Ergod. Th. & Dynam. Sys. (1990), **10**, 523-529
Printed in Great Britain

The number of limit cycles of polynomial deformations of a Hamiltonian vector field

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(*Received* 21 *June* 1988 *and revised* 11 *April* 1989)

Abstract. We prove that the lowest upper bound for the number of limit cycles of small nonconservative polynomial deformations of degree n of the Hamiltonian vector field

$$X_H = y\frac{\partial}{\partial x} + (x^2-1)\frac{\partial}{\partial y}$$

is $n-1$.

A consequence is that the lowest upper bound for the number of limit cycles of generic n-parameter deformations of cusps is $n-1$.

1. *Introduction*

Let X_H be the Hamiltonian vector field $X_H = y\,\partial/\partial x + (x^2-1)\,\partial/\partial y$. The vector field X_H has two singular points: a saddle point s and an elliptic point e. Let H be a Hamiltonian inducing the vector field X_H and let $h_e = H(e)$, $h_s = H(s)$ be the values of the Hamiltonian in the singular points. The level set of the Hamiltonian $H^{-1}(h_s)$ contains a separatrix loop $\gamma(h_s)$ and for each $h \in [h_e, h_s)$ there exists a compact connected component $\gamma(h)$ of $H^{-1}(h)$ contained in the domain G bounded by $\gamma(h_s)$. The component $\gamma(h)$, $h \in (h_e, h_s]$ is homeomorphic to a circle and we orient it clockwise.

Let $\mathcal{P}_n$ be the space of polynomial vector fields of degree less than or equal to n. For a vector field

$$Y = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} \in \mathcal{P}_n,$$

we denote by $\omega_Y = -B\,dx + A\,dy$ the one form dual to Y. For $Y \in \mathcal{P}_n$ we define the *Abelian integral* I_Y *associated with* Y by:

$$I_Y(h) = \int_{\gamma(h)} \omega_Y, \; h \in [h_e, h_s]. \tag{1}$$

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Let N be a compact neighbourhood of $\bar{G}$ and let $X: \varepsilon \mapsto X_\varepsilon$, $\varepsilon \in \mathbf{R}$, be a C^∞-family of C^∞-vector fields defined on N, of the form

$$X_\varepsilon = X_H + \varepsilon Y + o(\varepsilon), \quad Y \in \mathcal{P}_n, \quad \varepsilon \in \mathbf{R}. \tag{2}$$

We call X a *polynomial deformation* of X_H *of degree* $\leq n$. X is a *nonconservative deformation* of X_H if

$$I_Y \not\equiv 0. \tag{3}$$

We prove:

THEOREM 1. *Let X be a nonconservative polynomial deformation of degree $\leq n$ of X_H. Then there exists $E > 0$ such that the vector fields X_ε, $|\varepsilon| < E$, have at most $n-1$ limit cycles in N.*

Furthermore, there exists a polynomial vector field $Y \in \mathcal{P}_n$, $I_Y \not\equiv 0$, such that for a polynomial deformation X of X_H, of the form (2), *there exists $E > 0$ such that the vector fields X_ε for $0 < |\varepsilon| < E$, have $n-1$ limit cycles in N.*

To formulate Theorem 2 we need the notion of a cusp introduced by Roussarie in [**8**].

A *cusp* is a germ in $0 \in \mathbf{R}^2$ of a vector field X_0 satisfying the condition that its 2-jet $j^2 X_0(0)$ is differentiably equivalent to

$$y\frac{\partial}{dx} + (x^2 + \beta xy)\frac{\partial}{\partial y}, \quad \beta \in \mathbf{R}.$$

THEOREM 2. *Generically every n-parameter deformation of a cusp has locally at most $n-1$ limit cycles and there exist generic n-parameter deformations of cusps having locally $n-1$ limit cycles.*

2. *Preliminaries*

Let X be a nonconservative polynomial deformation of degree n of X_H. From (2) it follows that there exists $E > 0$ such that for $|\varepsilon| < E$ every vector field X_ε has precisely two singular points e_ε and s_ε in N and $(e_\varepsilon, s_\varepsilon) \to (e, s)$ for $\varepsilon \to 0$. By affine changes of coordinates close to the identity we can bring the singular points e_ε and s_ε into e and s.

We note that the form (2) and the nonconservative character of the deformation X are preserved by these changes of coordinates. Therefore, we can assume without loss of generality, that e and s are the only singular points in N of the vector fields X_ε, $|\varepsilon| < E$. It follows that all limit cycles in N of the vector fields X_ε, $|\varepsilon| < E$, cut the segment $\sigma = [e, s]$. For $|\varepsilon| < E$ we consider the Poincaré map P_ε defined on σ of the vector field X_ε. The limit cycles of X_ε are given by isolated fixed points of P_ε (different from e). That is by isolated zeros (different from e) of the difference function $\Delta_\varepsilon = P_\varepsilon - id$. We parametrize the points of σ by the values of the Hamiltonian. From the Perturbation lemma [**1**] it follows:

$$\Delta_\varepsilon(h) = \varepsilon I(h) + o(\varepsilon). \tag{4}$$

Here $I = I_Y$ is the Abelian integral associated, by (1), to the vector field Y appearing in (2) and $o(\varepsilon)$ is a C^∞-function on $[h_e, h_s) \times (-E, E)$, which is of order $o(\varepsilon)$ on any compact subset of $[h_e, h_s)$. We denote $\tilde{\Delta}_\varepsilon = \varepsilon^{-1}\Delta_\varepsilon$, $\varepsilon \neq 0$. We note that the functions $\tilde{\Delta}_\varepsilon$ for $\varepsilon \in (-E, E)$ and I are C^∞ on $[h_e, h_s)$ (I is actually analytic). Therefore, it follows from (4), that for each $\alpha \in (h_e, h_s)$, restricting $E > 0$ if necessary, we have that the number of isolated zeros of $\tilde{\Delta}_\varepsilon$, $0 < |\varepsilon| < E$, on $[h_e, \alpha]$ is less than or equal to the number of zeros of I on $[h_e, \alpha]$.

On the contrary, since the functions $\tilde{\Delta}_\varepsilon$ and I are in general not C^∞ in $h = h_s$, it can happen that for every $\varepsilon \neq 0$, $|\varepsilon|$ small, there exist r zeros $h_i(\varepsilon)$ $i = 1, \ldots, r$ of $\tilde{\Delta}_\varepsilon$ tending to h_s for $\varepsilon \to 0$, where r is greater than the multiplicity of h_s as a zero of I. Therefore, the zeros $h_i(\varepsilon)$, $i = 1, \ldots, r$ of $\tilde{\Delta}_\varepsilon$ do not correspond to the zeros of I on $[h_e, h_s)$, nor can be included in zeros of I on $[h_e, h_s]$. However, a bound for the number r of zeros of $\tilde{\Delta}_\varepsilon$ tending to h_s, for $\varepsilon \to 0$, can be given in function of the principal term of the asymptotic development of I in the point h_s. We have the following lemma due to Roussarie.

LEMMA 1 [7]. *The Abelian integral I has an asymptotic development of the form*

$$\sum_{i \geq 0} B_i t^i + A_i t^{i+1} \log t, \tag{5}$$

where

$$t = h_s - h. \tag{6}$$

Furthermore, there exist $\alpha > 0$, $E > 0$ *such that if* $I \sim B_m t^m$, $B_m \neq 0$ ($I \sim A_m t^{m+1} \log t$, $A_m \neq 0$) *then for* $0 < |\varepsilon| < E$ *the function* $\tilde{\Delta}_\varepsilon$ *has at most* $r = 2m$ ($r = 2m + 1$ resp.) *zeros on the interval* $(0, \alpha)$.

It can be shown that for every $Y \in \mathcal{P}_n$ the associated Abelian integral $I = I_Y$ can be extended analytically from the segment $[h_e, h_s)$ to the domain $D = C \backslash \{h \geq h_s\}$ [3], [6]. We denote by $\mathcal{I}_n$ the space of Abelian integrals defined by $\mathcal{I}_n = \{I_Y : Y \in \mathcal{P}_n\}$. We have the following bound due to Petrov of the number of zeros of Abelian integrals.

LEMMA 2 [6]. *Let* $I \in \mathcal{I}_n$, $I \not\equiv 0$. *Then I has at most n zeros in D.*

To prove Theorem 1 we shall improve the bound for the number of zeros of Abelian integrals in function of the principal part of the asymptotic development (5).

3. *The Main Lemma*

LEMMA 3. *Let* $I \in \mathcal{I}_n$, $I \not\equiv 0$ *and let r be defined by Lemma* 1. *Then I has at most* $n - r$ *zeros in D.*

Furthermore, there exists an Abelian integral $I \in \mathcal{I}_n$ *having* $n-1$ *distinct simple zeros in the interval* (h_e, h_s).

To prove Lemma 3 we first perform the change of coordinates (6) to bring the point h_s into $t_s = 0$. The point h_e is mapped to $t_e = h_s - h_e$, the interval (h_e, h_s) to $(0, t_e)$ and the domain D to $\tilde{D} = C \backslash \{t \leq 0\}$. We consider the Abelian integrals $I = I(t) \in \mathcal{I}_n$ as functions on $\tilde{D}$. Obviously the change of coordinates (6) carries over the zeros of $I(h)$ on D to zeros of $I(t)$ on $\tilde{D}$ preserving their number.

Furthermore, we use the fact that there exists an isomorphism $\mathbf{\Phi}$ of the vector spaces $\mathbf{R}^n$ and $\mathcal{I}_n$, $\mathbf{\Phi}:\mathbf{R}^n \to \mathcal{I}_n$ [5]. The isomorphism $\mathbf{\Phi}$ is given by:

$$\mathbf{\Phi}(\alpha_0, \ldots, \alpha_{[(n-1)/2]}; \beta_0, \ldots, \beta_{[n/2]-1}) = I \in \mathcal{I}_n,$$

where,

$$I(t) = \sum_{i=0}^{[(n-1)/2]} \alpha_i t^i I_0 + \sum_{i=0}^{[n/2]-1} \beta_i t^i I_1. \tag{7}$$

Here I_0, $I_1 \in \mathcal{I}_n$ are the Abelian integrals associated with the one-forms $\omega_0 = y\,dx$ and $\omega_1 = xy\,dx$. We define a norm $\| \ \|$ on $\mathcal{I}_n$ as the norm induced by $\mathbf{\Phi}$ from the Euclidean norm on $\mathbf{R}^n$. We note that convergence in the norm $\| \ \|$ on $\mathcal{I}_n$ is equivalent to uniform C^∞ convergence on compact subsets of $\tilde{D}$.

Furthermore, by a direct calculation or from [2] it follows that the coefficients a_0, b_0, a_1 and b_1 of the asymptotic developments of I_0, I_1

$$\begin{aligned} I_0 &= a_0 + b_0 t \log t + o(t \log t) \\ I_1 &= a_1 + b_1 t \log t + o(t \log t), \end{aligned} \tag{8}$$

satisfy

$$a_0 \neq 0,\ a_1 \neq 0, \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} \neq 0. \tag{9}$$

Proof of Lemma 3. For $I \in \mathcal{I}_n$, $I \neq 0$ let r be defined by Lemma 1. Let x be the number of zeros of I in $\tilde{D}$ and let $K \subset \tilde{D}$ be a compact neighbourhood of the set of zeros of I. There exists $\varepsilon > 0$ such that every Abelian integral $J \in \mathcal{I}_n$ satisfying $\|J - I\| < \varepsilon$ has x zeros in K. We will construct an Abelian integral $I^m \in \mathcal{I}_n$ satisfying $\|I^m - I\| < \varepsilon$ having r distinct zeros t_i, $i = 1, \ldots, r$, $t_i \in \tilde{D} \backslash K$. Thus I^m will have at least $x + r$ zeros in $\tilde{D}$. So applying Lemma 2 to $I^m(t)$ we will obtain $x + r \leq n$, thus proving the first statement of the lemma.

To construct I^m let us assume first that

$$I \sim B_m t^m, \quad B_m \neq 0. \tag{10}$$

We will obtain $I^m \in \mathcal{I}_n$ as a result of m successive perturbations of I, each time creating two new zeros. From (8), (9) and (10) it follows that $m \leq [(n-1)/2]$. Therefore, the Abelian Integral I^1 of the form

$$I^1 = \alpha_1 t^{m-1} I_0 + \beta_1 t^{m-1} I_1 + I, \tag{11}$$

is an element of $\mathcal{I}_n$. We will show that there exists $T > 0$, such that for $0 < t < T$ taking $t_1 = t$, $t_2 = 2t$, there exists a unique solution (α_1, β_1) of the system

$$\begin{aligned} \alpha_1 t_1^{m-1} I_0(t_1) + \beta_1 t_1^{m-1} I_1(t_1) &= -I(t_1) \\ \alpha_1 t_2^{m-1} I_0(t_2) + \beta_1 t_2^{m-1} I_1(t_2) &= -I(t_2). \end{aligned} \tag{12}$$

That is the Abelian integral I^1 defined by (11), where (α_1, β_1) is the unique solution of (12) satisfies

$$I^1(t_1) = I^1(t_2) = 0. \tag{13}$$

Furthermore, we will show that taking $t>0$ sufficiently small we can satisfy

$$\|I^1 - I\| < \varepsilon/m. \tag{14}$$

Indeed, the determinant of the system (12) is given by

$$W_{m-1}(t) = \begin{vmatrix} t^{m-1}I_0(t) & t^{m-1}I_1(t) \\ (2t)^{m-1}I_0(2t) & (2t)^{m-1}I_1(2t) \end{vmatrix}. \tag{15}$$

From (8), (9) and (15) it follows

$$W_{m-1}(t) = 2^{m-1}(a_0b_1 - a_1b_0)t^{2m-1}\log t + o(t^{2m-1}\log t), \quad a_0b_1 - a_1b_0 \neq 0. \tag{16}$$

So there exists $T>0$ such that $W_{m-1}(t)\neq 0$ for $0<t<T$, implying that the system (12) has a a unique solution (α_1, β_1) for $0<t<T$. We claim that $(\alpha_1, \beta_1)\to 0$, for $t\to 0$. Indeed, from (12), using (8) and (10) we have

$$\begin{aligned} \alpha_1 W_{m-1}(t) &= t^{m-1}I(2t)I_1(t) - (2t)^{m-1}I(t)I_1(2t) \\ &= B_m 2^{m-1}a_1 t^{2m-1} + o(t^{2m-1}). \end{aligned} \tag{17}$$

Now (16) and (17) imply that $\alpha_1\to 0$ as $t\to 0$. Analogously $\beta_1\to 0$ as $t\to 0$. So taking $0<t<T$ sufficiently small we can satisfy (14) and $t_1, t_2\in \tilde{D}\backslash K$. Thus I^1 has at least $x+2$ zeros in $\tilde{D}$. Let us note that the asymptotic behavior of I^1 in $t=0$ is given by

$$I^1 \sim B^1_{m-1}t^{m-1}, \quad B^1_{m-1}\neq 0.$$

Repeating this argument (replacing I by I^1) after m steps we obtain an Abelian integral $I^m\in \mathcal{I}_n$ having at least $x+2m$ zeros in $\tilde{D}$.

If the asymptotic behavior of I in $t=0$ is given by

$$I \sim A_m t^{m+1}\log t, \quad A_m \neq 0, \tag{18}$$

it is easy to see that there exists a small perturbation $I^0\in\mathcal{I}_n$ of I of the form

$$I^0 = \alpha_0 t^m I_0 + I,$$

having one additional zero $t_0>0$ close to $t=0$. So we reduce case (18) to case (10).

We now prove the second statement of the lemma. Suppose first that $n=2k+1$. Let $I=t^kI_0\in\mathcal{I}_n$. We note that I has one zero (in the point $t=t_e$) in $\tilde{D}$. Proceeding as in the proof of the first statement of the lemma we obtain an Abelian integral $I^k\in\mathcal{I}_n$ having $n=2k+1$ distinct zeros in $\tilde{D}$. They are all simple by Lemma 2. Moreover, we can achieve that the $2k$ new created zeros $t_i>0$, $i=1,\ldots,2k$ are arbitrarily close to zero, so $t_i\in(0, t_e)$. Returning to the old variable $h=h_s-t$ we obtain $n-1=2k$ simple distinct zeros $h_i=h_s-t_i$, $i=1,\ldots,n-1$ of $I(h)$ satisfying $h_i\in(h_e, h_s)$.

If $n=2k$, then it is easy to see that there exists an Abelian integral $I\in\mathcal{I}_n$ of the form $I=\alpha_1 t^{k-1}I_0+\beta_1 t^{k-1}I_1$ such that the asymptotic behavior in zero of I is given by $I\sim A_{k-1}t^k\log t$. Now proceeding analogously as in the case $n=2k+1$ we finish the proof of the lemma.

4. *Proofs of the Theorems* 1 *and* 2 *and a concluding Remark*

Proof of Theorem 1. Let $Y\in\mathcal{P}_n$ be a polynomial vector field and let X be a nonconservative deformation of X_H of the form (2). We assume that e and s are

https://doi.org/10.1017/S0143385700005721 Published online by Cambridge University Press

the only singular points of the vector fields X_ε, $|\varepsilon|<E$, in N. The number of limit cycles of a vector field X_ε, $|\varepsilon|<E$, is given by the number of isolated zeros of $\tilde{\Delta}_\varepsilon$ on (h_e, h_s). Let $I=I_Y\not\equiv 0$ be the Abelian integral associated to Y and let r be defined as in Lemma 1.

By Lemma 1 there exist $\alpha\in(h_e, h_s)$ such that reducing $E>0$ if necessary, we have that the function $\tilde{\Delta}_\varepsilon$, $0<|\varepsilon|<E$ has at most r isolated zeros in the interval (α, h_s).

On the other hand by Lemma 3 the Abelian integral I has at most $n-r$ isolated zeros on $[h_e, h_s)$. Therefore, reducing $E>0$ if necessary, from (4) it follows that for $0<|\varepsilon|<E$ the function $\tilde{\Delta}_\varepsilon$ has at most $n-r$ isolated zeros on the interval $[h_e, \alpha]$. Thus $\tilde{\Delta}_\varepsilon$ has at most n isolated zeros on the interval $[h_e, h_s)$. Since for $0<|\varepsilon|<E$ the function $\tilde{\Delta}_\varepsilon$ has at least a simple trivial zero in $h=h_e$, the first statement of the theorem follows.

The second statement of the theorem follows from the second statement of Lemma 3 using (4).

Proof of Theorem 2. We note first that Theorem 1 remains true if we let the vector field Y in (2) vary through a compact subset of $\mathcal{P}_n$ (with the C^∞-topology). The proof of Theorem 2 is obtained by replacing [**8**, Theorem 12] in [**8**] by this version of Theorem 1 and noting the relations preceeding [**8**, Theorem 9].

Remark. Let us consider more generally a Hamiltonian vector field of the form

$$X_H = y\frac{\partial}{\partial x}+P(x)\frac{\partial}{\partial y}, \tag{19}$$

where $P(x)$ is a polynomial of degree p. We assume that X_H has a separatrix loop γ passing through a saddle point s, and that all trajectories in the domain G bounded by γ are homeomorphic to circles except one elliptic singular point e. Let N be a compact neighbourhood of $\bar{G}$, which contains no other singular points.

We conjecture that the lowest uniform upper bound for the number of limit cycles in N of small nonconservative polynomial deformations of degree $\leq n$ of X_H given by (19) is equal to the lowest uniform upper bound for the number of nontrivial zeros of the associated Abelian integrals. Here, we consider the Abelian integrals as analytic functions defined in a neighbourhood $D\subset \mathbf{C}$ of $[h_e, h_s)$.

We note that from a general result of Varchenko [**9**] it follows that such a uniform upper bound for the number of zeros of Abelian integrals associated to nonconservative polynomial deformations of degree $\leq n$ exists.

By [**5**] we have that for the space of Abelian integrals under consideration there exists an isomorphism analogous to (7). The only point missing for the proof of the conjecture is that certain properties (of specific Abelian integrals arising in the generalization of (7)) generalizing (8) are satisfied. For $p=3$ this is verified in [4].

Acknowledgements. I would like to thank Professor R. Roussarie for introducing me to the problem and Ch. Bonatti for some suggestions helping clarify the exposition.

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