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ON THE EXISTENCE OF GLOBAL WEAK SOLUTIONS TO AN INTEGRABLE TWO-COMPONENT CAMASSA–HOLM SHALLOW-WATER SYSTEM

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Abstract In this paper, we investigate the existence of global weak solutions to an integrable two-component Camassa–Holm shallow-water system, provided the initial data $u_0(x)$ and $\rho_0(x)$ have end states $u_\pm$ and $\rho_\pm$, respectively. By perturbing the Cauchy problem of the system around rarefaction waves of the well-known Burgers equation, we obtain a global weak solution for the system under the assumptions $u_- \leqslant u_+$ and $\rho_- \leqslant \rho_+$.

Keywords: integrable two-component Camassa–Holm shallow-water system; rarefaction wave; global weak solutions

2010 *Mathematics subject classification:* Primary 35G25; 35L05

1. Introduction

In this paper, we consider the integrable two-component Camassa–Holm shallow-water system

$$\left.\begin{aligned} u_t - u_{txx} + 3uu_x &= 2u_x u_{xx} + uu_{xxx} - \sigma\rho\rho_x, && t>0,\ x\in\mathbb{R},\\ \rho_t + (u\rho)_x &= 0, && t>0,\ x\in\mathbb{R},\\ u(0,x) &= u_0(x), && x\in\mathbb{R},\\ \rho(0,x) &= \rho_0(x), && x\in\mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where $\sigma = \pm 1$. System (1.1) was initially introduced in [**43**] as a tri-Hamiltonian system and was recently derived in the context of shallow-water theory [**12**,**35**,**37**]. The variable $u(x,t)$ describes the horizontal velocity of the fluid, and the variable $\rho(x,t)$ denotes the horizontal deviation of the surface from equilibrium, all measured in dimensionless units [**12**]. It is an integrable system, and the inverse scattering has been developed in the recent paper [**31**]. Its geometric properties are studied in [**24**]. The case $\sigma = 1$ corresponds to the situation in which the acceleration due to gravity acts downwards [**12**].

System (1.1) with $\sigma = -1$ is identified with the first negative flow of the Ablowitz–Kaup–Newell–Segur hierarchy and has peakon and multi-kink solutions [**5**,**26**]. System (1.1) also has other physical backgrounds. It describes the closure of the kinetic moments of the single-particle probability distribution for geodesic motion on the symplectomorphisms in Vlasov plasma models [**32**] and is also summoned in a type of matching procedure called metamorphosis in the large-deformation diffeomorphic approach to image matching [**33**]. The mathematical properties of (1.1) have been studied in many works: see [**4**, **5**, **12**, **25**, **30**, **35**, **44**, **46**].

For $\rho \equiv 0$, (1.1) becomes the Camassa–Holm equation, modelling the unidirectional propagation of shallow-water waves over a flat bottom. Here $u(t, x)$ stands for the fluid velocity at time t in the spatial x-direction [**3**, **15**, **21**, **34**, **36**, **38**]. The Camassa–Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [**17**, **18**]. It has a bi-Hamiltonian structure [**6**, **27**] and is completely integrable [**3**]. Also, there is a geometric interpretation of (1.1) in terms of geodesic flow on the diffeomorphism group of the circle [**14**, **39**]. Recently, it was claimed in [**40**] that the equation might be relevant to the modelling of tsunami; see also the discussion in [**13**].

The Cauchy problem and initial-boundary-value problem for the Camassa–Holm equation have been studied extensively [**9**, **10**, **19**, **22**, **23**, **41**, **45**, **49**]. It has been shown that this equation is locally well-posed [**9**, **10**, **19**, **41**, **45**] for initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. More interestingly, it has global strong solutions [**7**, **9**, **10**] and finite-time blow-up solutions [**7**–**11**, **19**, **41**, **45**]. On the other hand, it has global weak solutions in $H^1(\mathbb{R})$ [**1**, **2**, **16**, **48**].

For $\rho \not\equiv 0$, the Cauchy problems of (1.1) with $\sigma = -1$ and with $\sigma = 1$ have been discussed in [**25**] and [**12**], respectively. A new global existence result and several new blow-up results of strong solutions to (1.1) with $\sigma = 1$ were obtained in [**28**]. The obtained results in [**28**] were sharp and improved considerably on the recent results in [**12**]. The existence of global weak solutions to (1.1) with $\sigma = 1$ was proved recently in [**29**].

In this paper, we further study the existence of global weak solutions to (1.1) with $\sigma = 1$, provided that the initial data $u_0(x)$ and $\rho_0(x)$ have end states $u_\pm$ and $\rho_\pm$, respectively. By perturbing the Cauchy problem around rarefaction waves of the well-known Burgers equation and obtaining some *a priori* estimates of approximate solutions, we prove the existence of global weak solutions to (1.1) with $\sigma = 1$ under the assumptions $u_- \leqslant u_+$ and $\rho_- \leqslant \rho_+$, respectively. The recent result in [**29**] is the special case of our result with $u_- = u_+ = 0$ and $\rho_- = \rho_+ = 1$.

The paper has the following structure. In § 2, we present some lemmas to the perturbed Cauchy problem of (1.1) with $\sigma = 1$ around rarefaction waves. In § 3, we present an existence result of global weak solutions to (1.1) with $\sigma = 1$.

2. Preliminaries

In this section, we present the local well-posedness for the perturbed Cauchy problem of (1.1) with $\sigma = 1$ in $H^2(\mathbb{R}) \times H^1(\mathbb{R})$, the precise blow-up scenarios and global existence results for strong solutions to the perturbing system, and several useful lemmas, which will be used in the following.

For given w_- and w_+, we consider

$$\left.\begin{aligned} w_t + ww_x &= 0, & t>0,\ x\in\mathbb{R},\\ w(0,x) = w_0(x) &= \tfrac{1}{2}(w_+ + w_-) + \tilde{w}\tanh(x), & x\in\mathbb{R}, \end{aligned}\right\} \tag{2.1}$$

where $\tilde{w} = \frac{1}{2}(w_+ - w_-)$.

Lemma 2.1 (see [50]). *Assume that $\tilde{w} = \frac{1}{2}(w_+ - w_-) \geqslant 0$. Equation (2.1) has a unique global smooth solution $w(t,x)$ satisfying the following.*

(i) $w_- \leqslant w(t,x) \leqslant w_+$, $0 \leqslant w_x(t,x) \leqslant \tilde{w}$ *for* $x\in\mathbb{R}$, $t>0$.

(ii) *For any $k\in\mathbb{N}^+$, $1\leqslant k\leqslant 4$, and p, $1\leqslant p\leqslant\infty$, there exists a positive constant $C_{k,p}$ such that*

$$\|\partial_x^k w(t,\cdot)\|_{L^p} \leqslant C_{k,p}\tilde{w} \quad \forall t\geqslant 0.$$

Note that if $p(x) := \frac{1}{2}\mathrm{e}^{-|x|}$, $x\in\mathbb{R}$, then $(1-\partial_x^2)^{-1}f = p * f$ for all $f\in L^p(\mathbb{R})$. System (1.1) with $\sigma = 1$ takes the form of a quasi-linear evolution equation of hyperbolic type:

$$\left.\begin{aligned} u_t + uu_x &= -\partial_x p * (u^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}\rho^2), & t>0,\ x\in\mathbb{R},\\ \rho_t + (u\rho)_x &= 0, & t>0,\ x\in\mathbb{R},\\ u(0,x) &= u_0(x), & x\in\mathbb{R},\\ \rho(0,x) &= \rho_0(x), & x\in\mathbb{R}. \end{aligned}\right\} \tag{2.2}$$

In this paper, we suppose that $\lim_{x\to\pm\infty} u(x) = u_\pm$, $\lim_{x\to\pm\infty}\rho(x) = \rho_\pm$ and that ϕ, φ are the solutions of (2.1) with initial data $\phi_0(x) = \frac{1}{2}(u_+ + u_-) + \frac{1}{2}(u_+ - u_-)\tanh(x)$ and $\varphi_0(x) = \frac{1}{2}(\rho_+ + \rho_-) + \frac{1}{2}(\rho_+ - \rho_-)\tanh(x)$, respectively.

Letting $v = u - \phi$ and $\gamma = \rho - \varphi$, (2.2) takes the form

$$\left.\begin{aligned} v_t + vv_x &= -\partial_x p * ((v+\phi)^2 + \tfrac{1}{2}(v_x+\phi_x)^2 + \tfrac{1}{2}(\gamma+\varphi)^2) - (\phi v)_x,\\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t>0,\ x\in\mathbb{R},\\ \gamma_t + (v+\phi)\gamma_x &= -(v_x+\phi_x)\gamma - ((v+\phi)\varphi)_x + \varphi\varphi_x, \qquad t>0,\ x\in\mathbb{R},\\ v(0,x) &= u_0(x) - \phi_0(x), \qquad x\in\mathbb{R},\\ \gamma(0,x) &= \rho_0(x) - \varphi_0(x), \qquad x\in\mathbb{R}. \end{aligned}\right\} \tag{2.3}$$

We can obtain the following two lemmas using arguments similar to those in [**25**].

Lemma 2.2. *If $u_+ \geqslant u_-$, $u_0 - \phi_0 \in H^2(\mathbb{R})$ and $\rho_+ \geqslant \rho_-$, $\rho_0 - \varphi_0 \in H^1(\mathbb{R})$, then there exist a maximal $T = T(\|u_0-\phi_0\|_{H^2(\mathbb{R})} + \|\rho_0-\varphi_0\|_{H^1(\mathbb{R})}) > 0$ and a unique solution $z = \binom{v}{\gamma}$ to (2.3) with the initial data*

$$z_0 = \begin{pmatrix} v_0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} u_0 - \phi_0 \\ \rho_0 - \varphi_0 \end{pmatrix}$$

such that

$$z \in C([0,T); H^2(\mathbb{R})\times H^1(\mathbb{R})) \cap C^1([0,T); H^1(\mathbb{R})\times L^2(\mathbb{R})).$$

Moreover, the mapping $z_0 \to z(\cdot, z_0)\colon H^2(\mathbb{R})\times H^1(\mathbb{R}) \to C([0,T); H^2(\mathbb{R})\times H^1(\mathbb{R})) \cap C^1([0,T); H^1(\mathbb{R})\times L^2(\mathbb{R}))$ is continuous.

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Lemma 2.3. *Let $u_+ \geqslant u_-$, $\rho_+ \geqslant \rho_-$ and*

$$z_0 = \begin{pmatrix} v_0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} u_0 - \phi_0 \\ \rho_0 - \varphi_0 \end{pmatrix} \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$$

and let T be the maximal existence time of the solution $z = \binom{v}{\gamma}$ to (2.3) with the initial data z_0. Then, the corresponding solution z blows up in finite time if and only if

$$\limsup_{t \to T} \|v_x(t, \cdot)\|_{L^\infty(\mathbb{R})} = +\infty.$$

Remark 2.4. Let $u_+ \geqslant u_-$, $\rho_+ \geqslant \rho_-$ and

$$z_0 = \begin{pmatrix} v_0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} u_0 - \phi_0 \\ \rho_0 - \varphi_0 \end{pmatrix} \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$$

and let T be the maximal existence time of the solution $z = \binom{v}{\gamma}$ to (2.3) with the initial data z_0. Define $u = v + \phi$ and $\rho = \gamma + \varphi$. Since $\phi_t + \phi\phi_x = 0$ and $\varphi_t + \varphi\varphi_x = 0$, we have that $y = \binom{u}{\rho}$ is the strong solution of (2.2).

Consider the initial-value problem

$$\left.\begin{aligned} q_t &= v(t, q) + \phi(t, q), \quad && t \in [0, T), \\ q(0, x) &= x, && x \in \mathbb{R}, \end{aligned}\right\} \tag{2.4}$$

where v denotes the first component of the solution z to (2.3). Applying classical results in the theory of ordinary differential equations, one can obtain the following result on q, which is crucial in studying global existence.

Lemma 2.5 (see [12,25]). *Let $v, \phi \in C([0, T) \times \mathbb{R})$. Then, (2.4) has a unique solution $q \in C^1([0, T) \times \mathbb{R}; \mathbb{R})$. Moreover, the map $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$, with*

$$q_x(t, x) = \exp\left(\int_0^t (v_x + \phi_x)(s, q(s, x))\,\mathrm{d}s\right) > 0 \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

Lemma 2.6. *Let $u_+ \geqslant u_-$, $\rho_+ \geqslant \rho_-$ and*

$$z_0 = \begin{pmatrix} v_0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} u_0 - \phi_0 \\ \rho_0 - \varphi_0 \end{pmatrix} \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$$

and let T be the maximal existence time of the solution $z = \binom{v}{\gamma}$ to (2.3) with the initial data z_0. Then, we have that

$$(\gamma + \varphi)(t, q(t, x)) q_x(t, x) = \gamma_0(x) + \varphi_0(x) \quad \forall (t, x) \in [0, T) \times \mathbb{R}; \tag{2.5}$$

that is,

$$\rho(t, q(t, x)) q_x(t, x) = \rho_0(x) \quad \forall (t, x) \in [0, T) \times \mathbb{R},$$

where $\rho = \gamma + \varphi$.

Proof. Let $u = v + \phi$. By Remark 2.4, (u, ρ) satisfies (2.2). Differentiating (2.5) with respect to t, in view of (2.4) and (2.2), we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&(\rho(t, q(t, x))q_x(t, x)) \\ &= (\rho_t(t, q(t, x)) + \rho_x(t, q(t, x))q_t(t, x))q_x(t, x) + \rho(t, q(t, x))q_{xt}(t, x) \\ &= (\rho_t(t, q(t, x)) + \rho_x(t, q(t, x))u(t, q) + \rho(t, q)u_x(t, q))q_x(t, x) \\ &= 0.\end{aligned}$$

This completes the proof of the lemma. □

By Lemma 2.6 and Lemma 2.1, for any $t \in (0, T)$, we have that

$$\|\gamma(t, \cdot)\|_{L^\infty(\mathbb{R})} \leqslant \exp\biggl(\int_0^t \|u_x(s, \cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s\biggr)\|\rho_0\|_{L^\infty(\mathbb{R})} + |\rho_-| + |\rho_+|. \tag{2.6}$$

Lemma 2.7. *Let $u_+ \geqslant u_-$, $\rho_+ \geqslant \rho_-$ and*

$$z_0 = \begin{pmatrix} v_0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} u_0 - \phi_0 \\ \rho_0 - \varphi_0 \end{pmatrix} \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$$

and let T be the maximal existence time of the solution $z = \binom{v}{\gamma}$ to (2.3) with the initial data z_0. Then, there exists C_1 depending only on $u_\pm$ and $\rho_\pm$ such that

$$E(t) = \int_{\mathbb{R}} (v^2 + v_x^2 + \gamma^2)\,\mathrm{d}x \leqslant \mathrm{e}^{C_1 t}(1 + \|z_0\|^2_{H^1(\mathbb{R})\times L^2(\mathbb{R})}) - 1 \quad \forall t \in [0, T).$$

Moreover, we have that

$$\|v(t, \cdot)\|^2_{L^\infty(\mathbb{R})} \leqslant \tfrac{1}{2}\mathrm{e}^{C_1 t}(1 + \|v_0\|^2_{H^1(\mathbb{R})} + \|\gamma_0\|^2_{L^2(\mathbb{R})}) \quad \forall t \in [0, T)$$

and

$$\|u(t, \cdot)\|^2_{L^\infty(\mathbb{R})} \leqslant \mathrm{e}^{C_1 t}(1 + \|v_0\|^2_{H^1(\mathbb{R})} + \|\gamma_0\|^2_{L^2(\mathbb{R})}) + ((u_-)^2 + (u_+)^2) \quad \forall t \in [0, T),$$

where $u = v + \phi$.

Proof. Differentiating the first equation in (2.3) with respect to x and using the identity $\partial_x^2 p * f = p * f - f$, we have that

$$\begin{aligned}v_{tx} &+ (v + \phi)(v + \phi)_{xx} + \tfrac{1}{2}(v_x + \phi_x)^2 - \phi_x^2 - \phi\phi_{xx} \\ &= (v + \phi)^2 + \tfrac{1}{2}(\gamma + \varphi)^2 - p * ((v + \phi)^2 + \tfrac{1}{2}(v_x + \phi_x)^2 + \tfrac{1}{2}(\gamma + \varphi)^2).\end{aligned} \tag{2.7}$$

We write that $f=(v+\phi)^2+\frac{1}{2}(v_x+\phi_x)^2+\frac{1}{2}(\gamma+\varphi)^2$. Using (2.3) and (2.7), and integrating by parts, we get that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(1+E(t))\\
&\quad=2\int_{\mathbb{R}}(vv_t+v_xv_{xt}+\gamma\gamma_t)\,\mathrm{d}x\\
&\quad=2\int_{\mathbb{R}}[-v((v+\phi)v_x+\partial_xp*f+\phi_xv)\\
&\qquad\qquad+v_x(-(v+\phi)(v+\phi)_{xx}-\tfrac{1}{2}(v_x+\phi_x)^2+\phi_x^2+\phi\phi_{xx}+(v+\phi)^2)\\
&\qquad\qquad\qquad+\tfrac{1}{2}(\gamma+\varphi)^2-p*((v+\phi)^2+\tfrac{1}{2}(v_x+\phi_x)^2+\tfrac{1}{2}(\gamma+\varphi)^2)\\
&\qquad\qquad\qquad\qquad+\gamma(-((v+\phi)\gamma)_x-((v+\phi)\varphi)_x+\varphi\varphi_x)]\,\mathrm{d}x\\
&\quad=2\int_{\mathbb{R}}(-\tfrac{3}{2}\phi_xv^2-\tfrac{1}{2}\phi_xv_x^2-vv_x\phi_{xx}-\phi\phi_xv+\tfrac{1}{2}\phi_x^2v_x+\gamma v_x\varphi+\tfrac{1}{2}\varphi^2v_x\\
&\qquad\qquad\qquad\qquad-\tfrac{1}{2}\phi_x\gamma^2+(v+\phi)\varphi\gamma_x+\gamma\varphi\varphi_x)\,\mathrm{d}x\\
&\quad\leqslant2\int_{\mathbb{R}}(-vv_x\phi_{xx}-\phi\phi_xv+\tfrac{1}{2}\phi_x^2v_x+\gamma v_x\varphi-\varphi\varphi_xv+v\varphi\gamma_x\\
&\qquad\qquad\qquad\qquad-\phi_x\varphi\gamma-\phi\varphi_x\gamma+\gamma\varphi\varphi_x)\,\mathrm{d}x\\
&\quad\leqslant4(\|\phi\|_{W^{2,\infty}(\mathbb{R})}+\|\varphi\|_{L^\infty(\mathbb{R})})\biggl(E(t)+\int_{\mathbb{R}}(\phi_x^2+\varphi_x^2)\,\mathrm{d}x\biggr)\\
&\quad\leqslant C_1(u_\pm,\rho_\pm)(1+E(t)),
\end{aligned}$$

where we applied Lemma 2.1 and Hölder's inequality. By means of Gronwall's inequality and the above inequality, we have that

$$E(t)\leqslant\mathrm{e}^{C_1t}(1+\|v_0\|_{H^1(\mathbb{R})}^2+\|\gamma\|_{L^2(\mathbb{R})}^2)-1.$$

Using this inequality and Sobolev's imbedding theorem, we obtain that

$$\begin{aligned}
\|v(t,\cdot)\|_{L^\infty(\mathbb{R})}^2&\leqslant\tfrac{1}{2}\|v|_{H^1(\mathbb{R})}^2\leqslant\tfrac{1}{2}(\|v\|_{H^1(\mathbb{R})}^2+\|\gamma\|_{L^2(\mathbb{R})}^2)\\
&\leqslant\tfrac{1}{2}\mathrm{e}^{C_1t}(1+\|v_0\|_{H^1(\mathbb{R})}^2+\|\gamma_0\|_{L^2(\mathbb{R})}^2)\quad\forall t\in[0,T).
\end{aligned}$$

Applying Lemma 2.1 and the relation $u=v+\phi$, for any $t\in[0,T)$, we obtain that

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})}^2\leqslant\mathrm{e}^{C_1t}(1+\|v_0\|_{H^1(\mathbb{R})}^2+\|\gamma_0\|_{L^2(\mathbb{R})}^2)+((u_-)^2+(u_+)^2).$$

This completes the proof of the lemma. □

Lemma 2.8. *Let $u_+\geqslant u_-$, $\rho_+\geqslant\rho_-$ and*

$$z_0=\begin{pmatrix}v_0\\ \gamma_0\end{pmatrix}=\begin{pmatrix}u_0-\phi_0\\ \rho_0-\varphi_0\end{pmatrix}\in H^2(\mathbb{R})\times H^1(\mathbb{R})$$

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and let T be the maximal existence time of the solution

$$z = \begin{pmatrix} v \\ \gamma \end{pmatrix} = \begin{pmatrix} u - \phi \\ \rho - \varphi \end{pmatrix}$$

to (2.3) with the initial data z_0. If there exists α such that $\rho_0(x) \geqslant \alpha > 0$ for all $x \in \mathbb{R}$, then there exists C depending only on $u_\pm$ and $\rho_\pm$ such that, for any $t \in [0, T)$, we have that

$$\begin{aligned} \|u_x(t,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant \frac{1}{\alpha}(\|\gamma_0\|^2_{L^\infty(\mathbb{R})} + \|v_{0,x}\|^2_{L^\infty(\mathbb{R})} + C) \\ \times \exp(4(\mathrm{e}^{Ct}(1 + \|v_0\|^2_{H^1(\mathbb{R})} + \|\gamma_0\|^2_{L^2(\mathbb{R})}) + C)t) \end{aligned}$$

and

$$\begin{aligned} \|v_x(t,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant \frac{1}{\alpha}(\|\gamma_0\|^2_{L^\infty(\mathbb{R})} + \|v_{0,x}\|^2_{L^\infty(\mathbb{R})} + C) \\ \times \exp(4(\mathrm{e}^{Ct}(1 + \|v_0\|^2_{H^1(\mathbb{R})} + \|\gamma_0\|^2_{L^2(\mathbb{R})}) + C + 1)t) + u_+ - u_-. \end{aligned}$$

Proof. By Lemma 2.5, we know that $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$, with

$$q_x(t,x) = \exp\left(\int_0^t u_x(s, q(s,x))\,\mathrm{d}s\right) > 0 \quad \forall (t,x) \in [0,T) \times \mathbb{R}.$$

Then, we have that

$$\|u_x(t, q(t,\cdot))\|_{L^\infty(\mathbb{R})} = \|u_x(t,\cdot)\|_{L^\infty(\mathbb{R})} \quad \forall t \in [0,T). \tag{2.8}$$

Set $M(t,x) = u_x(t, q(t,x))$ and $N(t,x) = \rho(t, q(t,x))$. By Remark 2.4 and (2.4), we have that

$$\frac{\partial M}{\partial t} = (u_{tx} + u u_{xx})(t, q(t,x)), \qquad \frac{\partial N}{\partial t} = -NM \tag{2.9}$$

and

$$M_t = -\tfrac{1}{2}M^2 + u^2 + \tfrac{1}{2}N^2 - p * (u^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}\rho^2)(t,q). \tag{2.10}$$

In view of Lemma 2.7, we obtain that

$$\begin{aligned} 0 &\leqslant p * (u^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}\rho^2) \\ &\leqslant 2p * (v^2 + v_x^2 + \gamma^2) + 2p * (\phi^2 + \phi_x^2 + \varphi^2) \\ &\leqslant 2\|p\|_{L^\infty(\mathbb{R})}\|v^2 + v_x^2 + \gamma^2\|_{L^1(\mathbb{R})} + 2\|p\|_{L^1(\mathbb{R})}\|\phi^2 + \phi_x^2 + \varphi^2\|_{L^\infty(\mathbb{R})} \\ &\leqslant \mathrm{e}^{C_1 t}(1 + \|z_0\|^2_{H^1(\mathbb{R})\times L^2(\mathbb{R})}) + 2(|u_-| + |u_+| + |\rho_-| + |\rho_+|). \end{aligned}$$

If we write that $f(t,x) = u^2(t,q) - p * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2)(t,q)$, then

$$|f(t,x)| \leqslant 2\mathrm{e}^{C_1 t}(1 + \|v_0\|^2_{H^1(\mathbb{R})} + \|\gamma_0\|^2_{L^2}) + C_2 \quad \forall (t,x) \in [0,T) \times \mathbb{R} \tag{2.11}$$

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and

$$M_t = -\tfrac{1}{2}M^2 + \tfrac{1}{2}N^2 + f(t,x), \quad (t,x) \in [0,T) \times \mathbb{R}, \tag{2.12}$$

where $C_2 = 2(|u_-| + |u_+| + |\rho_-| + |\rho_+|) + (u_-)^2 + (u_+)^2$. By Lemmas 2.5 and 2.6, we know that $N(t,x)$ has the same sign, with $N(0,x) = \rho_0(x)$ for every $x \in \mathbb{R}$. Thus,

$$N(t,x)N(0,x) > 0 \quad \forall x \in \mathbb{R}.$$

Next, we consider the Lyapunov function

$$w(t,x) = N(0,x)N(t,x) + \frac{N(0,x)}{N(t,x)}[1 + M^2(t,x)], \quad (t,x) \in [0,T) \times \mathbb{R}, \tag{2.13}$$

first introduced in [**12**]. By Sobolev's imbedding theorem, we have that

$$\begin{aligned} 0 < w(0,x) &= N^2(0,x) + 1 + M^2(0,x) \\ &\leqslant 2\gamma_0^2(x) + 2\varphi^2 + 1 + 2v_{0,x}^2(x) + 2\phi_x^2 \\ &\leqslant 2\|\gamma_0\|_{L^\infty(\mathbb{R})}^2 + 2\|v_{0,x}\|_{L^\infty(\mathbb{R})}^2 + C_3, \end{aligned} \tag{2.14}$$

where C_3 depends only on $u_\pm$ and $\rho_\pm$. Differentiating (2.13) with respect to t and using (2.9)–(2.12), we obtain that

$$\begin{aligned} \frac{\partial w}{\partial t}(t,x) &= 2\frac{N(0,x)}{N(t,x)}M(t,x)(f(t,x) + \tfrac{1}{2}) \\ &\leqslant 4(\mathrm{e}^{C_1 t}(1 + \|v_0\|_{H^1(\mathbb{R})}^2 + \|\gamma_0\|_{L^2(\mathbb{R})}^2) + C_2 + 1)\frac{N(0,x)}{N(t,x)}(1 + M^2) \\ &\leqslant 4(\mathrm{e}^{C_1 t}(1 + \|v_0\|_{H^1(\mathbb{R})}^2 + \|\gamma_0\|_{L^2(\mathbb{R})}^2) + C_2 + 1)w(t,x). \end{aligned}$$

By Gronwall's inequality, the above inequality and (2.14), we have that

$$\begin{aligned} w(t,x) &\leqslant w(0,x)\exp(4(\mathrm{e}^{C_1 t}(1 + \|v_0\|_{H^1(\mathbb{R})}^2 + \|\gamma_0\|_{L^2}^2) + C_2 + 1)t) \\ &\leqslant (2\|\gamma_0\|_{L^\infty(\mathbb{R})}^2 + 2\|v_{0,x}\|_{L^\infty(\mathbb{R})}^2 + C_3) \\ &\qquad \times \exp(4(\mathrm{e}^{C_1 t}(1 + \|v_0\|_{H^1(\mathbb{R})}^2 + \|\gamma_0\|_{L^2(\mathbb{R})}^2) + C_2 + 1)t) \end{aligned} \tag{2.15}$$

for all $(t,x) \in [0,T) \times \mathbb{R}$. On the other hand,

$$w(t,x) \geqslant 2\sqrt{N^2(0,x)(1 + M^2)} \geqslant 2\alpha|M(t,x)| \quad \forall (t,x) \in [0,T) \times \mathbb{R}.$$

Thus,

$$\begin{aligned} |M(t,x)| &\leqslant \frac{1}{2\alpha}w(t,x) \\ &\leqslant \frac{1}{2\alpha}(2\|\gamma_0\|_{L^\infty(\mathbb{R})}^2 + 2\|v_{0,x}\|_{L^\infty(\mathbb{R})}^2 + C_3) \\ &\qquad \times \exp(4(\mathrm{e}^{C_1 t}(1 + \|v_0\|_{H^1(\mathbb{R})}^2 + \|\gamma_0\|_{L^2(\mathbb{R})}^2) + C_2 + 1)t) \end{aligned}$$

for all $(t,x) \in [0,T) \times \mathbb{R}$. Then, by (2.15) and the above inequality, we have that

$$\begin{aligned}\|u_x(t,\cdot)\|_{L^\infty(\mathbb{R})} &= \|u_x(t,q(t,\cdot))\|_{L^\infty(\mathbb{R})}\\ &\leqslant \frac{1}{2\alpha}(2\|\gamma_0\|^2_{L^\infty(\mathbb{R})} + 2\|v_{0,x}\|^2_{L^\infty(\mathbb{R})} + C_3)\\ &\qquad \times \exp(4(\mathrm{e}^{C_1 t}(1+\|v_0\|^2_{H^1(\mathbb{R})} + \|\gamma_0\|^2_{L^2(\mathbb{R})}) + C_2 + 1)t).\end{aligned}$$

Noting that $v = u - \phi$, in view of Lemma 2.1, we have that

$$\begin{aligned}\|v_x(t,\cdot)\|_{L^\infty(\mathbb{R})} &\leqslant \frac{1}{2\alpha}(2\|\gamma_0\|^2_{L^\infty(\mathbb{R})} + 2\|v_{0,x}\|^2_{L^\infty(\mathbb{R})} + C_3)\\ &\qquad \times \exp(4(\mathrm{e}^{C_1 t}(1+\|v_0\|^2_{H^1(\mathbb{R})} + \|\gamma_0\|^2_{L^2(\mathbb{R})}) + C_2 + 1)t) + u_+ - u_-.\end{aligned}$$

Take $C = \max\{C_1, C_2, C_3\}$. This completes the proof of the lemma. □

From Lemmas 2.3 and 2.8, we have the following result.

Lemma 2.9. *Let $u_+ \geqslant u_-$, $\rho_+ \geqslant \rho_-$ and*

$$z_0 = \begin{pmatrix} v_0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} u_0 - \phi_0 \\ \rho_0 - \varphi_0 \end{pmatrix} \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$$

and let T be the maximal existence time of the solution

$$z = \begin{pmatrix} v \\ \gamma \end{pmatrix} = \begin{pmatrix} u - \phi \\ \rho - \varphi \end{pmatrix}$$

to (2.3), with the initial data z_0. If there exists α such that $\rho_0(x) \geqslant \alpha > 0$ for all $x \in \mathbb{R}$, then the corresponding strong solution $z = \binom{v}{\gamma}$ to (2.3) exists globally in time.

Remark 2.10. If there exists $\alpha < 0$ such that $\rho_0(x) \leqslant \alpha$ for any $x \in \mathbb{R}$, then the conclusion of Lemma 2.9 also holds true.

Finally, we give a useful lemma, which will be used in § 3.

Lemma 2.11 (see [47]). *Assume that X, B, Y are Banach spaces and that $X \subset B \subset Y$ with compact imbedding $X \hookrightarrow B$. If F is bounded in $L^\infty(0,T;X)$ and $\partial F/\partial t$ is bounded in $L^r(0,T;Y)$, where $r > 1$, then F is relatively compact in $C(0,T;B)$.*

3. The existence of global weak solutions

In this section, we first establish the global existence of approximate strong solutions to the perturbed system. By acquiring the precompactness of approximate solutions, we then prove the existence of the global weak solutions to the perturbed system. Finally, using Lemma 2.1, we obtain the existence of the global weak solutions to (1.1) with $\sigma = 1$.

Before giving the precise statements of the main result, we first introduce the definition of a weak solution to the Cauchy problem (1.1) with $\sigma = 1$.

Definition 3.1. If $y = \binom{u}{\rho}$ satisfies (2.2), $u(t,\cdot) \to u_0$ and $\rho(t,\cdot) \to \rho_0$ as $t \to 0^+$ in the sense of distributions, then y is called a weak solution to (1.1) with $\sigma = 1$.

The main result of this paper can be stated as follows.

Theorem 3.2. *Let $u_+ \geqslant u_-$, $\rho_+ \geqslant \rho_- \geqslant \alpha > 0$ and*

$$z_0 = \begin{pmatrix} v_0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} u_0 - \phi_0 \\ \rho_0 - \varphi_0 \end{pmatrix} \in (H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})) \times (L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})).$$

If there exists $0 < \beta \leqslant \alpha$ such that $\gamma_0(x) \geqslant \beta - \alpha$ for almost everywhere (a.e.) $x \in \mathbb{R}$, then (1.1) with $\sigma = 1$ has a weak solution. Moreover,

$$u \in L^\infty_{\text{loc}}(\mathbb{R}_+; W^{1,\infty}(\mathbb{R})) \quad \textit{and} \quad \rho \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^\infty(\mathbb{R})).$$

Proof. The proof of the theorem is divided into three steps.

Step 1 (the global existence of approximate solutions). Let

$$z_0 = \begin{pmatrix} v_0 \\ \gamma_0 \end{pmatrix} \in (H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})) \times (L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}))$$

and assume that there exist $\alpha \geqslant \beta > 0$ such that $\rho_+ \geqslant \rho_- \geqslant \alpha$ and $\gamma_0(x) \geqslant \beta - \alpha$ for a.e. $x \in \mathbb{R}$.

Define

$$z_0^n := \begin{pmatrix} \chi_n * v_0 \\ \chi_n * \gamma_0 \end{pmatrix} = \begin{pmatrix} v_0^n \\ \gamma_0^n \end{pmatrix} \in H^2(\mathbb{R}) \times H^1(\mathbb{R}) \quad \text{for } n \geqslant 1,$$

where $\{\chi_n\}_{n\geqslant 1}$ are the mollifiers

$$\chi_n(x) := \left(\int_{\mathbb{R}} \chi(\xi)\,\mathrm{d}\xi \right)^{-1} n\chi(nx), \quad x \in \mathbb{R},\ n \geqslant 1,$$

where $\chi \in C_c^\infty(\mathbb{R})$ is defined by

$$\chi(x) = \begin{cases} \mathrm{e}^{1/(x^2-1)}, & |x| < 1, \\ 0, & |x| \geqslant 1. \end{cases}$$

In view of $\chi_n(x) \geqslant 0$ for all $x \in \mathbb{R}$ and $\|\chi_n\|_{L^1(\mathbb{R})} = 1$, we get that

$$\rho_0^n(x) = \chi_n * \gamma_0(x) + \varphi_0(x) \geqslant \beta - \alpha + \alpha = \beta > 0 \quad \forall x \in \mathbb{R}.$$

Obviously, we have that

$$z_0^n \to z_0 \quad \text{in } H^1(\mathbb{R}) \times L^2(\mathbb{R}) \text{ as } n \to \infty \tag{3.1}$$

and

$$\|z_0^n\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \leqslant \|z_0\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})}. \tag{3.2}$$

By Lemma 2.9, we obtain that the corresponding solution $z^n = (v^n, \gamma^n) \in C(\mathbb{R}_+; H^2(\mathbb{R}) \times H^1(\mathbb{R}))$ to (2.3), with initial data z_0^n, exists globally in time.

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Remark 3.3. By Lemma 2.7 and (3.2), we have that

$$\begin{aligned}\|v^n(t,\cdot)\|^2_{H^1(\mathbb{R})}+\|\gamma^n(t,\cdot)\|^2_{L^2(\mathbb{R})} &\leqslant \mathrm{e}^{C_1 t}(1+\|z_0^n\|^2_{H^1(\mathbb{R})\times L^2(\mathbb{R})})\\ &\leqslant \mathrm{e}^{C_1 t}(1+\|z_0\|^2_{H^1(\mathbb{R})\times L^2(\mathbb{R})}) \quad \forall t\in\mathbb{R}_+ \end{aligned}\tag{3.3}$$

and

$$\|v^n(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\leqslant \tfrac{1}{2}\mathrm{e}^{C_1 t}(1+\|v_0\|^2_{H^1(\mathbb{R})}+\|\gamma_0\|^2_{L^2(\mathbb{R})}) \quad \forall t\in\mathbb{R}_+. \tag{3.4}$$

Step 2 (the precompactness of approximate solutions). We define $P^n(t,x) =: p*((v^n+\phi)^2+\frac{1}{2}(v^n_x+\phi_x)^2+\frac{1}{2}(\gamma^n+\varphi)^2)(t,x) = p*((u^n)^2+\frac{1}{2}(u^n_x)^2+\frac{1}{2}(\rho^n)^2)(t,x)$ and let T be any fixed time in the following text.

Lemma 3.4. *There exist a pair of subsequences $\{z^{n_k},P^{n_k}\}\subset\{z^n,P^n\}$ and a pair of functions*

$$z\in L^\infty((0,T);(W^{1,\infty}(\mathbb{R})\cap H^1(\mathbb{R}))\times(L^2(\mathbb{R})\cap L^\infty(\mathbb{R}))),\qquad \bar{P}\in L^\infty((0,T);W^{1,\infty}(\mathbb{R}))$$

such that

$$z^{n_k}\rightharpoonup z \quad \text{weakly in } H^1((0,T)\times\mathbb{R})\times L^2((0,T)\times\mathbb{R}) \text{ as } n_k\to\infty, \tag{3.5}$$

$$v^{n_k}\to v \quad \text{uniformly on each compact subset of } \mathbb{R}_+\times\mathbb{R} \text{ as } n_k\to\infty \tag{3.6}$$

and

$$P^{n_k}\to \bar{P} \quad \text{uniformly on each compact subset of } \mathbb{R}_+\times\mathbb{R} \text{ as } n_k\to\infty. \tag{3.7}$$

Proof. By Lemma 2.7, we can easily obtain that $\{z^n(t,x)\}$ is uniformly bounded in $L^\infty((0,T);H^1(\mathbb{R})\times L^2(\mathbb{R}))$.

We claim that the sequence $\{v^n\}$ is uniformly bounded in $H^1((0,T)\times\mathbb{R})$ for fixed $T>0$. Indeed, $v^n_t\in L^2((0,T)\times\mathbb{R})$, in view of (3.3), (3.4), and we have that

$$\|v^n v^n_x\|_{L^2((0,T)\times\mathbb{R})}\leqslant \|v^n\|_{L^\infty((0,T)\times\mathbb{R})}\|v^n_x\|_{L^2((0,T)\times\mathbb{R})}\leqslant \mathrm{e}^{C_1 t}(1+\|z_0\|^2_{H^1(\mathbb{R})\times L^2(\mathbb{R})})$$

and

$$\begin{aligned}&\|\partial_x p*((u^n)^2+\tfrac{1}{2}(u^n_x)^2+\tfrac{1}{2}(\rho^n)^2)\|^2_{L^2((0,T)\times\mathbb{R})}\\ &\qquad\leqslant \|p_x\|^2_{L^2(\mathbb{R})}\int_0^T \|(2(v^n)^2+(v^n_x)^2+(\gamma^n)^2)(t,\cdot)\|^2_{L^1(\mathbb{R})}\,\mathrm{d}t\\ &\qquad\quad + \|p\|^2_{L^1(\mathbb{R})}\int_0^T (4\|\phi\phi_x(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\|\varphi\varphi_x(t,\cdot)\|^2_{L^2(\mathbb{R})})\,\mathrm{d}t\\ &\qquad\quad + \|p_x\|^2_{L^2(\mathbb{R})}\int_0^T \|\phi_x^2(t,\cdot)\|^2_{L^1(\mathbb{R})}\,\mathrm{d}t\\ &\qquad\leqslant C(T,\|z_0\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})},u_\pm,\rho_\pm),\end{aligned}$$

where we used Lemma 2.1 and $\|p_x\|^2_{L^2(\mathbb{R})}\leqslant 1$, $\|p_x\|^2_{L^1(\mathbb{R})}\leqslant 1$ and $\|p\|^2_{L^1(\mathbb{R})}=1$. By Lemmas 2.1 and 2.7, we have that

$$\begin{aligned}\|(\phi v)_x\|_{L^2((0,T)\times\mathbb{R})} &\leqslant \|v_x\phi\|_{L^2((0,T)\times\mathbb{R})}+\|v\phi_x\|_{L^2((0,T)\times\mathbb{R})}\\ &\leqslant C(T,\|z_0\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})},u_\pm).\end{aligned}$$

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Then, by the first equation of (2.2), we obtain that $\{v_t^n\}$ is uniformly bounded in $L^2((0,T)\times\mathbb{R})$.

From Lemma 2.7, we can easily get that $\{\gamma^n\}$ is uniformly bounded in $L^2((0,T)\times\mathbb{R})$. Therefore, there exist $z=\binom{v}{\gamma}\in L^\infty((0,T);H^1(\mathbb{R})\times L^2(\mathbb{R}))$ and a subsequence $\{z^{n_k}(t,x)\}$ such that

$$z^{n_k}\rightharpoonup z \quad \text{weakly in } H^1((0,T)\times\mathbb{R})\times L^2((0,T)\times\mathbb{R}) \text{ as } n_k\to\infty.$$

By Lemma 2.11, there exist $v\in C((0,T);L^\infty_{\mathrm{loc}}(\mathbb{R}))$ and a subsequence $\{v^{n_k}(t,x)\}$ such that $\{v^{n_k}(t,x)\}$ is compact in $C((0,T);L^\infty_{\mathrm{loc}}(\mathbb{R}))$, that is, $\{v^{n_k}(t,x)\}$ converges to $v(t,x)$ uniformly on each compact subset of $\mathbb{R}_+\times\mathbb{R}$ as $k\to\infty$. Moreover, $v(t,x)\in C((0,T)\times\mathbb{R})\cap L^\infty((0,T);H^1(\mathbb{R}))$.

From Lemmas 2.1, 2.7 and 2.8 and (2.6), in view of $v_0\in W^{1,\infty}(\mathbb{R})$ and $\gamma_0\in L^\infty(\mathbb{R})$, we have that there exists an increasing function $M(T)>0$ such that, for any $(t,x)\in(0,T)\times\mathbb{R}$,

$$|u^n(t,x)|,|u_x^n(t,x)|,|\rho^n(t,x)|,|v^n(t,x)|,|v_x^n(t,x)|,|\gamma^n(t,x)|\leqslant M(T). \tag{3.8}$$

Then, we get that $z\in L^\infty((0,T);W^{1,\infty}(\mathbb{R})\times L^\infty(\mathbb{R}))$.

Next, we turn to the compactness of $\{P^n\}$. For fixed $t\in(0,T)$, in view of the fact that $\|p\|_{L^2(\mathbb{R})}\leqslant\|p\|_{L^1(\mathbb{R})}=1$ and (3.8), we have that

$$\begin{aligned}\|P^n(t,\cdot)\|_{L^\infty(\mathbb{R})}&\leqslant\|p*((u^n)^2+\tfrac12(u_x^n)^2+\tfrac12(\rho^n)^2)\|_{L^\infty(\mathbb{R})}\\&\leqslant\|p\|_{L^1(\mathbb{R})}\|((u^n)^2+\tfrac12(u_x^n)^2+\tfrac12(\rho^n)^2)(t,\cdot)\|_{L^\infty(\mathbb{R})}\\&\leqslant 3[M(T)]^2.\end{aligned}\tag{3.9}$$

In a similar way, in view of the fact that $\|\partial_x p\|_{L^2(\mathbb{R})}\leqslant\|\partial_x p\|_{L^1(\mathbb{R})}=1$ and (3.8), we get that

$$\|\partial_x P^n(t,\cdot)\|_{L^\infty(\mathbb{R})}\leqslant 3[M(T)]^2. \tag{3.10}$$

Therefore, $\{P^n\}$ is uniformly bounded in $L^\infty((0,T);W^{1,\infty}(\mathbb{R}))$.

Moreover, by Remark 2.4, we obtain that

$$u_{tx}^n+u^nu_{xx}^n+\tfrac12(u_x^n)^2=(u^n)^2+\tfrac12(\rho^n)^2-P^n$$

and

$$\rho_t^n+(u^n\rho^n)_x=0.$$

Therefore, we have that

$$\begin{aligned}\frac{\partial P^n}{\partial t}&=p*(2u^nu_t^n+u_x^nu_{tx}^n+\rho^n\rho_t^n)\\&=p*(2u^n(-u^nu_x^n-\partial_xP^n)+\rho^n(-u_x^n\rho^n-u^n\rho_x^n))\\&\quad+p*(u_x^n(-u^nu_{xx}^n-\tfrac12(u_x^n)^2+(u^n)^2+\tfrac12(\rho^n)^2-P^n))\\&=I_1+I_2+I_3.\end{aligned}\tag{3.11}$$

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Next, we estimate I_1, I_2 and I_3. I_1 can be written as

$$
\begin{aligned}
I_1 &= p * (2u^n(-u^n u_x^n - \partial_x P^n) + \rho^n(-u_x^n \rho^n - u^n \rho_x^n)) \\
&= p * (2u^n(-u^n u_x^n - \partial_x P^n) - u_x^n (\rho^n)^2) - p * (\rho^n u^n \rho_x^n) \\
&= p * (2u^n(-u^n u_x^n - \partial_x P^n) - u_x^n (\rho^n)^2) + \frac{1}{4}\int_{\mathbb{R}} (\mathrm{e}^{-|x-y|} u^n)_y (\rho^n)^2 \,\mathrm{d}y \\
&= p * (2u^n(-u^n u_x^n - \partial_x P^n) - \tfrac{1}{2} u_x^n (\rho^n)^2) + \frac{1}{4}\int_{\mathbb{R}} \mathrm{e}^{-|x-y|} u^n \operatorname{sgn}(x-y) (\rho^n)^2 \,\mathrm{d}y.
\end{aligned}
$$

Using (3.8)–(3.10) and Hölder's inequality, we get that

$$
\begin{aligned}
\|I_1\|_{L^\infty((0,T)\times\mathbb{R})} &\leqslant \|p\|_{L^1(\mathbb{R})} \|2u^n(-u^n u_x^n - \partial_x P^n) - \tfrac{1}{2} u_x^n (\rho^n)^2\|_{L^\infty(\mathbb{R})} \\
&\quad + \|u^n (\rho^n)^2\|_{L^\infty(0,T)\times\mathbb{R}} \|p\|_{L^1(\mathbb{R})} \\
&\leqslant 10[M(T)]^3,
\end{aligned}
$$

where we used that $\|p\|_{L^2(\mathbb{R})} \leqslant \|p\|_{L^1(\mathbb{R})} = 1$. Since $-u^n u_x^n u_{xx}^n - \frac{1}{2}(u_x^n)^3 = -\frac{1}{2}(u^n (u_x^n)^2)_x$, it follows that

$$
\begin{aligned}
I_2 &= p * (u_x^n(-u^n u_{xx}^n - \tfrac{1}{2}(u_x^n)^2 + (u^n)^2 + \tfrac{1}{2}(\rho^n)^2 - P^n)) \\
&= p * (u_x^n((u^n)^2 + \tfrac{1}{2}(\rho^n)^2 - P^n)) - \tfrac{1}{2} p * (u^n (u_x^n)^2)_x \\
&= p * (u_x^n((u^n)^2 + \tfrac{1}{2}(\rho^n)^2 - P^n)) - \tfrac{1}{2} \partial_x p * (u^n (u_x^n)^2).
\end{aligned}
$$

Using (3.8)–(3.10) and Hölder's inequality, we obtain that

$$
\begin{aligned}
\|I_2\|_{L^\infty((0,T)\times\mathbb{R})} &\leqslant \|p\|_{L^1(\mathbb{R})} \|(u_x^n((u^n)^2 + \tfrac{1}{2}(\rho^n)^2 - P^n))(t,\cdot)\|_{L^\infty(\mathbb{R})} \\
&\quad + \tfrac{1}{2}\|u^n (u_x^n)^2\|_{L^\infty(0,T)\times\mathbb{R}} \|\partial_x p\|_{L^1(\mathbb{R})} \\
&\leqslant 5(M(T))^3,
\end{aligned}
$$

where we used that $\|\partial_x p\|_{L^1(\mathbb{R})} = \|p\|_{L^1(\mathbb{R})} = 1$.

By (3.11) and the above estimates, we deduce that $\{\partial P^n/\partial t\}$ is uniformly bounded in $L^2((0,T); L^\infty(\mathbb{R}))$. Thus, again by Lemma 2.11, there exist $\bar{P} \in C((0,T); L^\infty(\mathbb{R}))$ and a subsequence $\{P^{n_k}(t,x)\}$ such that $\{P^{n_k}(t,x)\}$ is weakly compact in $C((0,T); L^\infty(\mathbb{R}))$ and $\{P^{n_k}(t,x)\}$ converges to $\bar{P}(t,x)$ uniformly on each compact subset of $\mathbb{R}_+ \times \mathbb{R}$ as $k \to \infty$. Moreover, $\bar{P}(t,x) \in L^\infty((0,T); W^{1,\infty}(\mathbb{R}))$. This completes the proof of the lemma. □

Step 3 (the existence of global weak solutions). By Lemma 3.4, we have that, for any $T > 0$,

$$
v^{n_k}\gamma^{n_k} \rightharpoonup v\gamma \quad \text{weakly in } L^2((0,T)\times\mathbb{R}) \tag{3.12}
$$

and

$$
v^{n_k} v_x^{n_k} \rightharpoonup v v_x \quad \text{weakly in } L^2((0,T)\times\mathbb{R}). \tag{3.13}
$$

Remark 3.5. By the above argument, we see that, for any fixed $T > 0$, there exists a pair of subsequences $\{(v_x^{n_k})^2\} \subset \{(v_x^n)^2\}$ and $\{(\gamma^{n_k})^2\} \subset \{(\gamma^n)^2\}$ converging weakly in $L^r((0,T) \times \mathbb{R})$ for all $1 < r < \infty$, i.e. there exists a pair of functions $\bar{v}_x^2 \in L^r((0,T) \times \mathbb{R})$ and $\bar{\gamma}^2 \in L^r((0,T) \times \mathbb{R})$ such that

$$(v_x^{n_k})^2 \rightharpoonup \bar{v}_x^2 \quad \text{and} \quad (\gamma^{n_k})^2 \rightharpoonup \bar{\gamma}^2 \quad \text{weakly in } L^r((0,T) \times \mathbb{R}).$$

Moreover, we have that

$$\begin{aligned} v_x^{n_k} &\rightharpoonup v_x \quad \text{weakly in } L^p((0,T) \times \mathbb{R}) \quad \text{and} \quad v_x^{n_k} \rightharpoonup v_x \quad \text{weakly}^* \text{ in } L^\infty(\mathbb{R}_+; L^2(\mathbb{R})), \\ \gamma^{n_k} &\rightharpoonup \gamma \quad \text{weakly in } L^p((0,T) \times \mathbb{R}) \quad \text{and} \quad \gamma^{n_k} \rightharpoonup \gamma \quad \text{weakly}^* \text{ in } L^\infty(\mathbb{R}_+; L^2(\mathbb{R})), \end{aligned}$$

where $p \geqslant 2$. Furthermore, we have

$$v_x^2(t,x) \leqslant \bar{v}_x^2(t,x) \quad \text{and} \quad \gamma^2(t,x) \leqslant \bar{\gamma}^2(t,x) \quad \text{a.e. on } (\mathbb{R}_+ \times \mathbb{R}). \tag{3.14}$$

Lemma 3.6. *In the sense of distributions on* $\mathbb{R}_+ \times \mathbb{R}$,

$$\begin{aligned} \frac{\partial v_x^2}{\partial t} + \frac{\partial}{\partial x}((v+\phi)v_x^2) &= (\bar{v}_x^2 + \bar{\gamma}^2)v_x - v_x^3 - \phi_x v_x^2 \\ &\quad + 2((v+\phi)^2 - \bar{P} + \tfrac{1}{2}\phi_x^2 + \varphi\gamma + \tfrac{1}{2}\varphi^2 - \phi_{xx}v)v_x, \end{aligned} \tag{3.15}$$

$$\frac{\partial \gamma^2}{\partial t} + \frac{\partial}{\partial x}((v+\phi)\gamma^2) = -(v_x + \phi_x)\gamma^2 - 2(((v+\phi)\varphi)_x)\gamma + 2\varphi\varphi_x\gamma, \tag{3.16}$$

$$\begin{aligned} \frac{\partial \bar{v}_x^2}{\partial t} + \frac{\partial}{\partial x}((v+\phi)\bar{v}_x^2) &= -\phi_x\bar{v}_x^2 + \overline{\gamma^2 v_x} + \phi_x^2 v_x \\ &\quad + 2(v+\phi)^2 v_x + 2\varphi\overline{\gamma v_x} + \varphi^2 v_x - 2\bar{P}v_x - 2\phi_{xx}vv_x \end{aligned} \tag{3.17}$$

and

$$\frac{\partial}{\partial t}\bar{\gamma}^2 + \frac{\partial}{\partial x}((v+\phi)\bar{\gamma}^2) = -\overline{v_x\gamma^2} - \phi_x\bar{\gamma}^2 - 2\varphi\overline{v_x\gamma} - 2\phi_x\varphi\gamma - 2(v+\phi)\varphi_x\gamma + 2\varphi\varphi_x\gamma \tag{3.18}$$

hold.

Proof. Note that z^{n_k} is the solution of (2.3). Differentiating the first equation in (2.3) with respect to x and using the relation $\partial_x^2 p * f = p * f - f$, we have that

$$v_{tx}^{n_k} + ((v^{n_k}+\phi)v_x^{n_k})_x = \tfrac{1}{2}(v_x^{n_k})^2 + \tfrac{1}{2}\phi_x^2 + (v^{n_k}+\phi)^2 + \tfrac{1}{2}(\gamma^{n_k}+\varphi)^2 - \phi_{xx}v^{n_k} - P^{n_k}. \tag{3.19}$$

By the second equation of (2.3), we get that

$$\gamma_t^{n_k} + ((v^{n_k}+\phi)\gamma^{n_k})_x + ((v^{n_k}+\phi)\varphi)_x = \varphi\varphi_x. \tag{3.20}$$

By (3.5)–(3.7), (3.12), (3.13) and Remark 3.5, we infer from (3.19), (3.20) that

$$\frac{\partial v_x}{\partial t} + \frac{\partial}{\partial x}((v+\phi)v_x) = \tfrac{1}{2}\bar{v}_x^2 + \tfrac{1}{2}\bar{\gamma}^2 + (v+\phi)^2 - \bar{P} + \tfrac{1}{2}\phi_x^2 + \varphi\gamma + \tfrac{1}{2}\varphi^2 - \phi_{xx}v \tag{3.21}$$

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and

$$\frac{\partial\gamma}{\partial t}+\frac{\partial}{\partial x}((v+\phi)\gamma)+((v+\phi)\varphi)_x=\varphi\varphi_x \tag{3.22}$$

in the sense of distributions on $\mathbb{R}_+\times\mathbb{R}$. Define $v_{n,x}(t,x):=(v_x(t,\cdot)*\chi_n)(x)$ and $\gamma_n(t,x):=(\gamma(t,\cdot)*\chi_n)(x)$. According to [**20**, Lemma II.1], we may deduce from (3.21), (3.22) that $v_{n,x}$ and γ_n solve

$$\begin{aligned}\frac{\partial v_{n,x}}{\partial t}+(v+\phi)\frac{\partial v_{n,x}}{\partial x}&=(-v_x^2-\phi_x v_x+\tfrac{1}{2}(\bar{v}_x^2+\bar{\gamma}^2))*\chi_n\\&\quad+((v+\phi)^2-\bar{P}+\tfrac{1}{2}\phi_x^2+\varphi\gamma+\tfrac{1}{2}\varphi^2-\phi_{xx}v)*\chi_n+\tau_n\end{aligned} \tag{3.23}$$

and

$$\frac{\partial\gamma_n}{\partial t}+(v+\phi)\frac{\partial\gamma_n}{\partial x}+(v_x\gamma)*\chi_n+(((v+\phi)\varphi)_x-\varphi\varphi_x)*\chi_n=\sigma_n, \tag{3.24}$$

where the errors τ_n and σ_n tend to zero in $L^1_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R})$. Multiplying (3.23) and (3.24) by $2v_{n,x}$ and $2\gamma_n$, respectively, we get that

$$\begin{aligned}&\frac{\partial v_{n,x}^2}{\partial t}+\frac{\partial}{\partial x}((v+\phi)v_{n,x}^2)\\&\quad=((\bar{v}_x^2+\bar{\gamma}^2-2v_x^2-2\phi_x v_x)*\chi_n)v_{n,x}+(v_x+\phi_x)v_{n,x}^2\\&\qquad+2(((v+\phi)^2-\bar{P}+\tfrac{1}{2}\phi_x^2+\varphi\gamma+\tfrac{1}{2}\varphi^2-\phi_{xx}v)*\chi_n+\tau_n)v_{n,x}\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}&\frac{\partial\gamma_n^2}{\partial t}+\frac{\partial}{\partial x}((v+\phi)\gamma_n^2)\\&\quad=-2((v_x\gamma+\phi_x\gamma+((v+\phi)\varphi)_x-\varphi\varphi_x)*\chi_n)\gamma_n+(v_x+\phi_x)\gamma_n^2+2\sigma_n\gamma_n.\end{aligned} \tag{3.26}$$

Using (3.8), we can send $n\to\infty$ in (3.25) and (3.26) to obtain (3.15) and (3.16).

On the other hand, multiplying (3.19) and (3.20) by $2v_x^{n_k}$ and $2\gamma^{n_k}$, respectively, we get that

$$\begin{aligned}&\frac{\partial}{\partial t}(v_x^{n_k})^2+\frac{\partial}{\partial x}((v^{n_k}+\phi)(v_x^{n_k})^2)\\&\quad=(v_x^{n_k}+\phi_x)(v_x^{n_k})^2-(v_x^{n_k})^2v_x^{n_k}\\&\qquad+(-2v_x^{n_k}\phi_x+\phi_x^2+2(v^{n_k}+\phi)^2+(\gamma^{n_k})^2+2\gamma^{n_k}\varphi+\varphi^2-2P^{n_k}-2\phi_{xx}v^{n_k})v_x^{n_k}\end{aligned}$$

and

$$\frac{\partial}{\partial t}(\gamma^{n_k})^2+\frac{\partial}{\partial x}((v^{n_k}+\phi)(\gamma^{n_k})^2)=-(v_x^{n_k}+\phi_x)(\gamma^{n_k})^2-2((v^{n_k}+\phi)\varphi)_x\gamma^{n_k}+2\varphi\varphi_x\gamma^{n_k}.$$

Once more using Remark 3.5 and (3.12), (3.13), we can send $k\to\infty$ in the above two equalities to obtain (3.17) and (3.18). □

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Lemma 3.7.

$$\lim_{t\to 0^+}\int_{\mathbb{R}} v_x^2(t,x)\,\mathrm{d}x = \lim_{t\to 0^+}\int_{\mathbb{R}} \bar{v}_x^2(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} v_{0,x}^2(x)\,\mathrm{d}x \tag{3.27}$$

and

$$\lim_{t\to 0^+}\int_{\mathbb{R}} \gamma^2(t,x)\,\mathrm{d}x = \lim_{t\to 0^+}\int_{\mathbb{R}} \bar{\gamma}^2(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} \gamma_0^2(x)\,\mathrm{d}x \tag{3.28}$$

hold.

Proof. By Lemmas 3.4 and 2.8, for any $T>0$, we have that $v^n \in L^\infty((0,T);H^1(\mathbb{R}))$, $\{v_t^n\}$ is uniformly bounded in $L^\infty((0,T);L^2(\mathbb{R}))$ and $v^n \in C([0,T];H^1(\mathbb{R}))$. Then, in view of [**42**, Appendix C] and the proof of Lemma 3.4, we have that $\{v^n\}$ contains a subsequence denoted, again, by $\{v^{n_k}\}$ that converges to v weakly in $H^1(\mathbb{R})$ uniformly in t. This implies that v is weakly continuous from $(0,T)$ into $H^1(\mathbb{R})$, i.e.

$$v \in C^w([0,T];H^1(\mathbb{R})). \tag{3.29}$$

Similarly, since $\gamma^n \in L^\infty((0,T);L^2(\mathbb{R}))$ and, for all $t\in(0,T)$,

$$\begin{aligned}
\|\gamma_t^n(t,\cdot)\|_{H^{-1}(\mathbb{R})} &= \sup_{\|f\|_{H^1(\mathbb{R})}} \int_{\mathbb{R}} (-((v^n+\phi)\gamma^n)_x - ((v^n+\phi)\varphi)_x + \varphi\varphi_x) f\,\mathrm{d}x \\
&= \sup_{\|f\|_{H^1(\mathbb{R})}} \int_{\mathbb{R}} ((v^n+\phi)\gamma^n f_x + v f_x\varphi - \phi_x\varphi f - \phi\varphi_x f + \varphi\varphi_x f)\,\mathrm{d}x \\
&\leqslant \|v^n+\phi\|_{L^\infty(\mathbb{R})}\|\gamma^n\|_{L^2(\mathbb{R})} + \|\varphi\|_{L^\infty(\mathbb{R})}\|v\|_{L^2(\mathbb{R})} + \|\varphi\|_{L^\infty(\mathbb{R})}\|\phi_x\|_{L^2(\mathbb{R})} \\
&\quad + \|\phi\|_{L^\infty(\mathbb{R})}\|\varphi_x\|_{L^2(\mathbb{R})} + \|\varphi\|_{L^\infty(\mathbb{R})}\|\varphi_x\|_{L^2(\mathbb{R})} \\
&\leqslant C(T, \|z_0\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})}, u_\pm, \rho_\pm),
\end{aligned}$$

where we applied Lemmas 2.1 and 2.8, it follows that $\{\gamma_t^n\}$ is uniformly bounded in $L^\infty((0,T);H^{-1}(\mathbb{R}))$. Then, again by [**42**, Appendix C], we have that $\{\gamma^n\}$ contains a subsequence denoted, once more, by $\{\gamma^{n_k}\}$ that converges to γ weakly in $L^2(\mathbb{R})$ uniformly in t. This implies that γ is weakly continuous from $(0,T)$ into $L^2(\mathbb{R})$, i.e.

$$\gamma \in C^w([0,T];L^2(\mathbb{R})). \tag{3.30}$$

Then, by (3.29) and (3.30), we get that

$$\gamma(t,\cdot) \rightharpoonup \gamma_0 \quad \text{and} \quad v_x(t,\cdot) \rightharpoonup v_{0,x} \quad \text{weakly in } L^2(\mathbb{R}) \text{ as } t\to 0^+.$$

Thus, we have that

$$\liminf_{t\to 0^+}\int_{\mathbb{R}} \gamma^2(t,x)\,\mathrm{d}x \geqslant \int_{\mathbb{R}} \gamma_0^2(x)\,\mathrm{d}x$$

and

$$\liminf_{t\to 0^+}\int_{\mathbb{R}} v_x^2(t,x)\,\mathrm{d}x \geqslant \int_{\mathbb{R}} v_{0,x}^2(x)\,\mathrm{d}x.$$

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Therefore, we deduce that

$$\begin{aligned}\liminf_{t\to 0^+}\int_{\mathbb{R}}(v_x^2(t,x)x+\gamma^2(t,x))\,\mathrm{d}x &\geqslant \liminf_{t\to 0^+}\int_{\mathbb{R}} v_x^2(t,x)\,\mathrm{d}x+\liminf_{t\to 0^+}\int_{\mathbb{R}}\gamma^2(t,x)\,\mathrm{d}x\\ &\geqslant \int_{\mathbb{R}}(v_{0,x}^2(x)+\gamma_0^2(x))\,\mathrm{d}x. \qquad (3.31)\end{aligned}$$

Moreover, from Lemma 2.8 we have that

$$\begin{aligned}&\int_{\mathbb{R}}(v^2(t,x)+\bar{v}_x^2(t,x)+\bar{\gamma}^2(t,x))\,\mathrm{d}x\\ &\qquad\leqslant \liminf_{n_k\to\infty}\int_{\mathbb{R}}((v^{n_k})^2(t,x)+(v_x^{n_k})^2(t,x)+(\gamma^{n_k})^2(t,x))\,\mathrm{d}x\\ &\qquad= \mathrm{e}^{C_1t}\biggl(1+\liminf_{n_k\to\infty}\int_{\mathbb{R}}((v_0^{n_k})^2(x)+(v_{0,x}^{n_k})^2(x)+(\gamma_0^{n_k})^2(x))\,\mathrm{d}x\biggr)-1\\ &\qquad= \mathrm{e}^{C_1t}+\mathrm{e}^{C_1t}\int_{\mathbb{R}}(v_0^2(x)+v_{0,x}^2(x)+\gamma_0^2(x))\,\mathrm{d}x-1.\end{aligned}$$

Thus, we have that

$$\limsup_{t\to 0}\int_{\mathbb{R}}(v^2(t,x)+\bar{v}_x^2(t,x)+\bar{\gamma}^2(t,x))\,\mathrm{d}x\leqslant\int_{\mathbb{R}}(v_0^2(x)+v_{0,x}^2(x)+\gamma_0^2(x))\,\mathrm{d}x.$$

Using the continuity of v and that

$$\lim_{t\to 0^+}\int_{\mathbb{R}}v^2(t,x)\,\mathrm{d}x=\int_{\mathbb{R}}v_0^2(x)\,\mathrm{d}x,$$

we obtain that

$$\limsup_{t\to 0^+}\int_{\mathbb{R}}(\bar{v}_x^2(t,x)+\bar{\gamma}^2(t,x))\,\mathrm{d}x\leqslant\int_{\mathbb{R}}(v_{0,x}^2(x)+\gamma_0^2(x))\,\mathrm{d}x. \qquad (3.32)$$

In view of (3.14), (3.31) and (3.32), we get (3.27) and (3.28). □

Lemma 3.8.

$$\bar{v}_x^2(t,x)=v_x^2(t,x)\quad\textit{and}\quad\bar{\gamma}^2(t,x)=\gamma^2(t,x) \qquad (3.33)$$

hold a.e. on $\mathbb{R}_+\times\mathbb{R}$.

Proof. Subtracting (3.15) from (3.17), we have that

$$\begin{aligned}&\frac{\partial}{\partial t}(\bar{v}_x^2-v_x^2)+\frac{\partial}{\partial x}((v+\phi)[\bar{v}_x^2-v_x^2])\\ &\qquad=(\overline{\gamma^2v_x}-\bar{\gamma}^2v_x)-\phi_x(\bar{v}_x^2-v_x^2)+2\varphi(\overline{v_x\gamma}-v_x\gamma)-(\bar{v}_x^2-v_x^2)v_x\\ &\qquad\leqslant(\overline{\gamma^2v_x}-\bar{\gamma}^2v_x)+2\varphi(\overline{v_x\gamma}-v_x\gamma)-(\bar{v}_x^2-v_x^2)v_x, \qquad (3.34)\end{aligned}$$

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where we used (3.14) and $\phi_x \geqslant 0$, as guaranteed by Lemma 2.1. Subtracting (3.16) from (3.18), we get that

$$\begin{aligned}\frac{\partial}{\partial t}(\bar{\gamma}^2-\gamma^2)+\frac{\partial}{\partial x}((v+\phi)[\bar{\gamma}^2-\gamma^2]) &= -(\overline{v_x\gamma^2}-v_x\gamma^2)-\phi_x(\bar{\gamma}^2-\gamma^2)-2\varphi(\overline{v_x\gamma}-v_x\gamma)\\ &\leqslant -(\overline{v_x\gamma^2}-v_x\gamma^2)-2\varphi(\overline{v_x\gamma}-v_x\gamma), \end{aligned} \tag{3.35}$$

where, again, we applied (3.14) and $\phi_x \geqslant 0$.

Adding (3.34) and (3.35), and integrating over $(\varepsilon, t)\times\mathbb{R}$, we obtain that

$$\begin{aligned}\int_{\mathbb{R}}(\bar{v}_x^2-v_x^2+\bar{\gamma}^2-\gamma^2)(t,x)\,\mathrm{d}x-\int_{\mathbb{R}}(\bar{v}_x^2-v_x^2+\bar{\gamma}^2-\gamma^2)(\varepsilon,x)\,\mathrm{d}x&\\ \leqslant \int_{\varepsilon}^{t}\int_{\mathbb{R}}|v_x|(\bar{v}_x^2-v_x^2+\bar{\gamma}^2-\gamma^2)(s,x)\,\mathrm{d}x\,\mathrm{d}s.&\end{aligned}$$

Letting $\varepsilon \to 0$ and using Lemma 3.8 and (3.8), this yields that

$$\int_{\mathbb{R}}(\bar{v}_x^2-v_x^2+\bar{\gamma}^2-\gamma^2)(t,x)\,\mathrm{d}x \leqslant M(T)\int_0^t\int_{\mathbb{R}}(\bar{v}_x^2-v_x^2+\bar{\gamma}^2-\gamma^2)(s,x)\,\mathrm{d}x\,\mathrm{d}s.$$

Using Gronwall's inequality and Lemma 3.8, we conclude that

$$\int_{\mathbb{R}}(\bar{v}_x^2-v_x^2+\bar{\gamma}^2-\gamma^2)(t,x)\,\mathrm{d}x \leqslant 0.$$

By (3.14), we obtain that

$$0 \leqslant \int_{\mathbb{R}}(\bar{v}_x^2-v_x^2+\bar{\gamma}^2-\gamma^2)(t,x)\,\mathrm{d}x \leqslant 0, \tag{3.36}$$

that is,

$$\int_{\mathbb{R}}(\bar{v}_x^2-v_x^2)(t,x)\,\mathrm{d}x = \int_{\mathbb{R}}(\bar{\gamma}^2-\gamma^2)(t,x)\,\mathrm{d}x = 0.$$

This implies that (3.33) holds. □

From (3.3)–(3.5), (3.11), (3.12) and (3.33), we infer that z satisfies (2.3) in $D'((0,T)\times\mathbb{R})$ and $z \in C^w([0,T]; H^1(\mathbb{R})\times L^2(\mathbb{R}))$ for any $T>0$.

Let $u = v+\phi$ and let $\rho = \gamma+\varphi$. Since $\phi_t+\phi\phi_x = 0$ and $\varphi_t+\varphi\varphi_x = 0$, we deduce that (u,ρ) satisfies (2.2) in $D'((0,T)\times\mathbb{R})$. Moreover, $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; W^{1,\infty}(\mathbb{R}))$ and $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\infty(\mathbb{R}))$.

This completes the proof of Theorem 3.2. □

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