Re-nnd SOLUTIONS OF THE MATRIX EQUATION $AXB = C$

DRAGANA S. CVETKOVIĆ-ILIĆ

(Received 22 August 2006; revised 28 October 2007)

Communicated by J. J. Koliha

Abstract

In this article we consider Re-nnd solutions of the equation $AXB = C$ with respect to $X$, where $A$, $B$, $C$ are given matrices. We give necessary and sufficient conditions for the existence of Re-nnd solutions and present a general form of such solutions. As a special case when $A = I$ we obtain the results from a paper of Groß (‘Explicit solutions to the matrix inverse problem $AX = B$’, Linear Algebra Appl. 289 (1999), 131–134).


Keywords and phrases: matrix equation, Hermitian part, Re-nnd solutions.

1. Introduction

Let $\mathbb{C}^{n \times m}$ denote the set of complex $n \times m$ matrices. Here $I_n$ denotes the unit matrix of order $n$. By $A^*$, $R(A)$, rank$(A)$ and $N(A)$, we denote the conjugate transpose, the range, the rank and the null space of $A \in \mathbb{C}^{n \times m}$.

The Hermitian part of $X$ is defined as $H(X) = (1/2)(X + X^*)$. We say that $X$ is Re-nnd (Re-nonnegative definite) if $H(X) \geq 0$ and $X$ is Re-pd (Re-positive definite) if $H(X) > 0$.

The symbol $A^-$ stands for an arbitrary generalized inner inverse of $A$, that is, $A^-$ satisfies $AA^-A = A$. By $A^\dagger$ we denote the Moore–Penrose inverse of $A \in \mathbb{C}^{n \times m}$, that is, the unique matrix $A^\dagger \in \mathbb{C}^{m \times n}$ satisfying

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

For some important properties of generalized inverses see [5, 6, 16] and [15].
Many authors have studied the well-known equation

$$AXB = C,$$  \hspace{1cm} (1.1)

with the unknown matrix $X$, such that $X$ belongs to some special class of matrices. For example, in [18] and [7] the existence of reflexive and anti-reflexive, with respect to a generalized reflection matrix $P$, solutions of the matrix equation (1.1) was considered, while in [9, 14, 17, 19] necessary and sufficient conditions for the existence of symmetric and antisymmetric solutions of the equation (1.1) were investigated.

The Hermitian nonnegative definite solutions for the equation $AXA^* = B$ were investigated by Khatri and Mitra [14], Baksalary [4], Dai and Lancaster [10], Groß [12], Zhang and Cheng [23] and Zhang [24].

Wu [21] studied Re-pd solutions of the equation $AX = C$, and Wu and Cain [22] found the set of all complex Re-nnd matrices $X$ such that $XB = C$ and presented a criterion for Re-nndness. Groß [11] gave an alternative approach, which simultaneously delivers explicit Re-nnd solutions and gave a corrected version of some results from [22]. Some results from [22] were extended in the paper of Wang and Yang [20], in which the authors presented criteria for $2 \times 2$ and $3 \times 3$ partitioned matrices to be Re-nnd, found necessary and sufficient conditions for the existence of Re-nnd solutions of the equation (1.1) and derived an expression for these solutions. In the paper of Dajić and Koliha [3], a general form of Re-nnd solutions of the equation $AX = C$ is given for the first time, where $A$ and $C$ are given operators between Hilbert spaces. In addition to these papers many other papers have dealt with the problem of finding the Re-nnd and Re-pd solutions of some other forms of equations.

In this paper, we first consider the matrix equation

$$AXA^* = C,$$

where $A \in \mathbb{C}^{n \times m}$, $B \in \mathbb{C}^{n \times n}$, and find necessary and sufficient conditions for the existence of Re-nnd solutions. Also, we present a general form of these solutions. Using this result, we obtain necessary and sufficient conditions for the equation

$$AXB = C,$$

where $A \in \mathbb{C}^{n \times m}$, $B \in \mathbb{C}^{m \times n}$ and $C \in \mathbb{C}^{n \times n}$, to have a Re-nnd solution. This way, the results of [22] and [11] follow as a corollary and a general form of those solutions is given in addition. As far as the author is aware, this is the first time necessary and sufficient conditions for the existence of a Re-nnd solution of the equation $AXB = C$ have been given in terms of g-inverses.

Now, we state some well-known results which are used frequently in the next section.

**Theorem 1.1** Ben-Israel and Greville [5]. Let $A \in \mathbb{C}^{n \times m}$, $B \in \mathbb{C}^{p \times r}$ and $C \in \mathbb{C}^{n \times r}$. Then the matrix equation

$$AXB = C,$$
Solutions to $AXB = C$

is consistent if and only if, for some $A^-, B^-$,

$$AA^-CB^-B = C,$$

in which case the general solution is

$$X = A^-CB^- + Y - A^-AYBB^-,$$

for arbitrary $Y \in \mathbb{C}^{m \times p}$.

The following result was derived by Albert [1] for block matrices, by Cvetković-Ilić et al. [8] for $C^*$ algebras, and by Dajić and Koliha [3] for operators between different Hilbert spaces. Here, we give the basic version proved in [1].

**Theorem 1.2.** Let $M \in \mathbb{C}^{(n+m) \times (n+m)}$ be a Hermitian block matrix given by

$$M = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix},$$

where $A \in \mathbb{C}^{n \times n}$ and $D \in \mathbb{C}^{m \times m}$. Then, $M \geq 0$ if and only if

$$A \geq 0, \quad AA^\dagger B = B, \quad D - B^*A^\dagger B \geq 0.$$

Anderson and Duffin [2] define the parallel sum of matrices for a pair of matrices of the same order as

$$A : B = A(A + B)^-B.$$

It is clear that for this definition to be meaningful, the expression $A(A + B)^-B$ must be independent of the choice of the g-inverse $(A + B)^-$. Hence, a pair of matrices $A$ and $B$ will be said to be parallel summable if $A(A + B)^-B$ is invariant under the choice of the inverse $(A + B)^-$, that is, if

$$\mathcal{R}(A) \subseteq \mathcal{R}(A + B) \land \mathcal{R}(A^*) \subseteq \mathcal{R}(A^* + B^*),$$

or, equivalently,

$$\mathcal{R}(B) \subseteq \mathcal{R}(A + B) \land \mathcal{R}(B^*) \subseteq \mathcal{R}(A^* + B^*). \quad (1.2)$$

Note that

$$\mathcal{R}(A) \subseteq \mathcal{R}(B) \iff BB^-A = A.$$

By [13, Theorem 2.1],

$$\mathcal{R}(A) \subseteq \mathcal{R}(B) \iff AA^* \leq \lambda^2BB^* \quad \text{for some } \lambda \geq 0,$$

so, for the nonnegative definite matrices $A$ and $B$, we have that

$$A \leq A + B \iff \mathcal{R}(A^{1/2}) \subseteq \mathcal{R}((A + B)^{1/2}),$$
which implies \( \mathcal{R}(A) \subseteq \mathcal{R}((A + B)^{1/2}) \) or, equivalently,

\[
(A + B)^{1/2}((A + B)^{1/2})^\dagger A = A.
\]

Now,

\[
(A + B)(A + B)^\dagger = ((A + B)^{1/2}((A + B)^{1/2})^\dagger)^2 A = A,
\]

which is equivalent to \( \mathcal{R}(A) \subseteq \mathcal{R}(A + B) \).

Hence, nonnegative definite matrices \( A \) and \( B \) are parallel summable. Furthermore, in [2] it was proved that for a pair of parallel summable matrices the following expression holds:

\[
A : B = B : A,
\]

that is,

\[
A(A + B)^-B = B(A + B)^-A. \tag{1.3}
\]

2. Results

The next result was first proved by Wu and Cain [22] and later derived in a different way by Groß [11]. It gives necessary and sufficient conditions for the matrix equation \( AX = C \) to have a Re-nnd solution \( X \), where \( A, C \) are given matrices of suitable size and presents a possible explicit expression for \( X \).

**Theorem 2.1.** Let \( A \in \mathbb{C}^{n \times m}, \ C \in \mathbb{C}^{n \times m} \). There exists a Re-nnd matrix \( X \in \mathbb{C}^{m \times m} \) satisfying \( AX = C \) if and only if \( AA^\dagger C = C \) and \( AC^* \) is Re-nnd.

From the proof of this theorem we can see that

\[
X_0 = A^\dagger C - (A^\dagger C)^* + A^\dagger AC^*(A^\dagger)^*,
\]

is one of Re-nnd solutions of \( AX = C \). Also, in [11] the author mentions that any matrix of the form

\[
X = X_0 + (I - A^\dagger A)Y(I - A^\dagger A),
\]

with \( Y \in \mathbb{C}^{m \times m} \) which is Re-nnd is also a Re-nnd solution of \( AX = C \), in the case where such solutions exist, but he did not present a general form of such solutions. Our main aim is to generalize these results to the equation \( AXB = C \) and to present a general form of Re-nnd solutions of it.

First, we consider the equation

\[
AXA^* = C, \tag{2.1}
\]

and find necessary and sufficient conditions for the existence of Re-nnd solutions.

The next auxiliary result presents a general form of a solution \( X \) of (2.1) which satisfies \( H(X) = 0 \).
LEMMA 2.2. If $A \in \mathbb{C}^{n \times m}$, then $X \in \mathbb{C}^{m \times m}$ is a solution of the equation

$$AXA^* = 0,$$  
\hspace*{1cm} (2.2)

which satisfies $H(X) = 0$ if and only if

$$X = W(I - A^\dagger A) - (I - A^\dagger A)W^*,$$  
\hspace*{1cm} (2.3)

for some $W \in \mathbb{C}^{m \times m}$.

PROOF. Denote by $r = \text{rank}(A)$. Let us suppose that $X$ is a solution of the equation (2.2) and $H(X) = 0$. Using a singular value decomposition of $A = U^* \text{Diag}(D, 0)V$, where $U \in \mathbb{C}^{n \times n}$, $V \in \mathbb{C}^{m \times m}$ are unitary and $D \in \mathbb{C}^{r \times r}$ is an invertible matrix, we have that

$$A^\dagger = V^* \text{Diag}(D^{-1}, 0)U \quad \text{and} \quad X = V^* \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} V,$$

for some $X_1 \in \mathbb{C}^{r \times r}$ and $X_4 \in \mathbb{C}^{(m-r) \times (m-r)}$.

From $AXA^* = 0$ we obtain that $X_1 = 0$ and, by $H(X) = 0$, that $X_3 = -X_2^*$ and $H(X_4) = 0$. Hence,

$$X = V^* \begin{bmatrix} 0 & X_2 \\ -X_2^* & X_4 \end{bmatrix} V.$$

Taking into account that $H(X_4) = 0$, for

$$W = V^* \begin{bmatrix} I & X_2 \\ 0 & (1/2)X_4 \end{bmatrix} V,$$

we have that

$$X = W(I - A^\dagger A) - (I - A^\dagger A)W^*.$$

In the other direction we can easily check that for arbitrary $W \in \mathbb{C}^{m \times m}$, $X$ defined by (2.3) is a solution of the equation (2.2) which satisfies $H(X) = 0$.

THEOREM 2.3. Let $A \in \mathbb{C}^{n \times m}$, $C \in \mathbb{C}^{n \times n}$ be given matrices such that the equation (2.1) is consistent and let $r = \text{rank } H(C)$. There exists a Re-nnd solution of the equation (2.1) if and only if $C$ is Re-nnd. In this case the general Re-nnd solution is given by

$$X = A^\equiv C(A^\equiv)^* + (I - A^\equiv A)UU^*(I - A^\equiv A)^* + W(I - A^\dagger A) - (I - A^\dagger A)W^*,$$  
\hspace*{1cm} (2.4)

with

$$A^\equiv = A^- + (I - A^- A)Z(H(C)^{1/2})^-,$$  
\hspace*{1cm} (2.5)

where $A^-$ and $(H(C)^{1/2})^-$ are arbitrary but fixed generalized inverses of $A$ and $H(C)^{1/2}$, respectively, and $Z \in \mathbb{C}^{m \times n}$, $U \in \mathbb{C}^{m \times (m-r)}$, $W \in \mathbb{C}^{m \times m}$ are arbitrary matrices.
PROOF. If $X$ is a Re-nnd solution of the equation (2.1), then
\[ AH(X)A^* = H(C) \geq 0. \]

In the other direction, if $C$ is Re-nnd, then $X_0 = A^{-}C(A^{-})^*$ is a Re-nnd solution of the equation (2.1).

Let us prove that a representation of the general Re-nnd solution is given by (2.4). If $X$ is defined by (2.4), then $X$ is Re-nnd and $AXA^* = AA^{-}C(AA^{-})^* = C$.

If $X$ is an arbitrary Re-nnd solution of (2.1), then $H(X)$ is a Hermitian nonnegative-definite solution of the equation $AZA^* = H(C)$, so, by [12, Theorem 1],
\[ H(X) = H(A^{-}C(A^{-})^* + (I - A^{-}A)UU^*(I - A^{-}A)^*), \]

where $A^{-}$ is given by (2.5), for some $Z \in \mathbb{C}^{m\times n}$ and $U \in \mathbb{C}^{m\times (m-r)}$.

Note that,
\[ H(X) = H(A^{-}C(A^{-})^* + (I - A^{-}A)UU^*(I - A^{-}A)^*), \]

implying
\[ X = A^{-}C(A^{-})^* + (I - A^{-}A)UU^*(I - A^{-}A)^* + Z, \]

where $H(Z) = 0$ and $AZA^* = 0$. Using Lemma 2.2, we have that
\[ Z = W(I - A^\dagger A) - (I - A^\dagger A)W^*, \]

for some $W \in \mathbb{C}^{m\times n}$. Hence, we obtain that $X$ has a representation as in (2.4).

Now, let us consider the equation
\[ AXB = C, \quad (2.6) \]

where $A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{n\times n}$ are given matrices and find necessary and sufficient conditions for the existence of a Re-nnd solution.

Without loss of generality we may assume that $n = m$ and that matrices $A$ and $B$ are both nonnegative definite. This follows from the fact that whenever the equation (2.6) is consistent then $X$ is a solution of that equation if and only if $X$ is a solution of the equation $A^*AXB^* = A^*CB^*$. Hence, from now on, we assume that $A$ and $B$ are nonnegative-definite matrices from the space $\mathbb{C}^{n\times n}$.

The next theorem is the main result of this paper which presents necessary and sufficient conditions for the equation (2.6) to have a Re-nnd solution.

**Theorem 2.4.** Let $A, B, C \in \mathbb{C}^{n\times n}$ be given matrices such that $A$ and $B$ are nonnegative definite and the equation (2.6) is consistent. There exists a Re-nnd solution of (2.6) if and only if
\[ T = B(A + B)^{-}C(A + B)^{-}A, \]

where
is Re-nnd, where \((A + B)^-\) is a g-inverse of \(A + B\). In this case a general Re-nnd solution is given by

\[
X = (A + B)^\top (C + Y + Z + W) ((A + B)^\top)^\ast \\
+ (I - (A + B)^\top (A + B))UU^\ast (I - (A + B)^\top (A + B))^\ast \\
+ Q(I - (A + B)^\top (A + B)) - (I - (A + B)^\top (A + B))Q^\ast, 
\]

(2.7)

where \(Y, Z, W\) are arbitrary solutions of the equations

\[
Y(A + B)^\top B = C(A + B)^\top A, \\
A(A + B)^\top Z = B(A + B)^\top C, \\
A(A + B)^\top W(A + B)^\top B = T, 
\]

(2.8)

such that \(C + Y + Z + W\) is Re-nnd, \((A + B)^\top\) is defined by

\[(A + B)^\top = (A + B)^\top + (I - (A + B)^\top (A + B))P(H(C + Y + Z + W)^{1/2})^\top,\]

where \(U \in \mathbb{C}^{n\times(n-r)}\), \(Q \in \mathbb{C}^{n\times n}\), \(P \in \mathbb{C}^{n\times n}\) are arbitrary, \(r = \text{rank}(C + Y + Z + W)\).

**Proof.** Denote by

\[
E = (A + B)^\top B, \quad F = C(A + B)^\top A, \\
K = A(A + B)^\top, \quad L = B(A + B)^\top C. 
\]

Now, equations (2.8) are equivalent to

\[
YE = F, \quad KZ = L, \quad KWE = T. 
\]

(2.9)

Using (1.2), (1.3) and the fact that \(E\) is g-invertible and \(E^- = B^- (A + B)\), we have that

\[
FE^- E = C(A + B)^\top AB^- (A + B)(A + B)^\top B \\
= C(A + B)^\top AB^- B = CB^- B(A + B)^\top AB^- B \\
= CB^- A(A + B)^\top BB^- B = CB^- A(A + B)^\top B \\
= CB^- B(A + B)^\top A = C(A + B)^\top A = F, 
\]

which implies that the equation \(YE = F\) is consistent. In a similar way, we can prove that the other two equations from (2.9) are consistent. Furthermore, \(T^\ast = F^\ast E = KL^\ast\) is Re-nnd which implies, by Theorem 2.1, that the first two equations from (2.9) have Re-nnd solutions.

Now, suppose that the equation (2.6) has a Re-nnd solution \(X\). Then

\[
H(T) = H(B(A + B)^\top AXB(A + B)^\top A) \\
= (B(A + B)^\top A)H(X)(B(A + B)^\top A)^\ast \geq 0. 
\]
Conversely, let $T$ be Re-nnd. We can check that

$$X_0 = (A + B)^-(C + Y + Z + W)(A + B)^-,$$  \hspace{1cm} (2.10)

is a solution of the equation (2.6), where $Y, Z, W$ are arbitrary solutions of the equations (2.9). This follows from

$$AX_0B = (A + B)(A + B)^-C(A + B)^-(A + B)$$

$$= (A + B)(A + B)^-AA^-CB^-B(A + B)^-(A + B)$$

$$= AA^-CB^-B = C.$$

Now, we have to prove that for some choice of $Y, Z, W$, the matrix $C + Y + Z + W$ is Re-nnd which would imply that $X_0$ is Re-nnd.

Let

$$Y = FE^- - (FE^-)^* + (E^-)^*F*E^- + (I - EE^-)^*(I - EE^-),$$


$$W = K^-TE^- - (I - K^-K)S - S(I - EE^-),$$

where $Q = (C^* - K^-T*E^-)(C^* - K^-T*E^-)^*$ and $S = K^-KC^* + C*EE^-$. Obviously, $Y, Z, W$ are solutions of the equations (2.9) and

$$H(Y) = (E^-)^*H(T)E^- + (I - EE^-)^*(I - EE^-),$$


$$H(W) = K^-TE^- + (E^-)^*T^*(K^-)^* - H(C*EE^- + K^-KC^* - 2K^-T*E^-).$$

Using


$$KC^*E = KL^* = T^*,$$

we compute,

$$H(C + Y + Z + W) = ((E^-)^* + K^-)H(T)((E^-)^* + K^-)^*$$

$$+ [(I - EE^-)^* (I - K^-K)]D \begin{bmatrix} I - EE^- \\ (I - K^-K)^* \end{bmatrix},$$

where

$$D = \begin{bmatrix} I & C - (E^-)^*T(K^-)^* \\ C^* - K^-T^*E^- & H(Q) \end{bmatrix}.$$
Hence, with such a choice of \( Y, Z, W \), it can be seen that \( X_0 \) defined by (2.10) is Re-nnd solution of (2.6). So, we proved the sufficient part of the theorem.

Let \( X \) be an arbitrary Re-nnd solution of (2.6). It is evident that \( Y = AXA, \ Z = BXB \) and \( W = BXA \) are solutions of (2.9), and that

\[
(A + B)X(A + B) = C + Y + Z + W,
\]

is Re-nnd. Now, using Theorem 2.3, we obtain that \( X \) has the representation (2.7).

Let us mention that, if we apply Theorem 2.4 to the equation \( AX = C \), we obtain [11, Theorem 1] as a corollary.

Note that if the equation \( AX = C \) is consistent then \( X \) is a solution of it if and only if \( A^*AX = A^*C \). By Theorem 2.4, we obtain that there exists a Re-nnd solution of the equation \( AX = C \) if and only if

\[
T = (A^*A + I)^{-1}A^*C(A^*A + I)^{-1}A^*A,
\]

is Re-nnd. Note that in this case \( (I + A^*A) \) is invertible matrix.

Let us prove that \( T \) is Re-nnd if and only if \( CA^* \) is Re-nnd.

By

\[
\]

we have that

\[
T = ((A^*A + I)^{-1}A^*)(CA^*)((A^*A + I)^{-1}A^*)^*,
\]

that is,

\[
H(T) = ((A^*A + I)^{-1}A^*)H(CA^*)((A^*A + I)^{-1}A^*)^*.
\]

From the last equality, \( H(CA^*) \geq 0 \) implies \( H(T) \geq 0 \).

Now, suppose that \( H(T) \geq 0 \). Owing to the consistence of the equation \( AX = C \), it follows that \( AA^*C = C \) which implies that

\[
(A^\dagger)(A^*A + I)T((A^\dagger)(A^*A + I))^* = (A^\dagger)A^*CA^*AA^\dagger = AA^\dagger CA^* = CA^*,
\]

that is,

\[
H(CA^*) = ((A^\dagger)(A^*A + I))H(T)((A^\dagger)(A^*A + I))^* \geq 0.
\]

References


DRAGANA S. CVETKOVIĆ-ILIĆ, Faculty of Sciences and Mathematics, Department of Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia
e-mail: gagamaka@ptt.yu