\bar{m}_i^*)} \right| \\ &= \left| \exp\left\{ \frac{G_i(\bar{m}_i + 1/c + d_i^{(c)}/c) - G_i(\bar{m}_i + d_i^{(c)}/c)}{1/c} \right\} - e^{G_i'(\bar{m}_i^*)} \right| \\ &\leq \varepsilon'. \end{aligned}$$

We can now show that, eventually as $c \rightarrow \infty$,

$$\begin{aligned}
 & |\Lambda_i^{(c)} - \Lambda_i^*| \\
 & \leq \mathbb{E} |\exp\{G_i^{(c)}(\bar{M}_i^{(c)} + 1) - G_i^{(c)}(\bar{M}_i^{(c)})\} - e^{G_i'(\bar{m}_i^*)}| \\
 & \leq \varepsilon' \mathbb{P}^{(c)}\left(\left|\frac{\bar{M}_i^{(c)}}{c} - \bar{m}_i^*\right| < J\varepsilon\right) + e^{G_i'(\bar{m}_i^*)} \mathbb{P}^{(c)}\left(\left|\frac{\bar{M}_i^{(c)}}{c} - \bar{m}_i^*\right| \geq J\varepsilon\right) \\
 & \quad + \mathbb{E}^{(c)} \exp\{G_i^{(c)}(\bar{M}_i^{(c)} + 1) - G_i^{(c)}(\bar{M}_i^{(c)})\} \mathbf{1}\left[\left|\frac{\bar{M}_i^{(c)}}{c} - \bar{m}_i^*\right| \geq J\varepsilon\right] \\
 & \leq \varepsilon' + e^{G_i'(\bar{m}_i^*)} \mathbb{P}^{(c)}\left(\inf_{m \in \mathcal{M}} \left\|\frac{M^{(c)}}{c} - m\right\| \geq \varepsilon\right) \\
 & \quad + \frac{B_{\check{G}^{(c)}}}{B_{G^{(c)}}} \check{\mathbb{P}}^{(c)}\left(\inf_{m \in \mathcal{M}} \left\|\frac{M^{(c)}}{c} - m\right\| \geq \varepsilon\right) \\
 & \leq \varepsilon' + e^{G_i'(\bar{m}_i^*)} \exp\{-c(\beta_{G,\varepsilon}^* - \beta_G^*) + c\varepsilon''\} + e^{c\varepsilon''} \exp\{-c(\beta_{G,\varepsilon}^* - \beta_G^*) + c\varepsilon''\} \\
 & \rightarrow \varepsilon' \quad \text{as } c \rightarrow \infty.
 \end{aligned}$$

In the second inequality we used substitution (45), and in the third inequality we applied (46) and let $2\varepsilon'' < \beta_{G,\varepsilon}^* - \beta_G^*$. As ε' is arbitrary, the result holds.

6. Conclusion

Previous work has considered the solution of the system problem by analysing differential equations. In this paper we have shown that this same notion of utility optimization can be solved by considering queueing networks with end-to-end control. This leads us to consider different interpretations of Kelly’s decomposition results where pricing is determined by delay. This work emphasises the duality between the flow through a network and its state, and also emphasises a wide variety of fairness that are provably achievable by end-to-end control.

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