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A REGULAR SPACE ON WHICH EVERY REAL-VALUED FUNCTION WITH A CLOSED GRAPH IS CONSTANT

BY

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ABSTRACT. An example is given of a regular space on which every real-valued function with a closed graph is constant. It was previously known that there are regular spaces on which every continuous function is constant. It is also shown here that there are regular spaces that support only constant real-valued continuous functions, but support non-constant real-valued functions with a closed graph.

1. **Introduction**. Let X and Y be topological spaces. A function $f : X \longrightarrow Y$ has a closed graph if $\{(x, f(x))|x \in X\}$ is closed in $X \times Y$. It is known that there are regular spaces on which every real-valued continuous function is constant. The purpose of this note is to construct a regular space on which every real-valued function with a closed graph is constant. In addition, it will be shown that there are regular spaces that support only constant real-valued continuous functions, but support non-constant real-valued with a closed graph. Throughout, R will be used to denote the real line with the usual topology.

The first example of a regular space on which every continuous real-valued function is constant was published by Hewitt in 1946 (see [9]). An example by Novak appeared in 1948 (see [13]). Since then other examples have appeared. See for example [1], [5], [8] and [10]. In particular, van Douwen [5] gives a clear and systematic method for constructing a regular space on which every real-valued continuous function is constant. In an interesting paper [10] which appeared in 1986, Iliadis and Tzannes investigate spaces on which every continuous function into a given range space, not necessarily the real line, is constant or locally constant.

Several papers have appeared on the points of discontinuity of functions with a closed graph. See for example [2], [4], [6], [11], [12] and [16]. The monograph by Hamlett and Harrington [5] is an excellent reference source on properties of functions with a closed graph and it also contains an extensive bibliography. In 1985, Doboš [4] showed that for a perfectly normal space X, a subset $F \subset X$ is closed and of the first category (in X) if and only if there exists a function $f : X \longrightarrow R$ with a closed graph such that the points of discontinuity of f coincide with F. The results which follow are part of an investigation into properties of real-valued function with a closed graph on spaces which are not perfectly normal. Further results are given in [3].

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2. **Preliminaries**. We state the following known results for later use.

THEOREM 2.1. *Let X be a normal topological space and let $f : X \longrightarrow R$ be a function with a closed graph. Then there exists a sequence of continuous functions $f_n : X \longrightarrow R$, $n = 1, 2, 3, \ldots$, such that $|f_n(x)| \leqq n$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$. (See* [11] *Theorem 6).*

LEMMA 2.2. *Let $f : X \longrightarrow Y$ be a function with a closed graph. If $K \subset Y$ is compact, then $f^{-1}(K)$ is closed in X. (See* [7] *page 6).*

LEMMA 2.3. *Let $f : X \longrightarrow Z$ be continuous. If $h : Z \longrightarrow Y$ has a closed graph, then $h \circ f$ has a closed graph (see* [16]).

Let $\mathcal{F}$ be the set of all real-valued functions with some property P on a topological space X.

NOTE 2.4. In what follows we will restrict our consideration to those properties P such that; (i) if h is a continuous function and f is any function with property P, then the composition, $f \circ h$, has property P, and (ii) if f has property P on a space X, then the restriction of f to any subspace of X also has property P.

Examples of classes of functions with a property P on a given space X are, the set of all continuous functions on X, the set of functions with closed graphs on X (by Lemma 2.3) and the set of all functions on X of Baire class $\alpha, \alpha < \Omega$, where Ω is the first uncountable ordinal.

Following van Douwen, two points a and b in X will be called twins for the family $\mathcal{F}$ of all real-valued functions with property P on X if $f(a) = f(b)$ for all $f \in \mathcal{F}$. Starting with an arbitrary regular space X with twins for the family $\mathcal{F}$ of all continuous real-valued functions on X, van Douwen in [5] gives a nice method for constructing a regular space on which every real-valued continuous function is constant. We now sketch van Douwen's method to show, how, starting with a regular space X with twins for the family $\mathcal{F}$ of all real-valued functions with property P on X, one can construct a regular space Y on which every real valued function with property P is constant.

VAN DOUWEN'S CONSTRUCTION. Let X be a regular space and a, $b \in X$ be twins for the set of all real-valued functions $\mathcal{F}$ with property P on X. Let I be an index set with the same cardinality as X. For every $s \in I$, let X_s be an homeomorphic copy of X. Let Z be the disjoint union of $\{X_s | s \in I\}$. V is open (closed) in Z if and only if $V \cap X_s$ is open (closed) in X_s for each $s \in I$. Let a_s and b_s denote the twins of X_s. Put $A = \{a_s | s \in I\}$ and $B = \{b_s | s \in I\}$. Let $g : A \longrightarrow Z \setminus \{A \cup B\}$ be a one-to-one onto function. Let Y be the decomposition of Z consisting of $\{B\}$ and the pairs $\{a_s, g(a_s)\}$ for $s \in I$. Y may be considered as the quotient space resulting from this decomposition where $q : Z \longrightarrow Y$ is the decomposition map.

Let f be any real-valued function on Y with property P. $f \circ q$ is a function from Z into R. Since q is continuous, $f \circ q$ has property P on Z by (i) of Note 2.4. By (ii) of Note 2.4, the restriction of $f \circ q$ to X_s has property P for each $s \in I$. Therefore, a_s, $b_s \in X_s$ are twins of $f \circ q$ for each $s \in I$. $f \circ q$ is constant on B and on each pair

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$\{a_s,\ g(a_s)\} \in Z$ for $s \in I$. Therefore $f \circ q$ is constant on Z. That is, f is constant on Y.

It was shown by van Douwen [5] that Y is regular (that is, T_3 and T_1). Therefore Y has the required properties.

3. **Functions with a closed graph on ordinal spaces**. The method we will use to construct a regular space Y on which every real-valued function with a closed graph is constant will be to first find a regular space X with twins for the set of all real valued functions with a closed graph on X. We will then construct the regular space Y by the condensation of twins as in Section 2. It is not difficult to see that the regular spaces, such as those given in [9] [14, page 109] and [17, page 134], which have twins for the set of continuous real valued functions, do not have twins for the set of real-valued functions with a closed graph. In this section we will begin the process of constructing a regular space X with twins for every real-valued function with a closed graph on X.

Let Ω denote the first uncountable ordinal and let $O = [0,\ \Omega]$ denote the set of all ordinals less than or equal to the first uncountable ordinal with the order topology. Let $O_0 = [0,\ \Omega)$.

LEMMA 3.1. *If $f : O_0 \longrightarrow R$ is a function with a closed graph, then there exists $x_0 \in O_0$ such that f is constant on $C = \{\beta \in O_0 | \beta > x_0\}$.*

PROOF. Since f has a closed graph and O_0 is normal, by Theorem 2.1, there exists a sequence of continuous real-valued functions f_n, $n = 1,\ 2,\ 3, \ldots,$ on O_0 such that $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in O_0$. For each integer n, there is an $x_n \in O_0$ such that f_n is constant on $\{\beta \in O_0 | \beta > x_n\}$. Let $x_0 = \sup x_n$. Then f is constant on $C = \{\beta \in O_0 | \beta > x_0\}$.

COROLLARY 3.2. *Let $O = [0, \Omega]$. If $f : O \longrightarrow R$ is a function with a closed graph, then there exists $x_0 \in O$, $x_0 < \Omega$, such that f is constant on $\{\beta \in O | \beta > x_0\}$.*

REMARK 3.3. Let $\{x_n | n = 1,\ 2, \ldots\}$ and $\{y_n | n = 1,\ 2, \ldots\}$ be two sequences in $O_0 = [0, \Omega)$. If $x_n \leqq y_n \leqq x_{n+1}$ for $n = 1,\ 2, \ldots,$ then both sequences converge to the same point in O_0.

LEMMA 3.3. *Let $T = [0, \Omega] \times [0, \Omega]$ and let $S = T - (\Omega, \Omega)$. If $f : S \longrightarrow R$ is a function with a closed graph, then f can be extended to a function $g : T \longrightarrow R$ such that the graph of g is closed, g is continuous at (Ω, Ω) and $g(x) = f(x)$ for all $x \in S$.*

PROOF. Let $\{(x_\alpha, y_\alpha)\}_{\alpha \in D}$ be a net of distinct points in S such that $x_a \longrightarrow \Omega$ and $y_\alpha \longrightarrow \Omega$. We will show that there exists a constant $c \in R$ such that $f((x_\alpha, y_\alpha)) \longrightarrow c$. Consider the following cases.

CASE 1. $x_\alpha = y_\alpha$ for all $\alpha \in D$. Since $\{(x,x) \in S | x < \Omega\}$ is homeomorphic to $[0, \Omega)$ and since f restricted to $\{(x,x) \in S | x < \Omega\}$ has a closed graph, there exists, by Lemma 3.1, an element $x_0 \in [0, \Omega)$ such that if $C = \{(x,x) \in S | x_0 < x < \Omega\}$,

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then f is identically equal to a constant, c, on C. Therefore, if $x_\alpha = y_\alpha$ for all $\alpha \in D$ and if $x_\alpha \longrightarrow \Omega$ and $y_\alpha \longrightarrow \Omega$, then $f(x_\alpha, y_\alpha) \longrightarrow c$.

CASE 2. Let $x_\alpha < y_\alpha < \Omega$, for every $\alpha \in D$, and for every $\beta \in [0, \Omega)$ let there be an at most countable subset K_β of D such that $x_\alpha = \beta$ for each $\alpha \in K_\beta$. Without loss of generality we may assume that $x_0 \leqq x_\alpha$ for all $\alpha \in D$, where f is identically equal to c on $\{(x,x) \in S | x_0 < x < \Omega\}$. Suppose $\{f(x_\alpha, y_\alpha)\}_{\alpha \in D}$ does not converge to c. Then there exists a subnet P of $\{(x_\alpha, y_\alpha)\}_{\alpha \in D}$ and a closed interval $J_1 \subset R$ such that $c \in \operatorname{int} J_1$ and $f(P) \cap J_1 = \emptyset$. Further, P is uncountable since no sequence in $[0, \Omega]$ converges to Ω. Also, for some closed and bounded interval $J_2 \subset J_1^c$, there is an uncountable subset $B \subset P \subset \{(x_\alpha, y_\alpha) | \alpha \in D\}$ such that $f(B) \subset J_2$. [Write J_1^c as a countable union of disjoint bounded intervals I_n, $n = 1, 2, 3, \ldots$, and let $B_n = \{(x_\alpha, y_\alpha) \in P | f(x_\alpha, y_\alpha) \in I_n\}$. Since $P = \bigcup B_n$ and P is uncountable, B_n must be uncountable for some n]. Since B is uncountable and since, for each $\beta \in [0, \Omega)$, there is an at most countable subset of K_β of D such that $(x_\alpha, y_\alpha) = (\beta, y_\alpha)$ for each $\alpha \in K_\beta$, it follows that if $B = \{(x_\gamma, y_\gamma) | \gamma \in D_1\}$, then $x_\gamma \longrightarrow \Omega$ and $y_\gamma \longrightarrow \Omega$. Select a sequence $\{(x_{\gamma_n}, y_{\gamma_n})\}_{n=1}^\infty$ from B as follows: Let $(x_{\gamma_1}, y_{\gamma_1})$ be any element of B. For each integer $n \geqq 1$, when $(x_{\gamma_n}, y_{\gamma_n}) \in B$ has been selected, select $(x_{\gamma_{n+1}}, y_{\gamma_{n+1}})$ where $x_{\gamma_{n+1}}$ is the smallest ordinal such that $(x_{\gamma_{n+1}}, y_{\gamma_{n+1}}) \in B$ for some $y_{\gamma_{n+1}}$ and $x_{\gamma_n} < y_{\gamma_n} \leqq x_{\gamma_{n+1}}$. Then, by Remark 3.3, $\{(x_{\gamma_n}, y_{\gamma_n})\}_{n=1}^\infty \longrightarrow (x,x)$, where $x_0 < x$. Also $\{f(x_{\gamma_n}, y_{\gamma_n}\} \subset J_2$, while $f(x,x) = c \in J_1$. J_2 is compact and (x,x) is a limit point of $f^{-1}(J_2)$, which is not in $f^{-1}(J_2)$. Therefore $f^{-1}(J_2)$ is not closed. By Lemma 2.2, this would contradict the fact that the graph of f is closed. Therefore $\{f(x_\alpha, y_\alpha)\}_{\alpha \in D}$ converges to c.

CASE 3. Let $x_\alpha < y_\alpha = \Omega$ for all $\alpha \in D$. Since $\{(\beta, \Omega) | \beta < \Omega\}$ is homeomorphic too $[0, \Omega)$ by Lemma 3.1, there exists $x_1 \in [0, \Omega)$ such that if $A = \{(\beta, \Omega) | x_i \leqq \beta < \Omega\}$, then f restricted to $\{(\beta, \Omega) | \beta < \Omega\}$ is equal to a constant, c_1, on A. Without loss of generality we may assume $x_0 \leqq x_1$ and $x_1 \leqq x_\alpha$ for all $\alpha \in D$. Also, for each β, where $x_1 \leqq \beta < \Omega$, f is identically equal to a constant, c_β, on some tail, A_β, of $\{(\beta, \gamma) | \gamma \leqq \Omega\}$. Since $A_\beta \cap A = (\beta, \Omega)$, and since $f(\beta, \Omega) = c_1$, it follows that $c_\beta = c_1$, for each $\beta \geqq x$. For each $\beta \in [x_1, \Omega)$, select $(x_\beta, y_\beta) \in A_\beta$ such that $x_\beta < y_\beta < \Omega$. It follows that $f(x_\beta, y_\beta) = c_1$, for each β and that $\{(x_\beta, y_\beta)\}_{\beta \geqq x_1}$ is a net in S such that $x_\beta \longrightarrow \Omega$, and $y_\beta \longrightarrow \Omega$. Now, by case 2, we see that $f(x_\beta, y_\beta) = c$, for all β. Therefore $c_1 = c$, and in this case $f(x_\alpha, y_\alpha) = c$ for all $\alpha \in D$.

CASE 4. If $\{(x_\alpha, y_\alpha)\}_{\alpha \in D}$ is any net in S such that $x_\alpha \leqq y_\alpha \leqq \Omega$, $x_\alpha \longrightarrow \Omega$ and $y_\alpha \longrightarrow \Omega$, then $\{(x_\alpha, y_\alpha)\}_{\alpha \in D}$ can be broken into at most three disjoint nets each of which would satisfy the conditions of either Case 1, 2 or 3 and the image of each net under f would therefore converge to c.

CASE 5. If $\{(x_\alpha, y_\alpha) | \alpha \in D\}$ is any net in S such that $y_\alpha \leqq x_\alpha \leqq \Omega$, then since $\{(x,y) \in S | x \leqq y \leqq \Omega\}$ is homeomorphic to $\{(x,y) \in S | y \leqq x \leqq \Omega\}$, it follows, as above, that $\{f(x_\alpha, y_\alpha)\}_{\alpha \in D}$ converges to c.

Put $g(x) = f(x)$ if $x \in S$ and put $g((\Omega, \Omega)) = c$, where f is identically equal to c on $C = \{(x,x) \in S | x_0 < x < \Omega\}$. $g(x)$ has the required properties.

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4. **The space Y**. First we will construct a regular space X with two points $a,\ b \in X$ such that a and b are twins for every real valued function on X with a closed graph. X will be a modification of the example in [17, problem 186] which is based on Hewitt's example [9].

EXAMPLE 4.1. Let $T = [0,\Omega]\times[0,\Omega]$ and let $S = T-(\Omega,\Omega)$. Let Z denote the set of positive and negative integers and form the product $S \times Z$. Form the quotient space K by identifying points in $S\times Z$ as follows: if n is odd, identify the corresponding points in the right edges of $S\times\{n\}$ and $S\times\{n+1\}$ and if n is even identify the corresponding points in the top edges of $S \times \{n\}$ and $S \times \{n+1\}$. Add to K two points a and b at 'infinity' and put $X = K\cup\{a,b\}$. The image of $S_n = S \times \{n\}$ in X is homeomorphic to S for each n. Let $U_n(a) = \{a\} \cup \bigcup_{m=n}^{\infty} S_m$ and $U_n(b) = \{b\} \cup \bigcup_{m=n}^{\infty} S_{-m}$ be neighbourhoods of a and b, respectively, for $n = 1,\ 2,\ 3,\ldots$. It follows as in [14] and [17] that X is a regular space which is not completely regular.

Let $f : X \longrightarrow R$ be any function with a closed graph. Show $f(a) = f(b)$. It follows from Lemma 3.1, Lemma 3.3 and the construction of X, that for every positive integer n, f is identically equal to some constant c, on both the right edge and top edge of S_n, except for at most a countable number of points. Now select a_n, a member of the top edge of S_n, for each positive integer n, such that $f(a_n) = c$. Then $a_n \longrightarrow a$. If $a \notin f^{-1}(c)$, then $f^{-1}(c)$ is not closed in X. This, by Lemma 2.2, contradicts the fact that the graph of f is closed. Therefore $f(a) = c$. Similarly $f(b) = c$ and $f(a) = f(b)$.

EXAMPLE 4.2. We now show there exists a regular space Y on which every real-valued function with a closed graph is constant. It follows from Lemma 2.3 that if $h : Z \longrightarrow Y$ is a function with a closed graph, then condition (i) of Note 2.4 is satisfied for any continuous function $f : X \longrightarrow Z$. It is easily seem that a function with a closed graph also satisfies condition (ii) of Note 2.4. Now let $\mathcal{F}$ be the family of all real-valued function on the space X of Example 4.1 with the property that, for each $f \in \mathcal{F}$, f has a closed graph. Then, by van Douwen's Construction as in Section 2, we can construct a space Y, starting with X, such that every real-valued function $f : Y \longrightarrow R$ with a closed graph is constant.

EXAMPLE 4.3. Let X, a and b be as in Example 4.1. We will show there exists a function $f : X \longrightarrow R$ of Baire class I such that $f(a) \neq f(b)$. Define $f : X \longrightarrow R$ by $f(a) = -1$, $f(b) = 1$ and $f(x) = 0$, if $x \in X$ and $x \neq a$, $x \neq b$. Let F be a closed set in R, clearly $f^{-1}(F)$ is a G_δ subset of X and f is of Baire class I.

5. In this section a regular space Y will be constructed with the property that Y supports only constant real-valued continuous functions, but it will be shown that Y supports non-constant real-valued functions with a closed graph. Y will be constructed using the method of van Douwen from Section 2 and will be based on an example of a regular space X, which is not completely regular, by Thomas [15]. An outline of Thomas' example follows. See [15] for further details and a geometric interpretation.

EXAMPLE 5.1. (Thomas). If $n = 0,\ \pm1,\ \pm2,\ \pm3,\ldots$, put $L(2n) = \{(2n,y) \in R^2 | 0 \leqq y < 1/2\}$. If $n = 0,\ \pm1,\ \pm2,\ \pm3,\ldots$, and $k = 2,\ 3\ 4,\ldots$, put $p(2n-1,k) =$

$(2n-1, 1-1/k) \in R^2$, and put $T(2n-1, k) = \{(2n-1 \pm t, 1-t-1/k) \in R^2 | t \in (0, 1-1/k]\}$. Let a, b be two points 'at infinity'. Put

$$X = \left\{ \bigcup_{n=-\infty}^{+\infty} L(2n) \right\} \cup \left\{ \bigcup_{n=-\infty}^{\infty} \bigcup_{k=2}^{\infty} \{T(2n-1,\ k) \cup p(2n-1,\ k)\} \right\} \cup \{a,\ b\}.$$

Topologize X as follows. If $x \in T(2n-1,\ k)$ for $n = 0,\ \pm 1,\ \pm 2,\ \pm 3, \ldots$, and $k = 2,\ 3, \ldots$, then x is open. If $x = p(2n-1,\ k)$, then every neighbourhood of x contains all but finitely many points of $T(2n-1,\ k)$. If $x = (2n,\ y) \in L(2n)$, then a neighbourhood of x consists of all but finitely many of the points of X with the same y-coordinate and with x-coordinate that differs from n by less than 1. If $x = a$ and c is a real number, then a subset of X which consists of all points with y-coordinate $> c$ is an open set containing a. If $x = b$ and c is a real number, then a subset of X which consists of all points with y-coordinate $< c$ is an open set containing b. It was shown in [15] that X is regular and if $f : X \longrightarrow R$ is continuous, then $f(a) = f(b)$.

Example 5.2. We now will give an example of a regular space Y on which every continuous function $g : Y \longrightarrow R$ is constant and an example of a non-constant function $f : Y \longrightarrow R$ with a closed graph. Let X be the space of Example 5.1. $a,\ b \in X$ are twins for every real-valued continuous function on X. Starting with this space, construct a regular space Y, as in Section 2, on which every continuous real-valued function is constant. In this construction, we may consider X_s for each $s \in I$, as

$$X_s = \left\{ \bigcup_{n=-\infty}^{+\infty} L(2n,\ s) \right\} \cup \left\{ \bigcup_{n=-\infty}^{+\infty} \bigcup_{k=2}^{+\infty} \{T(2n-1,\ k,\ s) \cup p(2n-1,\ k,\ s)\} \right\} \cup \{a_s,\ b_s\}.$$

For each $s \in I$, we may consider X_s as consisting of the countable family of lines $L(2n,\ s)$, for $n = 0, \pm 1, \pm 2, \ldots$, together with the countable family of lines $\{T(2n-1,\ k,\ s) \cup p(2n-1,\ k,\ s)\}$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$, $k = 2,\ 3, \ldots$, and the points a_s and b_s. Let $s_0 \in I$ be fixed. Define a function $h : X_{s_0} \longrightarrow R$ such that h is a one-to-one correspondence between the positive integers and the countable family of lines whose union, together with a_{s_0} and b_{s_0} equals X_{s_0}. Recall, Z is the disjoint union of $\{X_s | s \in I\}$. Define $f : Z \longrightarrow R$ as follows; for each fixed n, define f on $L(2n,\ s)$ such that $f(L(2n,\ s)) = h(L(2n,\ s_0))$ for each $s \in I$; for each fixed n and k, define f on $\{T(2n-1,\ k,\ s) \cup p(2n-1,\ k,\ s)\}$ such that $f(\{T(2n-1,\ k,\ s) \cup p(2n-1,\ k,\ s)\}) = h(\{T(2n-1,\ k,\ s_0) \cup p(2n-1,\ k,\ s_0)\})$, for each $s \in I$; put $f(b_s) = 0$ for each $s \in I$; and for each $s \in I$, put $f(a_s) = f(g(a_s))$, where g is the one-to-one function from A onto $Z \setminus \{A \cup B\}$ defined in Section 2. Since $f(B) = 0$ and $f(a_s) = f(g(a_s))$ for every $\{a_s,\ g(a_s)\}$ in the decomposition space Y, f may be considered as a mapping from Y into R. It will follow from the next three lemmas that the graph of f is closed.

Note that an open subset V of Z is said to be saturated relative to the decomposition space Y of Z if and only if V can be written as the union of elements of Y. That is, if and only if $q(V)$ is open in Y, where $q : Z \longrightarrow Y$ is the decomposition mapping. The space Y in the next three lemmas will be the space constructed in Example 5.2.

LEMMA 5.3. *Let J be a subset of the index set I (used in the construction of Y) and let $\{a_\alpha, g(a_\alpha)\}_{\alpha\in J}$ be a net in Y such that $\{g(a_\alpha)\}_{\alpha\in J} \subset \{T(2n_0-1,\ k_0,\ s) \cup p(2n_0-1,\ k_0,\ s)\}$ for n_0 and k_0 fixed and $s \in I$. If $\{a_\alpha, g(a_\alpha)\}_{\alpha\in J} \longrightarrow x$, then $x = \{a_\beta, g(a_\beta)\}$, where $g(a_\beta) \in \{T(2n_0-1,\ k_0,\ s) \cup p(2n_0-1,\ k_0,\ s)\}$ for $s \in I$.*

PROOF. Let n_0 and k_0 be fixed. Suppose $x \neq \{a_\beta, g(a_\beta)\}$ for $g(a_\beta) \in \{T(2n_0-1,\ k_0,\ s) \cup p(2n_0-1,\ k_0,\ s)\}$ and $s \in I$. There are then two cases to be considered: (i) $x = \{a_\beta, g(a_\beta)\}$, where $g(a_\beta) \notin \{T(2n_0-1,\ k_0,\ s) \cup p(2n_0-1,\ k_0,\ s)\}$ for $s \in I$; (ii) $x = \{B\}$.

CASE 1. Let $x = \{a_\beta, g(a_\beta)\}$, where $g(a_\beta) \notin \{T(2n_0-1,\ k_0,\ s) \cup p(2n_0-1,\ k,\ s)\}$ for $s \in I$. In this case, we will construct an open set $U \subset Y$ such that $\{a_\beta, g(a_\beta)\} \in U$ but $\{a_\alpha, g(a_\alpha)\} \notin U$ for all $\alpha \in J$. $g(a_\beta) \in X_{i_0}$ for some fixed $i_0 \in I$. There is an open set in Z, $U(g(a_\beta))$, such that $g(a_\beta) \in U(g(a_\beta)) \subset X_{i_0}$ and $U(g(_\beta)) \cap \{T(2n_0-1,\ k_0,\ s) \cup p(2n_0-1,\ k_0, s)\} = \phi$ for all $s \in I$. Similarly, there is an open set $U(a_\beta)$ in Z such that $a_\beta \in U(a_\beta) \subset X_\beta$, where $a_\beta \in X_\beta$, and $U(a_\beta) \cap \{T(2n_0-1,\ k_0,\ s) \cup p(2n_0-1,\ k_0,\ s)\} = \emptyset$ for all $s \in I$. Now for each $s \in I$, $s \neq \beta$, select an open set A_s, containing a_s, such that $A_s \subset X_s$ and $g(a_\alpha) \notin A_s$ for all $\alpha \in J$. This is possible since for each $s \in I$, A_s can be selected as a subset of $\{T(2n_0+1,\ k_0,\ s) \cup p(2n_0-1,\ k_0,\ s)\}^c$. Put $V_1 = U(g(a_\beta)) \cup U(a_\beta)$. Put $V_{n+1} = V_n \cup [\{A_s | g(a_s) \in V_n\}]$, for $n = 1, 2, 3, 4, \ldots$. Put $V = \bigcup_n V_n$. It follows from the choice of A_s and V_n, $n = 1, 2, \ldots$, that V is a saturated open set in Z. Clearly a_β and $g(a_\beta)$ are both members of V. Put $U = q(V)$, where q is the natural quotient map from Z onto Y. Then U is an open set in Y which contains $\{a_\beta, g(a_\beta)\}$ and it does not contain $\{a_\alpha, g(a_\alpha)\}$ for each $\alpha \in J$. Therefore $x \neq \{a_\beta, g(a_\beta)\}$, when $g(a_\beta) \notin \{T(2n_0-1,\ k_0,\ s) \cup p(2n_0-1,\ k_0,\ s)\}$ for $s \in I$.

CASE (ii). Suppose $x = \{B\}$. It can be shown in a way similar to (i) that $\{a_\alpha, g(a_\alpha)\}_{\alpha\in J}$ does not converge to x.

The only remaining possibility is that $x = \{a_\beta, g(a_\beta)\}$, where $g(a_\beta) \in \{T(2n_0-1,\ k_0,\ s) \cup p(2n_0-1,\ k_0,\ s)\}$ for $s \in I$.

LEMMA 2. *Let $\{a_\alpha, g(a_\alpha)\}_{\alpha\in J}$ be a net in Y such that $\{g(a_\alpha)\}_{\alpha\in J} \subset \{L(2n_0,\ s)\}$ where n_0 is fixed and $s \in I$. If $\{a_\alpha,\ g(a_\alpha)\} \longrightarrow x$, then $x = \{a_\beta, g(a_\beta)\}$ for some $\beta \in I$ and $g(a_\beta) \in \{L(2n_0, s)\}_{s\in I}$.*

PROOF. Similar to Lemma 1 and is omitted.

LEMMA 3. *If $f : Y \longrightarrow R$ is the function constructed in Example 5.2, then the graph of f is closed.*

PROOF. Let $\{(x_\alpha,\ f(x_\alpha))\}_{\alpha\in J}$ be any net in the graph of f such that $(x_\alpha,\ f(x_\alpha)) \longrightarrow (x,\ y) \in Y \times R$. We must show that $y = f(x)$. Since $x_\alpha \longrightarrow x$ and $f(x_\alpha) \longrightarrow y$, it follows from the definition of f that $y = m$, where m is a positive integer or zero, and, that $f(x_\alpha)$ is identically equal to m for all $\alpha \geqq \alpha(m)$. If $m = 0$, and $x_\alpha \in Y$ such that $f(x_\alpha) = 0$, then $x_\alpha = B$. Hence $f(x_\alpha) = f(B) = f(x) = 0 = y$ for all $\alpha \geqq \alpha(m)$.

If $m \neq 0$, then $\{x_\alpha\}_{\alpha\in J} = \{a_\alpha, g(a_\alpha)\}_{\alpha\in J} \subset f^{-1}(m)$ for all $\alpha \geqq \alpha(m)$. In this case put $T_m = \{g(a_s) \in Z - (A \cup B) \mid f(a_s, g(a_s)) = m\}$. Then for each m, there exists integers n_0 and k_0 such that **either** $T_m = \{T(2n_0 - 1,\ k_0,\ s) \cup p(2n_0 - 1,\ k_0,\ s)\}$ **or** $T_m = \{L(2n_0,\ s)\}$ where s ranges over I. In either case it follows that $\{g(a_\alpha)\} \in T_m$ for $\alpha \in J$ and $\alpha \geqq \alpha(m)$. Since $\{a_\alpha, g(a_\alpha)\} \longrightarrow x$, it follows from Lemmas 1 and 2 that $x = \{a_\beta, g(a_\beta)\}$ where $g(a_\beta) \in T_m$. Therefore $f(x) = m = y$ and the graph of f is closed.

REMARK. If the regular space Y constructed in Example 5.2 had been based on the regular spaces given in [14, page 109] and [17, problem 18G] which have twins, then it can be shown in a similar way that there exists a real valued function on Y which has a closed graph whose range is the set of non-negative integers.

REFERENCES

1. A. Armentrout, *A Moore space on which every real-valued continuous function is constant*, Proc. Amer. Math. Soc. **12** (1961), 106–109.

2. I. Baggs, *Functions with a closed graph*, Proc. Amer. Math. Soc. **43** (1974), 439–442.

3. ———, Nowhere dense sets and real-valued functions with a closed graph, *Internat. J. Math. and Math. Sci.* **12** (1989), 1–8.

4. J. Doboš, *On the set of points of discontinuity for functions with a closed graph*, Čas. Pěst. Mat. **110** (1985), 60–68.

5. E. K. van Douwen, *A regular space on which every continuous real-valued function is constant*, Niew Archief voor Wiskunde **20** (1972), 143–145.

6. R. V. Fuller, *Relations amongst continuous and various non-continuous functions*, Pac. J. Math. **25** (1968), 495–509.

7. T. R. Hamlett and L. L. Herrington, *The closed graph and P-closed graph properties in general topology*, Amer. Math. Soc. Contemporary Math. Volume 3, (1980).

8. H. Herrlich, *Wann sind alle statigen Abbildungen in Y konstant?*, Math. Zeitschr. **90** (1965), 152–154.

9. E. Hewitt, *On two problems of Urysohn*, Annals of Math. **47** (1946), 503–509.

10. S. Iliadis and V. Tzannes, *Spaces on which every continuous map into a given space is constant.* Can. J. Math. **38** (1986), 1281–1298.

11. P. Kostyroko, *A note on the functions with closed graphs*, Čas. Pěst. Mat. **94** (1969), 202–205.

12. P. E. Long, *Functions with closed graphs*, Amer. Math. Monthly **76** (1969), 930–932.

13. J. Novak, *Regular space on which every continuous function is constant*, Časopis Pěst. Mat. Fys. **73** (1948), 58–68.

14. L. A. Stein and J. A. Seebach, Jr., *Counterexamples in Topology*, Holt, Rinehart and Winston, Inc., 1970.

15. J. Thomas, *A regular space, not completely regular*, Amer. Math. Monthly, **76** (1969), 181–182.

16. T. Thompson, *Characterizing certain sets with functions having closed graphs*, Boll. del U.M.I., (4) **12** (1979), 327–329.

17. S. W. Willard, *General Topology*, Addison-Wesley, Reading, Mass., 1970.

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