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MEAN TRANSITION TIMES FOR THE EHRENFEST URN MODEL

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Abstract

For an Ehrenfest urn model, the mean transition time from one state to another is represented as a definite integral. Simple explicit expressions are given in some special cases.

1. Introduction

Consider the Ehrenfest urn model with two urns U_1 and U_2 and M balls in all. At each time-point $t = 0, 1, 2, \cdots$, one of the M balls is drawn at random and is moved from one urn to the other. Let E_n be the state 'n balls are in urn U_2' $(n = 0, 1, \cdots, M)$. The process starts from E_0; that is, all balls are initially in urn U_1. We want to find the average time m_n required for the transition $E_0 \rightarrow E_n$.

Matstoms (1988) has conjectured that, when $M = 2N$, then

(1) $$m_N = N \sum_{j=0}^{N-1} 1/(2j+1).$$

In Section 2 we derive an expression for m_n in the form of a definite integral. In Section 3 we show that Matstoms' conjecture is correct. In Section 4 we find m_M in explicit form.

2. Representation of m_n as an integral

Let T_k be the time required for the transition $E_k \rightarrow E_{k+1}$ $(0 \leqq k \leqq M-1)$ and set $e_k = E(T_k)$. Now $T_k = 1$ with probability $(M-k)/M$ and $T_k = 1 + T_{k-1} + T'_k$ with probability k/M. (Here T'_k has the same distribution as T_k and is independent of T_k.) Hence we obtain the recursive formula

$$e_k = (M + ke_{k-1})/(M-k) \quad (k = 1, 2, \cdots, M-1); \qquad e_0 = 1.$$

Our main idea in this note is to write e_k as an integral

(2) $$e_k = M \int_0^1 x^{M-k-1}(2-x)^k \, dx.$$

This expression is proved by induction in the recursive formula, performing a partial integration.

We obtain m_n by adding over k in (2) from 0 to $n-1$. Summing the resulting geometric series in the integrand and performing the substitution $x = 1 - y$ in the integral we find

(3) $$m_n = (M/2) \int_0^1 (1-y)^{M-n}\{(1+y)^n - (1-y)^n\}/y \, dy.$$

This is the representation we want. We remark in passing that $m_{n'} - m_n$ is the mean time required for the transition $E_n \rightarrow E_{n'}$ where $n < n'$.

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3. First special case: $M = 2N,\ n = N$

Let us assume that there is an even number, $2N$, of balls in U_1 at the start. We want to know how long it takes, on the average, until there are N balls in each urn. In this case (3) becomes

$$m_N = N\int_0^1 \{(1-y^2)^N - (1-y)^{2N}\}/y\,dy.$$

It is found successively that

$$\begin{aligned} m_N &= N\int_0^1 \{1-(1-y)^{2N}\}/y\,dy - N\int_0^1 \{1-(1-y^2)^N\}/y\,dy \\ &= N\int_0^1 \sum_{k=0}^{2N-1} (1-y)^k\,dy - N\int_0^1 y\sum_{j=0}^{N-1}(1-y^2)^j\,dy \\ &= N\sum_{k=0}^{2N-1} 1/(k+1) - (N/2)\sum_{j=0}^{N-1} 1/(j+1) \\ &= N\sum_{j=0}^{N-1} 1/(2j+1). \end{aligned}$$

Hence Matsoms' conjecture (1) is correct.

4. Second special case: $n = M$

We now want to know how long it takes, on the average, for all the M balls to be transferred from U_1 to U_2. Setting $n = M$ in (3) we find

$$m_M = (M/2)\int_0^1 \{(1+y)^M - (1-y)^M\}/y\,dy.$$

It is seen that $m_M/M - m_{M-1}/(M-1) = 2^{M-1}/M$. Hence

$$m_M = M\sum_{j=0}^{M-1} 2^j/(j+1). \tag{4}$$

It is instructive to compare (1) and (4). It is seen that the average time required to attain equal numbers of balls in the urns is quite short compared to that required to have all balls transferred from one urn to the other. For example, when $M = 10$, then $m_5 = 8{\cdot}9$ and $m_{10} = 1186{\cdot}5$. Note that in (4) the important terms are the last ones.

Reference

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