1. Introduction and statement of results

Let $\Gamma$ be a curve in $\mathbb{R}^2$ defined by $y = A(x)$. The Cauchy transform $C_A$ on $\Gamma$ is defined by the kernel

$$K(x, y) = \frac{1 + iA(y)}{(x - y) + i(A(x) - A(y))}.$$ 

When $A$ is a Lipschitz function, the $L^2$ boundedness of $C_A$ is well understood and several proofs of it have been produced (cf. [C, CJS, CMM, DJ, M]). If $A$ is a $C^1$-smooth function, then the local $L^2$ boundedness of $C_A$ is also well understood (cf. [FJR]). However, if $A$ is a smooth, not necessarily Lipschitz function, the question of global $L^2$ boundedness of $C_A$ has not been settled. In [KS], we observe that $C_A$ is not, in general, bounded on $L^2$ if $A$ is a smooth non-Lipschitz function, and prove that $C_A$ is bounded on $L^2$ if $A$ is either a polynomial of odd degree or an even polynomial. The purpose of this paper is to give a new proof of it and to extend the result to arbitrary polynomials.

**Theorem.** If $A$ is a polynomial, then the Cauchy transform on the curve $y = A(x)$ is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$.

In [KS], we used a continuously varying cut-off function to separate the singularities of $K(x, y)$ at $x = y$ and at $x = -y$ and the T1-theorem of David and Journé. In this paper, instead of using a cut-off function, we use a direct decomposition of the Cauchy kernel. If $A$ is a polynomial, then the Cauchy kernel $K(x, y)$ can be decomposed as
(1.1) \[ K(x, y) = \frac{1 + iA(y)}{x - y} + i(A(x) - A(y)) = \frac{1}{x - y} + i\frac{P(x, y)}{1 + iQ(x, y)} \]

where

(1.2) \[ Q(x, y) = \frac{A(x) - A(y)}{x - y} \quad \text{and} \quad P(x, y) = \frac{A'(y) - Q(x, y)}{x - y}. \]

Moreover, if \( A \) is an even polynomial, one can easily see that \( Q(x, y) = (x + y)R(x, y) \) for some polynomial \( R \). Define an operator \( T_A \) by

(1.3) \[ T_A f(x) = \int_{-\infty}^{\infty} \frac{P(x, y)}{1 + iQ(x, y)} f(y) dy. \]

Then, \( T_A = \mathcal{H} + iT_A \) where \( \mathcal{H} \) is the Hilbert transform. We prove that \( T_A \) is bounded on \( L^p(\mathbb{R}) \), \( 1 < p < \infty \). If \( A \) is a polynomial of odd degree, it is easy to prove the \( L^2 \) boundedness of \( T_A \) since \( Q(x, y) \) does not have any zero when \( x^2 + y^2 \) is large. If \( A \) is an even polynomial, then we compare \( T_A \) with a linear combination of the Hilbert transform and various operators defined in Section 3. We show that these operators are bounded on \( L^p(\mathbb{R}) \) and that the difference of \( T_A \) and a linear combination of the Hilbert transform and these operators can be estimated by the Hardy Littlewood maximal operator which is well known to be bounded on \( L^p \) (cf. [S]). If \( A \) is a polynomial of even degree, then there exists a change of variables \( \alpha(x) \) defined for large \( x \) such that \( A(\alpha(x)) \) becomes an even polynomial. By carefully studying the behavior of \( \alpha(x) \) for large \( x \), we are able to reduce matters to the case of even polynomials.

We organize this paper as follows; in Section 2, we prove some properties of \( Q(x, y) \) which will be used in later sections. In Section 3, we introduce some related operators and prove that they are bounded on \( L^p(\mathbb{R}) \). In Section 4, we prove that \( T_A \) is bounded on \( L^p(\mathbb{R}) \) if \( A \) is a polynomial of odd degree. In the final section, we prove that \( T_A \) is bounded on \( L^p(\mathbb{R}) \) if \( A \) is a polynomial of even degree.

We use a standard notation of \( A \leq B \) to imply that \( A \leq CB \) for some constant \( C \). \( A \approx B \) means that both \( A \leq B \) and \( A \geq B \) hold.

2. Preliminary on polynomials

Let \( A \) be a polynomial and let

\[ Q(x, y) = \frac{A(x) - A(y)}{x - y} \]
In this section, we collect some properties of $Q$ which will be used in later sections.

**Lemma 2.1.**

1. If $\deg A = 2n + 1$, then there exists a positive constant $r$ such that
   \[ |Q(x, y)| \geq x^{2n} + y^{2n} \]
   if $x^2 + y^2 \geq r$.

2. If $A$ is an even polynomial and $\deg A = 2n + 2$, then there exist a positive constant $r$ and a polynomial $R(x, y)$ such that
   \[ Q(x, y) = (x + y)R(x, y) \]
   and
   \[ |R(x, y)| \geq x^{2n} + y^{2n} \]
   if $x^2 + y^2 \geq r$.

**Proof.** For (1), note that
\[
\frac{x^{2n+1} - y^{2n+1}}{x - y} = \sum_{j=0}^{2n} x^{2n-j} y^j
= \frac{1}{2} (x^{2n} + y^{2n}) + \frac{1}{2} (x + y)^2 \sum_{j=1}^{n} x^{2(n-j)} y^{2(j-1)}
\geq \frac{1}{2} (x^{2n} + y^{2n}).
\]

Let $A(x) = \sum_{j=0}^{2n+1} a_j x^j$ ($a_{2n+1} \neq 0$). Then
\[
\left| \frac{A(x) - A(y)}{x - y} \right| \geq \frac{|a_{2n+1}|}{2} (x^{2n} + y^{2n}) - \sum_{j=1}^{2n} |a_j| (|x|^j + |y|^j)
\geq x^{2n} + y^{2n}
\]
if $x^2 + y^2$ is large.

For (2), we note that
\[
|x^{2j} - y^{2j}| = |x - y||x + y||x^{2j-2} + x^{2j-4}y^2 + \cdots + x^2y^{2j-4} + y^{2j-2}|
\approx |x - y||x + y|(x^{2j-2} + y^{2j-2}).
\]
Let $A(x) = \sum_{j=0}^{2n+2} a_j x^j$ ($a_{2n+1} \neq 0$). Then
\[ |A(x) - A(y)| \geq |a_{2n+1}| |x^{2n+2} + y^{2n+2}| - \sum_{j=1}^{2n} |a_j| |x^{2j} - y^{2j}| \]
\[ \geq |x - y| |x + y| \left[ |a_{2n+2}| \left( |x|^{2n} + |y|^{2n} \right) - C \left( |x|^{2n-2} + |y|^{2n-2} + 1 \right) \right] \]
\[ \geq |x - y| |x + y| \left( |x|^{2n} + |y|^{2n} \right) \]
for some constants \( C \) as long as \( x^2 + y^2 \) is large. This completes the proof.

The next lemma and corollary show that a polynomial of even degree is essentially the same as an even polynomial for our purpose.

**Lemma 2.2.** Let \( A \) be a polynomial of even degree. Then, there exist \( r > 0 \) and a smooth change of variable \( \alpha(x) \) on \( |x| > r \) such that

1. \( A(\alpha(x)) \) is an even polynomial,
2. \( \alpha(x) = x + \beta(x) \) where \( \beta(x) = O(1) \) and \( \beta'(x) = O(1/x) \) as \( x \to \infty \).

**Proof.** Let \( A(x) = \sum_{j=0}^{2n} a_j x^j \) and assume that \( a_{2n} = 1 \) without loss of generality. Choose \( r > 0 \) so that \( A \) is monotone if \( |x| > r \) and define \( \alpha \) on \( |x| > r \) by

\[ (2.1) \quad A(\alpha(x)) = \frac{A(x) + A(-x)}{2}. \]

Since \( A(x) \approx x^{2n} \) and \( A'(x) \approx x^{2n-1} \) if \( |x| > r \) by increasing \( r \) if necessary, one can easily see that \( \alpha(x) \approx x \) and \( \alpha'(x) \approx 1 \). From (2.1), we have

\[ \alpha(x)^{2n} + \sum_{j=0}^{2n-1} a_j \alpha(x)^j = x^{2n} + \sum_{j=0}^{n-1} a_{2j} x^{2j}. \]

It then follows that

\[ \alpha(x)^{2n} = x^{2n} + O(x^{2n-1}) \]
and

\[ 2n\alpha(x)^{2n-1}\alpha'(x) = 2nx^{2n-1} + O(x^{2n-2}). \]

It follows immediately from these relations that

\[ \alpha(x) = x + O(1) \quad \text{and} \quad \alpha'(x) = 1 + O\left(\frac{1}{x}\right) \quad \text{as} \quad x \to \infty. \]

This completes the proof.
Corollary 2.3. Let $\beta$ and $r$ be as above. Then,
\[ \left| \frac{\beta(x) - \beta(y)}{x - y} \right| \leq \frac{1}{|x| + |y|} \]
if $x^2 + y^2 \geq r$.

Proof. If $xy < 0$, then there is nothing to prove. Suppose that $xy > 0$ and that $y > x > 0$ without loss of generality. If $y > 2x$, then since $\beta(x) = O(1)$ as $x \to \infty$, we have
\[ \left| \frac{\beta(x) - \beta(y)}{x - y} \right| \leq \frac{1}{y} \leq \frac{1}{|x| + |y|}. \]

If $x < y < 2x$, then since $\beta'(x) = O(1/x)$ as $x \to \infty$, we have
\[ \left| \frac{\beta(x) - \beta(y)}{x - y} \right| = \left| \beta'(\xi) \right| \leq \frac{1}{\xi} \leq \frac{1}{x} \leq \frac{1}{|x| + |y|} \]
for some $x < \xi < y$. This completes the proof.

3. Related operators

In this section we introduce some related operators and show that they are bounded on $L^p(\mathbb{R})$ by comparing them with the Hardy Littlewood maximal operator and the Hilbert transform. Throughout this paper $M$ denotes the Hardy Littlewood maximal operator.

Proposition 3.1. Let $P(x, y)$ and $R(x, y)$ be smooth functions such that there exists a positive constant $r$ so that
\[ |P(x, y)| \leq |x|^n + |y|^n \quad \text{and} \quad |R(x, y)| \geq |x|^n + |y|^n \quad \text{if} \quad x^2 + y^2 \geq r. \]
Suppose that $0 \leq \alpha$ and $0 < \beta - \alpha \leq \gamma$. For $f \in C_0^\infty(\mathbb{R})$, define
\[ Uf(x) = \int_{-\infty}^{\infty} \frac{|x - y|^\alpha |P(x, y)|}{1 + |x - y|^\beta |R(x, y)|^\gamma} |f(y)| \, dy. \] (3.1)
If $\gamma \geq 1 + 1/n$, then
\[ |Uf(x)| \geq Mf(x) \] (3.2)
for every $x$. 

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Proof. For simplicity, we assume that $r = 1$. Write

$$Uf(x) = \int_{|x-y| \leq 1} + \int_{|x-y| > 1} := I + II.$$

Since $0 < \beta - \alpha$ and $\gamma \geq 1 + 1/n$, we have

$$|II| \leq \int_{|x-y| > 1} \frac{|x-y|^\alpha(|x| + |y|)^n}{1 + |x-y|^\beta(|x| + |y|)^n} |f(y)| \, dy$$

$$\leq \int_{1+|x-y|} \frac{(1 + |x|)^n}{1 + |x|^{n\gamma}} |f(y)| \, dy$$

$$+ \sum_{j=1}^{\infty} \int_{2^{-1}(1+|x|) < |x-y| \leq 2(1+|x|)} \frac{2^n(1 + |x|)^n}{2^{(\beta-\alpha+\gamma)n}(1 + |x|)^{\beta-\alpha+\gamma}} |f(y)| \, dy$$

$$\leq \left(1 + \sum_{j=1}^{\infty} 2^{(1+n-\beta+\alpha-\gamma)n}(1 + |x|)^{1+n-\beta+\alpha-\gamma} \right) Mf(x)$$

$$\leq Mf(x).$$

We now deal with $I$. If $|x| \leq 1$, then it is obvious that $|I| \leq Mf(x)$. Suppose that $|x| > 1$. Then

$$|I| \leq \sum_{j=1}^{\infty} \int_{2^{-1} < |x-y| < 2^{1+j}} \frac{2^{-\alpha}(|x| + |y|^n)}{1 + 2^{-\beta}(|x|^{n\gamma} + |y|^{n\gamma})} |f(y)| \, dy$$

$$\leq \sum_{j=1}^{\infty} \frac{|x|^{n2^{-\alpha+1+j}}}{1 + 2^{-\beta}|x|^{n\gamma}} Mf(x).$$

Pick $N$ so that $1 \leq |x|^{n2^{-N}} \leq 10$. Then we obtain

$$\sum_{j=N}^{\infty} \frac{|x|^{n2^{-\alpha+1+j}}}{1 + 2^{-\beta}|x|^{n\gamma}} \leq |x|^{n2^{-N+1}} \leq 20.$$

We also obtain

$$\sum_{j=1}^{N} \frac{|x|^{n2^{-\alpha+1+j}}}{1 + 2^{-\beta}|x|^{n\gamma}} \leq \sum_{j=1}^{N} 2^{(\beta-\alpha-1)n} |x|^{(1-\gamma)n} \leq 2^{-(\gamma-1)N} \sum_{j=1}^{N} 2^{(\beta-\alpha-1)n} \leq C$$

because $|x|^{(1-\gamma)n} \leq 2^{-(\gamma-1)n}$ and $\beta - \alpha \leq \gamma$. This completes the proof.

Remark 3.2. The condition on $\gamma$ in Proposition 3.1 are sharp in the sense that if $\gamma < 1 + 1/n$, then (3.2) does not hold.
PROPOSITION 3.3. Let $P(x, y)$ be a homogeneous polynomial of degree $2n - 1$. Let

\begin{equation}
(3.3)
Vf(x) = \int_{-\infty}^{\infty} \frac{|P(x, y)|}{x^{2n} + y^{2n}} |f(y)| \, dy
\end{equation}

for $f \in C_0^\infty(\mathbb{R})$. Then, $V$ extends to be an operator bounded on $L^p(\mathbb{R}), 1 < p < \infty$.

Proof. Let $f \in C_0^\infty(\mathbb{R})$. Let

\begin{equation}
V_1 f(x) = \int_{-\infty}^{\infty} \frac{|x|^{2n-1}}{x^{2n} + y^{2n}} |f(y)| \, dy.
\end{equation}

We will show that

\[ |V_1 f(x)| \leq Mf(x). \]

Assume $x \neq 0$.

\[
|V_1 f(x)| \leq |x|^{2n-1} \left( \int_{|y| \leq 3|x|} + \sum_{j=1}^{\infty} \int_{|x| < |x-y| \leq 2^j|x|} \frac{1}{x^{2n} + y^{2n}} |f(y)| \, dy \right)
\]

\[
\leq |x|^{2n-1} \left( \int_{|x| < |x-y| \leq 3|x|} x^{-2n} |f(y)| \, dy + \sum_{j=1}^{\infty} \int_{|x| < |x-y| \leq 2^j|x|} (2^j x)^{-2n} |f(y)| \, dy \right)
\]

\[
\leq \left( 1 + \sum_{j=1}^{\infty} 2^{(j-2n)} \right) Mf(x) \leq Mf(x).
\]

Hence $V_1$ extends to be an operator bounded on $L^p(\mathbb{R}), 1 < p < \infty$.

Since $|P(x, y)| \leq |x|^{2n-1} + |y|^{2n-1}$, we have

\[ |Vf| \leq V_1 f + V_1^* f \]

where $V_1^*$ is the adjoint of $V_1$. Hence Proposition 3.3 follows from $L^p$-boundedness of $V_1$ and $V_1^*$. This completes the proof.

COROLLARY 3.4. Let $P(x, y)$ and $R(x, y)$ be homogeneous polynomials of degree $2n$. Assume that

\[ |R(x, y)| \geq x^{2n} + y^{2n}. \]

For $f \in C_0^\infty(\mathbb{R})$, define

\[ Wf(x) = \int_{-\infty}^{\infty} \frac{P(x, y)}{(x-y)R(x, y)} f(y) \, dy. \]
Then, \( W \) extends to be an operator bounded on \( L^p(\mathbb{R}), 1 < p < \infty \).

**Proof.** Let \( a = P(x, x) / R(x, x) \). Then, we have

\[
\frac{P(x, y)}{(x - y)R(x, y)} - \frac{a}{x - y} = \frac{P(x, y) - aR(x, y)}{x - y} \frac{1}{R(x, y)}.
\]

Note that \((P(x, y) - aR(x, y))/(x - y)\) is a homogeneous polynomial of degree \(2n - 1\). Hence Corollary 3.4 follows from the \( L^p \)-boundedness of the Hilbert transform and Proposition 3.3. This completes the proof.

4. Cauchy transform I

Recall that \( T_A \) is the operator defined by the kernel

\[
\frac{P(x, y)}{1 + iQ(x, y)}
\]

where

\[
Q(x, y) = \frac{A(x) - A(y)}{x - y} \quad \text{and} \quad P(x, y) = \frac{A'(y) - Q(x, y)}{x - y}.
\]

and that

\[
\mathcal{C}_A = \mathcal{H} + iT_A
\]

where \( \mathcal{H} \) is the Hilbert transform. We now prove that the operator \( T_A \) is bounded on \( L^p(\mathbb{R}) \). We first deal with the case when \( \text{deg} A \) is odd in this section.

**Theorem 4.1.** Let \( P(x, y) \) and \( Q(x, y) \) be polynomials of degree \( 2n - 1 \) and \( 2n \), respectively. Assume that there exists a constant \( r \) such that

\[
|Q(x, y)| \geq x^{2n} + y^{2n}
\]

if \( x^2 + y^2 \geq r \). For \( f \in \mathcal{C}_{\infty}^w(\mathbb{R}) \), define

\[
Tf(x) = \int_{-\infty}^{\infty} \frac{P(x, y)}{1 + iQ(x, y)} f(y)dy.
\]

Then, \( T \) extends to be an operator bounded on \( L^p(\mathbb{R}), 1 < p < \infty \).

**Proof.** Let \( q \) be the conjugate of \( p \), and let
CAUCHY TRANSFORMS ON POLYNOMIAL CURVES

\[ k(x, y) = \frac{P(x, y)}{1 + iQ(x, y)}. \]

Then

\[
\int |Tf(x)|^p \, dx \leq \int_{|x| \leq r} |Tf(x)|^p \, dx + \int_{|x| > r} |Tf(x)|^p \, dx := I + II.
\]

If \(|x| \leq r\), then

\[
\int_{-\infty}^{\infty} |k(x, y)|^q \, dy \leq 1 + \int_{|y| > r} \frac{y^{(2n-1)q}}{1 + y^{2n}} \, dy \leq C.
\]

It then follows from the Hölder inequality that

\[
I \leq \int_{|x| \leq r} \left( \int |k(x, y)|^q \, dy \right)^{p/q} \, dx \left\| f \right\|_p^p \leq \left\| f \right\|_p^p.
\]

For II, observe that if \(|x| > r\), then

\[
|Tf(x)| \leq \int_{-\infty}^{\infty} \frac{|x|^{2n-1} + |y|^{2n-1}}{x^{2n} + y^{2n}} |f(y)| \, dy.
\]

Hence, it follows from the proof of Proposition 3.3 that

\[
\int_{|x| > r} |Tf(x)|^p \, dx \leq \left\| f \right\|_p^p.
\]

This completes the proof.

**Corollary 4.2.** If \(A\) is a polynomial of odd degree, then the Cauchy transform \(\mathcal{C}_A\) is bounded on \(L^p(\mathbb{R})\), \(1 < p < \infty\).

**Proof.** It follows from Theorem 4.1 and Lemma 2.1.

### 5. Cauchy transform II

In this section we prove that if \(A\) is a polynomial of even degree, then the operator \(T_A\) is bounded on \(L^p\). We first deal with the case when \(A\) is an even polynomial.

**Theorem 5.1.** Suppose that \(P(x, y) = P_0(x, y) + P_1(x, y)\) and \(R(x, y) = R_0(x, y) + R_1(x, y)\) satisfy the following
(1) $P_0(x, y)$ and $R_0(x, y)$ are homogeneous polynomials of degree $2n$.

(2) $|P_1(x, y)| \leq |x|^{2n-1} + |y|^{2n-1}$ and $|R_1(x, y)| \leq |x|^{2n-1} + |y|^{2n-1}$ if $x^2 + y^2 > r$ for some $r$.

(3) $|R(x, y)| \geq |x|^{2n} + |y|^{2n}$ if $x^2 + y^2 > r$.

For $f \in C_0^\infty(\mathbb{R})$, define

$$Tf(x) = \int_{-\infty}^{\infty} \frac{P(x, y)}{1 + i(x - y)R(x, y)} f(y) dy.$$  

Then, $T$ extends to be an operator bounded on $L^p(\mathbb{R})$, $1 < p < \infty$.

**Proof.** By a straightforward computation, we have

$$\left| \frac{P(x, y)}{1 + i(x - y)R(x, y)} - \frac{P_0(x, y)}{1 + i(x - y)R_0(x, y)} \right| \leq \frac{|P(x, y) - P_0(x, y)|}{1 + |x - y|^2} \left| \frac{R_0(x, y)}{R_0(x, y)} \right| |f(y)| dy \leq M|f(x)|.$$  

Hence we may assume $P(x, y)$ and $R(x, y)$ are homogeneous polynomials of degree $2n$. Note that

$$\frac{P(x, y)}{1 + i(x - y)R(x, y)} = \frac{P(x, y)}{1 + (x - y)^2 R(x, y)^2} + i \frac{(x - y)R(x, y)}{1 + (x - y)^2 R(x, y)^2}.$$  

The operator defined by the first kernel on the right hand side is proved to be bounded on $L^p(\mathbb{R})$ in Proposition 3.1. Let

$$T_1f(x) = \int_{-\infty}^{\infty} \frac{(x - y)R(x, y)P(x, y)}{1 + (x - y)^2 R(x, y)^2} f(y) dy.$$  

We compare $T_1$ with the operator $W$ defined in Corollary 3.4 with the same $P$ and $R$. Let $T_2f(x) = T_1f(x) - Wf(x)$. Then

$$T_2f(x) = \int_{-\infty}^{\infty} \frac{P(x, y)}{(x - y)R(x, y)(1 + (x - y)^2 R(x, y)^2)} f(y) dy.$$
With $a = P(x, x)/R(x, x)$, define
\[ T_af(x) = T_rf(x) - aH f(x) \]
where $H$ is the Hilbert transform. Then
\[
\frac{1}{x-y} \left( \frac{P(x, y)}{R(x, y)(1 + (x-y)^2 R(x, y)^2)} - a \right)
\]
where $E(x, y) = (P(x, y) - aR(x, y))/(x-y)$ is a polynomial of degree $2n-1$. Hence, by Proposition 3.3, it suffices to show $L^p$-boundedness of the operator $T_f$ defined by
\[
T_f(x) = \int_{-\infty}^{\infty} \frac{(x-y)R^2(x, y)}{1 + (x-y)^2 R^2(x, y)} f(y) dy.
\]
Let $\varphi(x) = \chi_{|x|>1}$ if $|x| \geq 1$ and $\varphi(x) = 1$ if $|x| < 1$. By similar estimates as above and Proposition 3.1, we obtain
\[
\left| T_f(x) - \int_{|x-y| \geq \varphi(x)} \frac{f(y)}{x-y} dy \right| \leq M f(x).
\]
Note that
\[
\left| \int_{|x-y| \geq \varphi(x)} \frac{f(y)}{x-y} dy \right| \leq \sup_{\varepsilon > 0} \left| \int_{|x-y| \geq \varepsilon} \frac{f(y)}{x-y} dy \right|.
\]
Since the right hand side is bounded on $L^2$ (cf. p. 42, [S]), the proof is completed.

**Corollary 5.2.** If $A$ is an even polynomial, then the Cauchy transform $\mathcal{C}_A$ is bounded on $L^p(R)$, $1 < p < \infty$.

**Proof.** It follows from Theorem 5.1 and Lemma 2.1.

We now deal with the case when $A$ is a polynomial of even degree.

**Theorem 5.3.** Let $A$ be a polynomial of even degree. Then, the operator $T_A$ is bounded on $L^p(R)$, $1 < p < \infty$.

**Proof.** Let $A(x) = \sum_{j=0}^{2n+2} a_j x^j$ and let $r$ be the number given in Lemma 2.2. It
suffices to prove
\[ I := \int_{|x| > r} \int_{|y| > r} \frac{P(x, y)}{1 + iQ(x, y)} f(y) \, dy \, dx \leq \| f \|_p. \]

In fact, the rest cases can be treated by the Hölder inequality since \(|Q(x, y)| \approx |x|^{2n+1} + |y|^{2n+1}\) if either \(|x| < r\) and \(|y| > 2r\), or \(|x| > 2r\) and \(|y| < r\). In order to estimate \(I\), we make changes of variables \(y = \alpha(s)\) and \(x = \alpha(t)\) defined in Lemma 2.2. Then, since \(\alpha'(s) \approx 1\), we have
\[
I \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{P(\alpha(t), \alpha(s))}{1 + iQ(\alpha(t), \alpha(s))} F(s) \, ds \, dt,
\]
where \(F(s) = f(\alpha(s))\alpha'(s)\chi\) while \(\chi\) is the characteristic function on \(\{ |\alpha(s)| > r \}\). Let \(B(t) = A(\alpha(t))\). Then, \(B(t)\) is an even polynomial and
\[
Q(\alpha(t), \alpha(s)) = \frac{B(t) - B(s)}{\alpha(t) - \alpha(s)}.
\]
Since \(\alpha(t) = t + O(1)\), we have
\[
P(\alpha(t), \alpha(s)) = P_0(t, s) + E(t, s)
\]
where \(P_0(t, s)\) is a homogeneous polynomial of degree \(2n\) and \(E(t, s) = O(|t|^{2n-1} + |s|^{2n-1})\) if \(|t| + |s|\) is large. Since
\[
|Q(\alpha(t), \alpha(s))| = \left| \frac{B(t) - B(s)}{\alpha(t) - \alpha(s)} \right| \approx |t + s| \left( |t|^{2n} + |s|^{2n} \right)
\]
for \(|t| + |s|\) large, it is already proved in Proposition 3.1 that the operator defined by the kernel \(E(t, s)/[1 + iQ(\alpha(t), \alpha(s))]\) is bounded on \(L^p\). For convenience, put
\[
k_0(t, s) = \frac{P_0(t, s)}{1 + iQ(\alpha(t), \alpha(s))} \quad \text{and} \quad k(t, s) = \frac{P_0(t, s)}{1 + i\frac{B(t) - B(s)}{t - s}}.
\]

Then, \(k(t, s)\) defines an operator bounded on \(L^p\) by Theorem 4.1. A straightforward computation gives
\[
k_0(t, s) - k(t, s) = \frac{iP_0(t, s)Q(\alpha(t), \alpha(s))}{[1 + iQ(\alpha(t), \alpha(s))]} \frac{\beta(t) - \beta(s)}{t - s} \left[ 1 + i\frac{B(t) - B(s)}{t - s} \right].
\]
where \( \alpha(t) = t + \beta(t) \) as defined in Lemma 2.1. It then follows from Corollary 2.3 that
\[
| k_0(t, s) - k(t, s) | \leq \frac{(| t |^{2n} + | s |^{2n})(| t |^{2n} + | s |^{2n})}{1 + | t + s |^2(| t |^{2n} + | s |^{2n})^2} \frac{1}{| t | + | s |} \left( \frac{1}{| t |^{2n}} + \frac{1}{| s |^{2n}} \right)
\]
Hence, \( | k_0(t, s) - k(t, s) | \) defines an operator bounded on \( L^p \) by Proposition 3.1. This completes the proof.

Finally, we have the main theorem of this paper.

**Theorem.** If \( A \) is a polynomial, then the Cauchy transform on the curve \( y = A(x) \) is bounded on \( L^p (\mathbb{R}) \), \( 1 < p < \infty \).

**REFERENCES**


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