Integer matrices obeying generalized incidence equations

Jennifer Wallis

We consider integer matrices obeying certain generalizations of the incidence equations for \((v, k, \lambda)\)-configurations and show that given certain other constraints, a constant multiple of the incidence matrix of a \((v, k, \lambda)\)-configuration may be identified as the solution of the equation.

We define \((v, k, \lambda)\)-configurations as usual (see [3]). If \(B\) is the \((0, 1)\) incidence matrix of a \((v, k, \lambda)\)-configuration and if \(A = bB\) where \(b\) is a positive integer, then

\[
\begin{align*}
AA^T &= b^2(k-\lambda)I + b^2\lambda J \\
AJ &= bkJ \\
\lambda(v-1) &= k(k-1),
\end{align*}
\]

with \(J\) as usual the matrix with every element +1, and \(I\) the identity matrix. Ryser [2] proved a partial converse:

**Lemma 1.** If \(A\) is a \(v \times v\) integer matrix satisfying equations (1) with \(b = 1\), then \(A\) is the incidence matrix of a \((v, k, \lambda)\)-configuration (and consequently has every entry 0 or 1).

One might conjecture, in view of the powerful theorems of Ryser [2] and Bridges and Ryser [1], that an integer matrix satisfying (1) would necessarily be \(b\) times the incidence matrix of a \((v, k, \lambda)\)-configuration. But the matrix
satisfies (1) with $b = 2$, $v = 7$, $k = 3$ and $\lambda = 1$. So we need other conditions on the matrix $A$ before we can ensure that every element is 0 or $b$. We shall prove:

**Theorem 2.** If $A$ is a $v \times v$ matrix of non-negative integers which satisfies (1), and if every entry of $A$ is less than or equal to $b$, then $A$ is $b$ times the incidence matrix of a $(v, k, \lambda)$-configuration.

The corresponding result for non-positive $A$ and negative $b$ also holds.

By similar methods we shall obtain a result about more general equations:

**Theorem 3.** Let $B$ be an integer matrix of order $v$ which satisfies

$$BB^T = (p-q)I + qJ$$

$$BJ = dJ$$

where $p$, $q$ and $d$ are constants and $d > 0$. Write $w$ and $z$ for the greatest and least elements of $B$ respectively, and $w = |w|$.

If

$$z \leq \frac{d}{v} = \delta \text{ and } z \leq \frac{w^2 + p}{d + wz},$$

then $\delta$ is an integer, $p = d\delta = v\delta^2$, and $B = \delta J$.

1. Proof of Theorem 2

**Lemma 4.** Let $B = \{b_{i,j}\}$ of order $v$ be a matrix of non-negative integers such that

$$\sum_{j=1}^{v} b_{i,j}^2 = p, \text{ } p \text{ a constant, for every } i,$$

and let
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$BJ = dJ$, $d$ a non-zero constant. If $b_{ij} \leq \frac{p}{d}$ for every $b_{ij}$, or if $b_{ij} \geq \frac{p}{d}$ for every non-zero $b_{ij}$, then every entry of $B$ is 0 or $\frac{p}{d}$.

Proof. \[ \sum_{j=1}^{v} b_{ij}^2 = p \quad \text{and} \quad \sum_{j=1}^{v} b_{ij} = d, \]
so

\[ d \sum_{j=1}^{v} b_{ij}^2 - p \sum_{j=1}^{v} b_{ij} = dp - dp = 0; \]

that is

\[ \sum_{j=1}^{v} b_{ij} (d b_{ij} - p) = 0. \]

From the data every term in this summation has the same sign, so every term is zero. So $b_{ij} = 0$ or $\frac{p}{d}$.

COROLLARY 5. If there is a matrix $B$ satisfying the conditions of Lemma 4, then $d|p$.

Corresponding results may be obtained for matrices of non-positive integers.

Proof of Theorem 2. The matrix $A$ satisfies the conditions of Lemma 4 with $p = b^2 k$ and $d = bk$. So every entry is 0 or $b (b = \frac{p}{d})$.

Consider $B = b^{-1} A$. $B$ is an integer matrix satisfying Lemma 1, so it is the incidence matrix of a $(v, k, \lambda)$-configuration, and we have the result.

2. Proof of Theorem 3

Proof of Theorem 3. Clearly $p = \sum_{i} b_{ij}^2$ implies $p \geq 0$; and $d > 0$ implies $p > 0$. Consider the class of matrices

$C_\alpha = B + \alpha J$

where $\alpha$ is an integer and $\alpha \geq \omega$. Every element of every member of this class is non-negative and
Then using Lemma 4, if every non-zero element of $C_\alpha$ is less than or equal to $\beta$,

$$\beta = \alpha + \frac{ad+p}{d+\omega}$$

then every element is 0 or $\beta$.

We show that the conditions on $z$ imply that every element is $\leq \beta$. For

$$z \leq \frac{wd+p}{d+\omega}$$

implies

$$z(d+\omega) \leq wd + p;$$

since $z \leq \frac{d}{v}$ we have

$$zd + zwv + \gamma zv \leq wd + p + \gamma zv \leq wd + p + \gamma d$$

for any integer $\gamma \geq 0$, so

$$z \leq \frac{(w+\gamma)d+p}{d+(w+\gamma)v}.$$  

This means (putting $\alpha = \omega + \gamma$) that for any admissible $\alpha$,

$$z + \alpha \leq \alpha + \frac{ad+p}{d+\omega};$$

but $z + \alpha$ is the greatest element of $C_\alpha$. Therefore, any element of $C_\alpha$ is 0 or $\alpha + \frac{ad+p}{d+\omega}$, so any element of $B$ is $-\alpha$ or $\frac{ad+p}{d+\omega}$.

Corollary 5 tells us that

$$A(\gamma) = \frac{(w+\gamma)d+p}{d+(w+\gamma)v} = \frac{d+p(w+\gamma)^{-1}}{d(w+\gamma)^{-1}+v}$$

is integral for all integers $\gamma \geq 0$. Therefore $\lim_{\gamma \to \infty} A(\gamma)$ must be an integer, so $\nu | d$. Write $d = \nu \delta$.
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\[ A(\gamma) = \frac{(\omega+\gamma)v\delta+p}{v\delta+(\omega+\gamma)v} \]

so \( v | p \). Write \( p = \epsilon v \):

\[ A(\gamma) = \frac{(\omega+\gamma)\delta+\epsilon}{\delta+(\omega+\gamma)} \cdot \]

Choose \( n \) any integer greater than \( \delta + \omega \). Then

\[ A(n-\delta-\omega) = \frac{(n-\delta)\delta+\epsilon}{n} \]

so \( n | (\epsilon-\delta^2) \). But this is true for every large enough \( n \); hence \( \epsilon = \delta^2 \). That is

\[ d = v\delta \]
\[ p = v\delta^2 \]

so

\[ p = d\delta = v\delta^2 \cdot \]

Then we have

\[ \frac{ad+p}{d+\omega} = \frac{v\delta(\alpha+\delta)}{v(\delta+\alpha)} = \delta \]

for any \( \gamma \), so every element of \( B \) is \(-\alpha\) or \( \delta \). Now the row sum of \( B \) is \( d = v\delta \) and the sum of the squares of the elements is \( p = v\delta^2 \); together these imply

\[ B = \delta J \]

where \( \delta = \frac{d}{v} \).

References


University of Newcastle,
New South Wales.