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FINITE GROUP WITH HALL COVERINGS

ENRICO JABARA and MARIA SILVIA LUCIDO

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Abstract

In this paper we describe the groups admitting a covering with Hall subgroups. We also determine the groups with a π_1-Hall subgroup, where π_1 is the connected component of the prime graph, containing the prime 2.

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1. Introduction

In this paper we study the Hall coverings, defined as follows. A *Hall covering* of a finite group G is a set $\mathscr{H} = \{H_1, H_2, \dots, H_r\}$ of proper Hall subgroups of G such that:

(a) $\bigcup_{i=1}^r H_i = G$ and
(b) either $|H_i| = |H_j|$ or $(|H_i|, |H_j|) = 1$ for $i, j = 1, \dots, r$.

If the elements of $\mathscr{H}$ all have order a prime power, then $\mathscr{H}$ is called a *Sylow covering of* G. The finite groups G with a Sylow covering have been studied independently by Higman [10] and Zacher [27, 28] in the case in which G is soluble, by Suzuki [25] in the case of a simple group G and by Brandl [2] in the general situation. This last paper has a missing case, which we consider here.

We want to study the groups which admit a Hall covering. It is clear that if a group G admits a Hall covering, then its prime graph is not connected. We shall also see that if G admits a Hall covering, than G has a π_1-Hall subgroup, where π_1 is the connected component of the prime graph of G containing the prime 2.

It is well known that G is a soluble group if and only if G has a π-Hall subgroup for any set of primes π. If G is not soluble, the existence of some Hall subgroups have been proved in several papers (see, for example, [9, 23, 8]). We prove the following theorem on the existence of a π_1-Hall subgroup. We suppose that the group G is not soluble and that G is not a Frobenius group. In fact if G is a non-soluble Frobenius group, the Frobenius complement is isomorphic to a direct product of $SL(2, 5)$ with a $\{2, 3, 5\}'$-group with cyclic Sylow subgroups. The Frobenius complements are π_1-Hall subgroups and they are all conjugate (see [12, page 387]).

THEOREM A. *Let G be a non-soluble group in which the prime graph is not connected. Suppose further that G is not a Frobenius group. Then G has a π_1-Hall subgroup if and only if $G/\operatorname{Fit}(G)$ is one of the groups in Table* 1.

We also classify the groups which admit a π-Hall subgroup for any connected subset π of $\pi(G)$ (see Corollary 3.6).

We prove the following theorem, describing the finite groups admitting a Hall covering.

THEOREM B. *Let G be a group in which the prime graph is not connected. Then G admits a Hall covering if and only if either*

(i) *G is a Frobenius or a 2-Frobenius group or*

(ii) *$G/\operatorname{Fit}(G)$ is isomorphic to one of the following groups*: $PSL(2, q)$, $PSL(3, 4)$, $PSL(3, q)$ *with* $(3, q-1) = 1$, $Sz(q)$, A_7, M_{22}, $M(q)$.

Another class of groups related to groups admitting Hall coverings is the class of *groups with a partition* (see [22, Section 3.5]) and the *CN-groups*, that is groups in which the centralizer of any non trivial element is nilpotent (see [7, Chapter 10]). We shall see how these groups are strictly related to nilpotent Hall coverings.

The results of this paper depend upon the classification.

2. Notation and preliminary results

All the groups considered in this paper are finite. If G is a group we denote by $\pi(G)$ the set of prime divisors of $|G|$. If $\mathscr{H}$ is a Hall covering of the group G, we define $\pi(\mathscr{H}) = \{\pi(H_i) \mid i = 1, 2, \dots, r\}$; then $\pi(\mathscr{H}) = \{\sigma_1, \sigma_2, \dots, \sigma_s\}$ with $\sigma_i \cap \sigma_j = \emptyset$ if $i \neq j$ (and obviously $s < r$). We suppose that if $i < j$, then $\min \sigma_i < \min \sigma_j$ (in particular if $2 \in \pi(G)$ then $2 \in \sigma_1$).

If G is a group, we define its *prime graph* $\Gamma(G) = \Gamma$ as follows: the set of vertices of Γ is $\pi(G)$ and two vertices p and q are connected ($p \sim q$) if and only if there exists in G an element of order pq. Let $\pi_1, \pi_2, \dots, \pi_t$ be the connected components of Γ

and let $t(G) = t$ be the number of such connected components; we suppose $2 \in \pi_1$, if $2 \in \pi(G)$. Then $\pi(G)$ is the disjoint union of the π_i, $i = 1, 2, \dots, t$. Moreover, if G admits a Hall covering $\mathscr{H}$, then any element of $\pi(\mathscr{H})$ is a disjoint union of certain connected components of $\Gamma(G)$, in particular, $2 \leq s \leq t$.

If $\mathscr{C}_1$ and $\mathscr{C}_2$ are two classes of groups, a group G is $\mathscr{C}_1$-by-$\mathscr{C}_2$ if G has a normal subgroup N with $N \in \mathscr{C}_1$ and $G/N \in \mathscr{C}_2$.

We denote by $\mathrm{Fit}(G)$ the Fitting subgroup of G, that is, the maximal normal nilpotent subgroup of G.

A group G is an *almost simple* group if there exists a simple non-abelian group S such that $S \leq G \leq \mathrm{Aut}(S)$.

Let p be an odd prime and $q = p^{2f}$. We denote by $M(q)$ the non split extension of $PSL(2, q)$, with $|M(q) : PSL(2, q)| = 2$.

A proper subgroup H of G is *isolated* (in G) if

(a) $H \cap H^g = 1$ for any $g \notin N_G(H)$;
(b) $C_G(h) \leq H$ for any $1 \neq h \in H$.

The notation for the simple groups follows the one of [5]. For the rest, the notation will be standard (see, for example, [7] and [12]).

A group is called 2-*Frobenius* if it has two normal subgroups N, K, with $N < K$, such that K and G/N are Frobenius groups.

The following results were proved in an unpublished paper of Gruenberg and Kegel, but they can be found in [26]

PROPOSITION 2.1 ([26]). *If G is a group whose prime graph has more than one connected component, then G has one of the following structures*:

(a) *G is a Frobenius or a 2-Frobenius group.*
(b) *G is simple.*
(c) *G is simple by π_1.*
(d) *G is π_1 by simple by π_1.*

Moreover, if G is not soluble and π_i is a component of $\Gamma(G)$ with $i > 1$, then G has an isolated π_i-Hall subgroup.

COROLLARY 2.2. *If G is a soluble group with a Hall covering, then $|\pi(\mathscr{H})| = 2 = t(G)$ and G is a Frobenius or a 2-Frobenius group.*

It is well known that G is soluble if and only if G has a π-Hall subgroup for any $\pi \subseteq \pi(G)$; moreover any two of them are conjugate. If we want to consider the general case, we first have to deal with the existence of $\pi_i(G)$-Hall subgroups of G. We recall the following:

PROPOSITION 2.3 ([8]). *If G has a π-Hall subgroup with $2 \notin \pi$, then the π-Hall subgroups are all conjugate.*

Therefore by Proposition 2.1 and Proposition 2.3, we know that if $t(G) \geq 2$, then there exists a π_i-Hall subgroup for any $i \geq 2$ and these are all conjugate.

We want to examine now the groups which admit a π_1-Hall subgroup.

3. π_1-Hall subgroups

By the preceding remarks, we can assume that

(∗) G is a non-soluble group in which the prime graph is not connected and G is not a Frobenius group.

The aim of this section is to prove the following:

PROPOSITION 3.1. *Let G be a group satisfying (∗). Then G has a π_1-Hall subgroup if and only if $G/\operatorname{Fit}(G)$ is one of the groups of Table* 1.

We begin with some general remarks.

LEMMA 3.2. *Let G be a group satisfying (∗).*

(i) *If R is the maximal normal soluble subgroup of G, then $R = \operatorname{Fit}(G) = O_{\pi_1}(G)$ and $G/\operatorname{Fit}(G)$ is isomorphic to an almost simple group. Moreover if S is the only simple non-abelian section of G, we have $\pi_i(G) = \pi_i(S)$, for $i \geq 2$.*

(ii) *G has a π_1-Hall subgroup if and only if $G/\operatorname{Fit}(G)$ has a π_1-Hall subgroup.*

PROOF. (i) It can be easily deduced by the results in the paper [26].

(ii) Let $F = \operatorname{Fit}(G)$. If $\overline{G} = G/F$ has a $\pi_1(\overline{G})$-Hall subgroup $\overline{H}$, then of course H is a $\pi_1(G)$-Hall subgroup of G, since $\pi(H) \subseteq \pi_1(G)$.

Let now H be a $\pi_1(G)$-Hall subgroup of G, then $F \leq H$. Otherwise $FH > H$ and FH is also a $\pi_1(G)$-subgroup of G, contradicting the maximality of H. Therefore H/F is a $\pi_1(G/F)$-Hall subgroup of G/F. □

The aim of the following sections is therefore to prove:

PROPOSITION 3.3. *If G is an almost simple group, then G has a π_1-Hall subgroup if and only if G is one of the groups of Table* 1.

In Tables 1 and 2, we suppose that r is an odd prime number, p is a prime number, $q = p^f$ and P is a Sylow p-subgroup of G. We use the notation of the Atlas [5].

We denote by H a $\pi_1(G)$-Hall subgroup of G and we write in the third column the structure of a representative of the conjugacy classes of the $\pi_1(G)$-Hall subgroups of G. In the last column we write some remarks concerning H. We also recall that $A_5 \cong PSL(2,4) \cong PSL(2,5)$ and $A_6 \cong PSL(2,9)$.

If G is $PSL(n, q)$, we denote by $P_{1'}$, $P_{n'}$ the maximal parabolic subgroups of type P_J, with J respectively $\Pi \setminus \{1\}$ and $\Pi \setminus \{n\}$, as described in Remark 1 in Section 4.

We observe that the following groups admit also a π-Hall subgroup, with π a set of primes strictly containing π_1.

3.1. Simple groups If G admits a Hall covering, then the number $t(G)$ of connected components of the prime graph $\Gamma(G)$ is greater than or equal to 2. We first suppose $t(G) = 2$, and therefore $\pi_i = \sigma_i$ for $i = 1, 2$.

We recall that a group is said to be *factorizable* by two proper subgroups A and B if $G = AB = BA$.

LEMMA 3.4. *Let G be a finite group with $t(G) = 2$. If G has a π_1-Hall subgroup A, then G is factorizable by A and another proper subgroup B such that $(|A|, |B|) = 1$.*

PROOF. If G has a π_1-Hall subgroup A, then by Proposition 2.1, G has also a π_2-Hall subgroup B. Then $(|A|, |B|) = 1$ and $|G| = |A||B|$, and therefore $G = AB$. □

Let now G be a simple group. Then by Lemma 3.4, G is factorizable by two proper subgroups A and B and we can assume A to be a π_1-Hall subgroup and B a π_2-Hall subgroup. We can therefore conclude by [1, Theorem 1.1] that G is one of the following:

(i) A_r, with $r \geq 5$ a prime and $r - 2$ not a prime, then $A \cong A_{r-1}$;

(ii) $PSL(r, q)$, with r an odd prime such that $(r, q - 1) = 1$ and either $G \cong PSL(5, 2)$ and $|B| = 5 \cdot 31$ or A is a maximal parabolic subgroup such that $PSL(r-1, q)$ is involved in A.

We observe that in the case $PSL(5, 2)$ with $|B| = 5 \cdot 31$, A is not a π_1-Hall subgroup because $5 \in \pi_1$.

We now suppose that G is a simple non-abelian group with $t(G) \geq 3$. We consider separately the case in which G is a sporadic or an alternating group and the case in which G is a simple group of Lie type. In the following we look for π-Hall subgroups of G, with π a set of primes in $\pi(G)$ containing π_1. We use the results in [26], without further reference.

Alternating groups Since $A_5 \simeq PSL(2, 5)$ and $A_6 \simeq PSL(2, 9)$, it is enough to consider A_r with $r \geq 7$, r and $r - 2$ primes. The maximal subgroups of the alternating groups have been classified (see for example [6, Theorem 5.2A]). The only cases in which A_r ($r \geq 2$, r and $r - 2$ primes) admits a $\{r - 2, r\}'$-Hall subgroup H is for $r = 7$. In fact, by point (i) of [6, Theorem 5.2A], we should have

$$H \leq (A_{r-3} \times A_3)\langle x \rangle, \quad \text{with } x \text{ of order } 2$$

but if $r > 7$, then $(A_{r-3} \times A_3)\langle x \rangle$ has index greater than $r(r - 2)$ in A_r.

TABLE 1.

G	Conditions	H	Remarks
A_7		$(A_4 \times A_3).2$	soluble
A_r	$r-2$ not a prime	A_{r-1}	simple
M_{11}		$3^2 : Q_8.2$	soluble
M_{22}		$2^4 : A_6$	
M_{23}		$PSL(3,4) : 2_2$ $2^4 : A_7$	
J_1		$2 \times A_5$	
$PSL(2,q)$	$q = 2^n$	P	nilpotent
$PSL(2,q)$	$q \equiv 1\ (4)$, $q \neq 13, 25, 61$	D_{q-1}	soluble
$PSL(2,q)$	$q \equiv -1\ (4)$, $q \neq 11, 23, 59$	D_{q+1}	soluble
$PSL(2,q)$	$q = 11, 13$	D_{12} A_4	soluble soluble
$PSL(2,q)$	$q = 23, 25$	D_{24} S_4, 2 classes	soluble soluble
$PSL(2,q)$	$q = 59, 61$	D_{60} A_5, 2 classes	soluble simple
$PSL(3,q)$	$q = 2^2$	P	nilpotent
$PSL(r,q)$	$(r, q-1) = 1$	$P_{1'}$ $P_{r'}$	
$Sz(q)$	$q = 2^f$, f odd	P	nilpotent
S_7		S_6	almost simple
S_r	$r-2$ not a prime	S_{r-1}	almost simple
$PSL(2,q)\langle\alpha\rangle$ $q = 2^n$	α field automorphism $\lvert\alpha\rvert = 2^m$	$N_G(P)$	soluble
$M(q)$		$D_{2(q-1)}$	soluble
$PSL(r,q)\langle\alpha\rangle$ $(r, q-1) = 1$	α field automorphism $\lvert\alpha\rvert = r^m$	$P_{1'}\langle\alpha\rangle$ $P_{r'}\langle\alpha\rangle$	

TABLE 2.

G	π	π-Hall subgroup	Remarks
A_7	$\pi_1 \cup \{5\}$	A_6	simple
M_{11}	$\pi_1 \cup \{5\}$	$M_{10} = M(9)$	almost simple
M_{23}	$\pi_1 \cup \{11\}$	M_{22}	simple
$PSL(3,q)$, $q = 2^2$	$\{2, 3\}$	$N_G(P)$	soluble
$PSL(2,q)$, $q = 2^n$	$\pi(q(q-1))$	$N_G(P)$	soluble
$PSL(2,7)$	$\pi(q^2-1)$	S_4	soluble
$PSL(2,11)$	$\pi(q^2-1)$	A_5, 2 classes	simple
$Sz(q)$	$\pi(q(q-1))$	$N_G(P)$	soluble

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Sporadic groups Let G be a sporadic group and $h = |G|/|G|_{\pi_1}$. If G has a π-Hall subgroup, with $\pi_1 \subseteq \pi \subset \pi(G)$, then it must have a maximal subgroup of order dividing h. Then using the Tables in [26] and the Atlas [5], it is easy to check that the following groups do not have maximal subgroups dividing h: M_{24}, J_3, J_4, HS, Suz, $O'N$, Ly, Co_2, F_{23}, Th. For the other sporadic groups we use the following arguments.

Let χ_2 be the non principal character of minimal degree of F'_{24}. Then $\deg(\chi_2) = 8671$. Since $17 \cdot 23 \cdot 29 = 11339$ (and any other character of F'_{24} has greater degree), then F'_{24} hasn't subgroups whose index divides $17 \cdot 23 \cdot 29$.

Let χ_2 be the non principal character of minimal degree of M. Then $\deg(\chi_2) > 41 \cdot 59 \cdot 71$ and therefore M hasn't subgroups whose index divides $41 \cdot 59 \cdot 71$.

Let χ_2 be the non principal character of minimal degree of BM. Then $\deg(\chi_2) > 31 \cdot 47$ and therefore BM hasn't subgroups whose index divides $31 \cdot 47$.

Simple groups of Lie type We now consider a finite simple group of Lie type defined over a field with $q = p^f$ elements. We recall that a *Singer cycle* of $PSL(n, q)$ is an element of order $(q^n - 1)/(q - 1)(n, q - 1)$.

If G is a simple group of Lie type with $t(G) \geq 3$, then G is one of the following (see [13, 14, 26]): $PSL(2, q)$, $PSL(3, 4)$, $E_7(2)$, $E_7(3)$, $E_8(q)$, $F_4(q)$ with q even, $G_2(q)$ with $q \equiv 0$ (3), $PSU(6, 2)$, $Sz(q)$, ${}^2D_p(3)$ with $p = 2^n + 1$, $n \geq 2$, ${}^2E_6(2)$, ${}^2F_4(q)$, $Ree(q)$.

We first observe that if G is $PSL(2, 2^n)$, $PSL(3, 4)$ or $Sz(q)$, then $\pi_1(G) = \{2\}$. Therefore a π_1-Hall subgroup is in fact a Sylow 2-subgroup. Also for $PSL(2, q)$, q odd, it is easy to see that a π_1-Hall subgroup exists and they are all conjugate.

We begin with an easy remark, which allows us to understand the structure of the maximal parabolic subgroups of a finite group of Lie type.

REMARK 1. Let J be a subset of the set Π of fundamental roots of the finite group of Lie type G and Φ_J be the set of roots which are integral combinations of roots in J. Let L_J be the subgroup of G generated by H and the root subgroups X_r, for all $r \in \Phi_J$. Then $P_J = U_J L_J$, $L_J \cap U_J = 1$ and U_J is an unipotent subgroup, by [4, Theorem 8.5.2]. If P_J is a maximal parabolic subgroup, then $J = \Pi \setminus \{i\}$, for some fundamental root i. Since H normalises any X_r, we have $L_J = \langle X_r : r \in \Phi_J \rangle H_i$, where H_i is the subgroup of G generated by $h_i(\lambda)$, $\lambda \in K^*$ (see [4, page 98]). We call $M_J = \langle X_r : r \in \Phi_J \rangle$.

We begin with a case by case analysis.

Let G be one of the groups listed in Table 3. We suppose that there exists K a π-Hall subgroup of G, with $\pi_1 \subseteq \pi$. We want to prove that K cannot be contained in any maximal subgroup of G, and therefore G does not admit any π_1-Hall subgroup. We use the Theorem of [17], observing that $|K| \geq q^{k(G)}$, where $q^{k(G)}$ is as defined in [17, Table 1], and also in our Table 2. If M is a maximal subgroup of G containing K,

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TABLE 3.

G	$q^{k(G)}$	$h(G)$
$E_7(q)$	q^{64}	q^7+1
$E_8(q)$	q^{110}	$(q^{12}-1)^2$
$F_4(q)$	q^{24}	$(q^6-1)^2$
$G_2(q)$	q^6	$(q^2-1)^2$
${}^2E_6(q)$	q^{37}	$(q^6-1)^2$

then

(1) M contains a p-Sylow subgroup of G ($q = p^f$),
(2) $|M| \geq q^{k(G)}$,
(3) $|M|$ is divisible by $h(G)$, where $h(G)$ is an integer, as listed in Table 2.

By [17, Theorem], M is either a parabolic subgroup or M is as in [17, Table 1]. The groups listed in [17, Table 1] do not contain a p-Sylow subgroup of G. Moreover, if we consider the maximal parabolic subgroups of G, we can easily check that no one of them has order divisible by $h(G)$. We conclude that G does not admit a π-Hall subgroup.

$PSU(6, 2)$ It can be checked in the Atlas [5] that there is no π-Hall subgroup, for $\pi_1 \subseteq \pi$.

${}^2D_n(3)$ It can be proved that if K is a maximal subgroup containing a 3-Sylow subgroup, then K is a parabolic subgroup (by [16] and some easy calculations). If we denote by i the ith node in the Dynkin diagram, then the isomorphism classes of the maximal parabolic subgroups are

$$\begin{aligned}
J &= \Pi \setminus \{i\} &&\Rightarrow\quad M_J \cong A_{i-1}(q) \times {}^2D_{n-i}(q), \text{ for } 1 \leq i \leq n-4,\\
J &= \Pi \setminus \{n-3\} &&\Rightarrow\quad M_J \cong A_{n-4}(q) \times {}^2A_3(q),\\
J &= \Pi \setminus \{n-2\} &&\Rightarrow\quad M_J \cong A_{n-3}(q) \times A_1(q^2),\\
J &= \Pi \setminus \{n-1\} &&\Rightarrow\quad M_J \cong A_{n-2}(q).
\end{aligned}$$

Since the p'-part of the order of the maximal parabolic subgroup P_J is $(q-1)|M_J|$, it can be easily seen that $q^{n-1}-1$ does not divide the order of any maximal parabolic subgroup of G, while $q^{n-1}-1$ should divide the order of a π-Hall subgroup of G, $\pi_1 \subseteq \pi$.

${}^2F_4(q)$ We know from [19], that the only maximal subgroups containing a p-Sylow subgroup of G are the maximal parabolic subgroups, and no one of these is divisible by q^6+1, which should divide the order of a π-Hall subgroup of G, $\pi_1 \subseteq \pi$.

${}^2G_2(q)$ We know from [15], that the only maximal subgroups containing a p-Sylow subgroup of G are the maximal parabolic subgroups, and no one of these is divisible by q^2+1, which should divide the order of a π-Hall subgroup of G, $\pi_1 \subseteq \pi$.

3.2. Almost simple groups The connected components of the prime graph of almost simple groups have been calculated in [18]. We therefore refer to [18], without further reference.

For the sporadic groups we refer again to [5]. For the alternating groups, it is easy to observe that if $G = S_r$ is the symmetric group over r elements, with r an odd prime, $r \geq 7$, then the stabiliser of an element is isomorphic to S_{r-1} and it is a π_1-Hall subgroup. Moreover the π_1-Hall subgroups are all conjugate.

If $S \cong PSL(2, q)$ and G contains a diagonal automorphism, then $\pi(q^2-1) \subseteq \pi_1(G)$ and $PGL(2, q)$ does not contain subgroups of order (divisible by) $q^2 - 1$.

If G contains a field automorphism α of order not a power of 2 and $q \neq 2$ or 3, then $\Gamma(G)$ is connected. If $|\alpha| = 2$ and $G = S\langle\alpha\rangle$, then $\pi_1(G) = \pi(q(q-1))$. If q is odd, then there is no $\pi_1(G)$-Hall subgroup in G, since there isn't a $\pi_1(G)$-Hall subgroup in S. If q is even, let B be the subgroup of S of the upper triangular matrices. We observe that B is fixed by α and therefore $\widetilde{B} = B\langle\alpha\rangle$ is a π_1-Hall subgroup of G. Moreover the π_1-Hall subgroups of G are all conjugate.

If f is an odd prime and $q = 2^f$ or $q = 3^f$, then $\pi_1(G) = \pi(f\, q(q+1)/(2, q-1))$. If K is a $\pi_1(G)$-Hall subgroup of G, then $K \cap S$ is a subgroup of S of order $q(q+1)/(2, q+1)$, which does not exist.

If q is odd and a square, that is $q = q_0^2$, for some $q_0 = p^n$, then there exists a non-split extension $M(q)$ of $PSL(2, q)$ of order 2, with $\Gamma(M(q)) = \Gamma(S)$. We observe that the order of a π_1-Hall subgroup of G should be $2(q-1)$ and therefore a π_1-Hall subgroup of S is $N_S(H) = N$ the normaliser of the diagonal group H. We also observe that H, and therefore N, is fixed by any automorphism of S. Then G has a π_1-Hall subgroup.

If $S = Sz(q)$ with $q = 2^f$, and G is a subgroup of its automorphism group, then $\Gamma(G)$ is always connected, except when f is a prime and $G = S\langle\alpha\rangle$, with α a field automorphism of order f. In this case $\pi_1(G) = \pi(2f\,(q + \sqrt{2q} + 1))$ or $\pi_1(G) = \pi(2f\,(q - \sqrt{2q} + 1))$ depending if $f \equiv 1, 7$ (8) or $f \equiv 3, 5$ (8). In both cases there should exists a $\pi(2(q \pm \sqrt{2q} + 1)$-Hall subgroup of S and this is not possible in any of the two cases (see [12] or [25]).

If $S \cong PSL(3, 4)$, it is easy to check (see [5]) that there is no π_1-Hall subgroup for any of the extensions.

If $S \cong PSL(r, q)$ with $(r, q-1) = 1$ and $q = p^f$, p a prime, then $\mathrm{Aut}(S) = S(\langle\varphi, \tau\rangle)$, where φ is a field automorphism of order f, and τ is the graph automorphism of order 2 of S.

If G contains a graph automorphism and $t(G) = 2$, then there is no $\pi_1(G)$-Hall subgroup in G. In fact, no $\pi_1(S)$-Hall subgroup of G is fixed by α, which interchanges the two conjugacy classes of parabolic subgroups.

If G contains a field automorphism of order a prime different from r, then $\Gamma(G)$ is connected. If $G = S\langle\alpha\rangle$ with α a field automorphism of order r, then $\pi_1(G) = \pi_1(S)$

and $C_S(\alpha) \cong PGL(3, q_0)$ if $q_0^3 = q$. We observe that there exists a π_1-Hall subgroup $\widetilde{P}_1$ of G, which is an extension of P_1, a π_1-Hall subgroup of S.

By the proof of Proposition 3.1 and Proposition 3.3, we also get the following corollaries.

COROLLARY 3.5. *Let G be a group and π be a set of primes such that $\pi_1 \subseteq \pi \subset \pi(G)$. Then*

(i) *G has a π-Hall subgroup if and only if G has a π_1-Hall subgroup*;

(ii) *if $\pi_1 \subset \pi$ and G satisfies $(*)$, then $G/\mathrm{Fit}(G)$ is isomorphic to one of the groups in Table 2.*

Let G be a group and π be a set of primes in $\pi(G)$. We say that π is *connected* if and only if there exists $i = 1, \ldots, t(G)$ such that $\pi \subseteq \pi_i$.

COROLLARY 3.6. *Let G be a group satisfying $(*)$. Then G has a π-Hall subgroup, for any connected subset π of $\pi(G)$ if and only if $G/\mathrm{Fit}(G)$ is isomorphic to one of the following groups: $PSL(2, q)$, $Sz(q)$, $PSL(3, 3)$, $PSL(3, 4)$, A_7, M_{11}, $PSL(2, 2^n)\langle\alpha\rangle$ with $|\alpha| = 2^m$, $M(q)$.*

PROOF. It is enough to examine the non-soluble groups H in Table 1. If G is a sporadic, alternating or symmetric group, then, for example, there does not exist a $\{2, 5\}$-Hall subgroup of G (for the symmetric groups see [9]). If $G = PSL(r, q)$, with $q = p^f$, then there does not exist a $\{p, t\}$-Hall subgroup for any prime t such that $(t, q(q-1)) = 1$, except for $PSL(3, 2) \cong PSL(2, 7)$, $PSL(5, 2)$ for which the statement holds with $t = 7$, and $PSL(3, 3)$, where a π_1-Hall subgroup is in fact a $\{2, 3\}$-Hall subgroup (see [23, Theorem 2.3.2]). □

4. Hall coverings

In this section we want to prove the following:

THEOREM 4.1. *Let G be a group satisfying $(*)$. Then G admits a Hall covering if and only if $G/\mathrm{Fit}(G)$ is isomorphic to one of the following groups: $PSL(2, q)$, $PSL(3, 4)$, $PSL(3, q)$ with $(3, q-1) = 1$, $Sz(q)$, A_7, M_{22}, $M(q)$.*

We begin with a lemma which allows us to reduce to the case of an almost simple group.

LEMMA 4.2. *Let G be a group satisfying $(*)$. Then G has a Hall covering if and only if $G/\mathrm{Fit}(G)$ has a Hall covering.*

PROOF. This is Lemma 3.5 (ii). □

We have proved in the preceding sections that if a group G has a Hall cover, then G has a π_1-Hall subgroup (see Corollary 3.5). It is therefore enough to examine the almost simple groups G belonging to Table 1.

Alternating groups Since $A_5 \simeq PSL(2, 5)$, we suppose $r \geq 7$. Then the element $(12)(34)(5 \cdots r)$ of order $2(r-4)$ fixes no point and therefore it cannot be contained in a subgroup of A_r isomorphic to A_{r-1}. Therefore A_r ($r \geq 7$ a prime) does not admit Hall coverings with $s = 2$.

It is easy to see that A_7 admits a Hall covering with $t = s = 3$.

Sporadic groups M_{11} contains elements of order 6 but no subgroups of index 55 or 5 or 11 contains such elements.

M_{22} does not contain subgroups of index $5 \cdot 7$, $5 \cdot 11$ or $5 \cdot 7 \cdot 11$ (see [5]). It can be easily seen that the $\{5, 7\}'$-Hall subgroups, together with the 5-Sylow and the 7-Sylow subgroups are a Hall covering of M_{22} with $t = s = 3$.

M_{23} contains elements of order 15, while none of its $\{2, 3, 5, 7\}$-subgroups contain elements of order 15. Therefore M_{23} does not admit Hall coverings.

J_1 contains elements of order 15 but the only π-Hall subgroups with $\{3, 5\} \subseteq \pi$, are isomorphic to $A_5 \times C$ with C a cyclic group of order 2.

$PSL(2, q)$ It is well known that $PSL(2, q)$ is a group with a partition and it admits a covering with π_i-Hall subgroups, for $i = 1, 2, 3$ (see [12]). Moreover if $3 < q \not\equiv 1(4)$, then the Borel subgroup of order $q(q-1)/(2, q-1)$ is a $\pi(q(q-1))$-Hall subgroup. Then, in this case, it also admits a partition with $\pi((q+1)/(2, q-1))$-Hall and $\pi(q(q-1))$-Hall subgroups. A subgroup containing a p-Sylow subgroup of G must be contained in a Borel subgroup, then the only other possibility is to have a $\pi(q^2-1)$-Hall subgroup. We are then in the case of G factorizable again and the only case we have to consider is $PSL(2, 11)$, with $A \cong A_5$ as a $\{2, 3, 5\}$-Hall subgroup. But there is an element of order 6 in $PSL(2, 11)$, which is not contained in any $\{2, 3, 5\}$-Hall subgroup.

$PSL(3, 4)$ In this case every π_i contains only a prime, and therefore there is a covering with the Sylow subgroups. We recall that $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$. Moreover a 2-Sylow subgroup must be contained in a parabolic subgroup. By the remark at the beginning of the proof, there exists three conjugacy classes of parabolic subgroups: one of order $2^6 \cdot 3$, which is not a $\{2, 3\}$-Hall, and two of order $2^6 \cdot 3 \cdot 5$. Moreover the only subgroups containing a Singer cycle are those of order 21. Therefore the only possibility is a Hall covering with $\sigma_1 = \pi_1 = \{2\}$ and $\sigma_2 = \pi_4 = \{7\}$. If H is a $\{3, 5\}$-Hall subgroup, then H should be contained in a maximal subgroup M with $M \cong A_6$ (see [5]). But A_6 hasn't a $\{3, 5\}$-Hall subgroup.

PSL(r, q) If $G = PSL(r,q)$, then $M = P_{1'}$ or $M = P_{r'}$ is a maximal parabolic subgroup, and also a π_1-Hall subgroup of G. Then $|M| = q^{r-1}(q-1)|SL(r-1,q)|$, since $(r, q-1) = 1$. Then M is a σ_1-Hall subgroup and $B = \langle x_r \rangle$ is a $\sigma_2 = \pi_2$-Hall subgroup of G, where x_r is a Singer cycle of order $(q^r-1)/(q-1)$. Moreover any σ_1-Hall subgroup is contained in a maximal subgroup and the only maximal subgroups with order divisible by $|G|_{\pi_1}$ are those isomorphic to M (see [20]). It can be proved (see [3, Proposition 3.3]) that if $r \geq 5$, there exists an element x in $PSL(r,q)$ of order

$$b = \frac{q^t-1}{q-1}\,\frac{q^{r-t}-1}{q-1}.$$

Moreover b does not divide the following products

$$\prod_{i=1,\dots,s}(q^{j_i}-1) \quad \text{for } 1 \leq j_i \leq r-1, \ \sum_{i=1,\dots,s} j_i = r-1,$$

and b does not divide $q^r - 1$. But then x does not belong neither to a π_1-Hall subgroup nor to a π_2-Hall subgroup. Therefore, also in this case, G cannot have a Hall covering.

If $r = 3$, then there are two coverings: with the conjugates of a Singer cycle and with one of the two classes of maximal parabolic subgroups of G:

$$\mathscr{H}_1 = \{P_{1'}^g, \langle x_3\rangle^g \mid g \in G\}, \quad \mathscr{H}_2 = \{P_{r'}^g, \langle x_3\rangle^g \mid g \in G\}.$$

This is proved in [3, Proposition 4.1 and Corollary 4.2].

Sz(q) By [12, Theorem 3.10, cap XI], the Suzuki groups admits a partition with π_i-Hall subgroups. Moreover, G admits a $\pi_1 \cup \pi_2$-Hall subgroup, which is a Frobenius group of order $q^2(q-1)$. Therefore, there are two kinds of coverings with Hall subgroups:

(i) $\pi_1, \pi_2, \pi_3, \pi_4$;
(ii) $\pi_1 \cup \pi_2, \pi_3, \pi_4$.

Almost simple groups Let G be an almost simple group which admits a Hall covering.

We recall that $\pi(G/S) \subseteq \pi(G)$, by [26, Theorem A (d)]. Therefore if $\mathscr{H} = \{H_1, H_2, \dots, H_r\}$ is a Hall covering of G, then $\mathscr{H}_S = \{H_1 \cap S, H_2 \cap S, \dots, H_r \cap S\}$ is a Hall covering of S. We only have to consider the almost simple non simple groups, that is groups G such that $S < G \leq \mathrm{Aut}(S)$, with S a simple non-abelian group admitting a Hall covering.

If $G = S_7$, then $\pi_1(G) = \{2,3,5\}$ and the only subgroup of index 7 of S_7 is isomorphic to S_6. But S_6 does not contain elements of order 10, as a $\{2,3,5\}$-Hall subgroup of S_7 should.

$PSL(2, q) \leq G \leq \mathrm{Aut}(PSL(2, q))$ We first consider the case in which $G = PSL(2, 2^n)\langle\alpha\rangle$ and α is a field automorphism of order 2. We recall that $C_S(\alpha) = PSL(2, q_0)$, where $q_0^2 = q$, while $C_B(\alpha) = B_0$ of order $q_0(q_0 - 1)$. Therefore there exists an element $x \in C_S(\alpha)$ of order $(q_0 + 1)$ such that $x \cdot \alpha$ has order $2(q_0 + 1)$ and is not contained in S. This element is not contained in any of the conjugate of $\widetilde{B}$, since there is no element of such order in $\widetilde{B}$, with $\widetilde{B}$ the π_1-Hall subgroup of G previously described.

If $G = M(q)$, then by the preceding Proposition, we have $\widetilde{N}$, a π_1-Hall subgroup of G. We observe that any element of G is contained in one of the π_i-Hall subgroups, and therefore we have the following covering:

$$(\cup_g \widetilde{N}^g) \cup (\cup_g \widetilde{P}^g) \cup (\cup_g \widetilde{T}^g),$$

P is a p-Sylow subgroup of G, and T is a (Singer) cycle of order $(q + 1)/2$.

$PSL(3, q)\langle\alpha\rangle$ By Proposition 3.1, there exists a π_1-Hall subgroup $\widetilde{P}_{1'}$. But there exists an element of order $3(q - 1)$ which is not contained in $\widetilde{P}_{1'}$. The same is true if we consider the other class $P_{3'}$ of π_1-Hall subgroups of S.

5. Further remarks

As already mentioned, the class of CN-groups is related to the groups admitting a Hall covering. It is not difficult to verify that if a group G admits a nilpotent Hall covering (that is a Hall covering in which all the subgroups of the covering are nilpotent) then G is a CN-group. It is also true that if G is a CN-group, then G admits a nilpotent Hall covering, using, for example, [7, Theorem 14.1.7].

We recall that the simple groups with a partition have been classified by Suzuki (see, for example, [22, Section 3.5]): they are $PSL(2, p^n)$, $p^n > 3$ and $Sz(2^{2n+1})$. They all admit a Hall covering, while the only simple CN-group without a partition is $PSL(3, 4)$.

The soluble CN-groups are known (see [7, Theorem 14.1.5]), while Suzuki proved that a simple CN-group is isomorphic to one of the following list (see [12, Remark XI.3.12.a]):

(i) $PSL(2, 2^n)$ with $n > 1$;
(ii) $PSL(2, p)$ with p Mersenne or Fermat prime;
(iii) $PSL(2, 9)$;
(iv) $PSL(3, 4)$;
(v) $Sz(2^{2n+1})$ with $n > 1$.

In the same paper [24, Theorem 4], Suzuki proved that a non-soluble CN-group is a CIT-group, that is a group of even order in which the centralizer of any involution is

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a 2-group. From the results of Higman [11], Suzuki [24] and Martineau [21], we also get:

THEOREM 5.1. *Let G be a non-soluble CN-group, then either*

(1) *G is isomorphic to simple groups on Suzuki's list or*

(2) *G is isomorphic to $M(9)$ or*

(3) *G has a non trivial normal 2-subgroup N and G/N is isomorphic to $PSL(2, 2^n)$ or to $Sz(2^{2n+1})$. Moreover, N is an elementary abelian group.*

REMARK 2. The group $M(9)$ is a CN-group and it also admits a Sylow covering. This case was missing in the paper [2] on the non-soluble groups in which any element has order a power of a prime.

We note that we do not use character theory to prove Theorem 5.1, as it is done in [2]. We use a more elementary fact, which can be found in [11].

LEMMA 5.2 ([11, Theorem 8.1]). *Let H be a group with a normal 2-subgroup T such that H/T is dihedral of order 6. Let h be an element of H of order 3 acting fixed point free on T, and let R be a Sylow 2-subgroup of H. Then*

(i) *T is of class at most 2;*

(ii) *if $|T| > 4$, the class of T is less than the class of any other subgroup of R of index 2.*

PROOF OF THEOREM 5.1. Let G be a non-soluble CN-group, then G has a nilpotent Hall covering. If G is simple, then G is in the Suzuki list. If G is almost simple, then applying Theorem 4.1 we get that G is isomorphic to $M(9)$ (see also [24, Theorem 3]).

By the above mentioned results of Suzuki, it is sufficient to prove the theorem for CIT-groups.

Let now N be the maximal normal soluble subgroup of G; then, if $\overline{G} = G/N$ we have $Z(\overline{G}) = 1$ and $O_{2'}(\overline{G}) = 1$. We suppose $N \neq 1$ and, by Lemma 3.2 (i), we know that $N = \mathrm{Fit}(G)$. We first prove that N is a 2-group. In fact N is nilpotent and we can therefore assume that it is an r-group. If $r \neq 2$ then any Sylow 2-subgroup $\overline{S}$ of $\overline{G} = G/N$ acts fixed point free over N. Then $\overline{S}$ is a cyclic or a generalized quaternion group (see [7, 10.3.1]). In the first case $\overline{G}$ has a normal 2-complement; in the second case by the Brauer-Suzuki Theorem (see [7, Chapter 12] and recall that $O_{2'}(\overline{G}) = 1$) we get $Z(\overline{G}) \neq 1$. In both cases we get a contradiction. Therefore N is a 2-group.

Since G is a CIT-group, any Sylow 2'-subgroup of G acts fixed point free over N, and it is therefore cyclic. This implies that $\overline{G}$ is isomorphic to $PSL(2, 2^n)$, $Sz(q)$ or $PSL(2, p)$ with p a Fermat or Mersenne prime and $p > 5$. If $\overline{G}$ is isomorphic to $PSL(2, p)$ with p Fermat or Mersenne prime and $p > 5$, a Sylow 2-subgroup $\overline{S}$ of $\overline{G}$ is dihedral of order at least 8. If $\overline{T}$ is an elementary abelian 2-subgroup of $\overline{G}$, then $|T| = 4$ and $\overline{H} = N_{\overline{G}}(\overline{T})$ is isomorphic to S_4. We can apply lemma 5.2 to the preimages H and T of $\overline{H}$ and $\overline{T}$ in G and R a Sylow 2-subgroup of H. In particular T has class 2, otherwise $T \le C_G(N) \le N$.

Let $\overline{T}^*$ be an elementary abelian subgroup of order 4 of $\overline{H}$, distinct from $\overline{T}$. If T^* is the preimage of $\overline{T}^*$ in G, then T and T^* are isomorphic. But T^* is a subgroup of index 2 of R and therefore, by Lemma 5.2, T^* has class strictly less than the one of T.

The actions of $H = PSL(2, 2^n) = SL(2, 2^n)$ or $H = Sz(2^{2n+1})$ over an elementary abelian group N are described respectively in [11, Theorem 8.2], and in the main theorem of [21]. The semidirect product $G = NH$ obtained by these actions is a CIT-group. □

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Dipartimento di Matematica Applicata
e Informatica
Università "Ca' Foscari" di Venezia
Via Torino 155
31073 Venezia Mestre
Italy
e-mail: jabara@dsi.unive.it

Dipartimento di Matematica
e Informatica
Università di Udine
Via delle Scienze 208
I-33100 Udine
Italy
e-mail: mslucido@dimi.uniud.it