ON THE ABSOLUTE SUMMABILITY FACTOR
OF FOURIER SERIES

Yasuo Okuyama

The purpose of this paper is to give a general theorem on the
absolute Riesz summability factor of Fourier series which implies
Matsumoto's Theorem [Tohoku Math. J. 8 (1956), 114-124] and to
deduce some results from the theorem.

1.

Let $\sum a_n$ be an infinite series and $s_n$ its $n$th partial sum. Let
$\{p_n\}$ be a sequence of positive numbers such that

$$ p_n = p_0 + p_1 + \ldots + p_n + \infty \quad \text{as} \quad n \to \infty. $$

If the sequence

$$ t_n = \frac{1}{p_n} \sum_{k=0}^{n} p_k s_k \quad (n = 0, 1, 2, \ldots) $$

is of bounded variation, that is, $\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty$, then the series

$\sum a_n$ is said to be summable $|R, p_n, 1|$. 

Let $f(t)$ be a periodic function with period $2\pi$ and integrable $(L)$
over $(-\pi, \pi)$. We assume without any loss of generality that the Fourier

Received 21 May 1981. I should like to express my sincerest thanks
to Professor Tsuchikura and Kanno for their kind and valuable suggestions. 
This research was partially supported by Grant - in - Aid for Scientific 
Research (No. 564048), Ministry of Education.

327
series of \( f(t) \) is given by

\[
(1.2) \quad \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t), \quad \text{say},
\]

and

\[
\int_{-\pi}^{\pi} f(t) dt = 0.
\]

We use the following notations:

\[
\phi_x(t) = \phi(t) = \frac{1}{2} [f(x+t) + f(x-t)];
\]

\[
\Phi(t) = \int_0^t |\phi(u)| du;
\]

\[
P_n(t) = \frac{\sin(n+\frac{1}{2})t}{2 \sin \frac{t}{2}};
\]

\[
L_0(t) = 1, \quad L_1(t) = \log t, \quad L_p(t) = L_1\left( L_{p-1}(t) \right) = \log \ldots \log t \quad \text{\( (p\ \text{times}) \)},
\]

\[
L_p^{(\epsilon)}(t) = L_1(t) \ldots L_{p-1}(t) [L_p(t)]^{1+\epsilon} \quad (\epsilon \geq 0, \ p = 1, 2, \ldots),
\]

where, if the right hand sides are not determined as positive numbers, we replace them by 1's.

Let \( \{\lambda_n\} \) be a monotone decreasing sequence. We put

\[
\Delta \lambda_n = \lambda_n - \lambda_{n+1}. \quad \text{Also we define a function} \quad \lambda(t) \quad \text{continuous in the interval} \quad (0, \infty) \quad \text{such that} \quad \lambda(n) = \lambda_n \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and} \quad \lambda(t) \quad \text{is linear for every non-integral} \quad t. \quad \text{Similarly} \quad p(t) \quad \text{is defined by the sequence} \quad \{p_n\} \quad \text{and we put}
\]

\[
P(t) = \int_0^t p(u) du.
\]

\( A \) denotes a positive absolute constant that is not always the same.

2.

Concerning the absolute Riesz summability factor of Fourier series, Matsumoto [6] proved the following theorem.
THEOREM A. The \( |R, P_n, 1| \) summability of the series \( \sum_{n=1}^{\infty} \lambda_n A_n(t) \)

at \( t = x \) is a local property where

<table>
<thead>
<tr>
<th>( p_n )</th>
<th>( \exp n^\delta )</th>
<th>( \exp(\log n)^\delta )</th>
<th>( \exp(\log \log n)^\delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_n )</td>
<td>( \frac{1}{n^\delta (\log n)^{1+\varepsilon}} )</td>
<td>( \frac{1}{(\log(n+1))^{\delta+\varepsilon}} )</td>
<td>( \frac{1}{(\log \log(n+1))^{\delta+\varepsilon}} )</td>
</tr>
<tr>
<td>( \delta )</td>
<td>( 0 &lt; \delta \leq 1 )</td>
<td>( 0 &lt; \delta )</td>
<td>( 0 &lt; \delta )</td>
</tr>
<tr>
<td>( \chi(1/t) )</td>
<td>( \frac{t}{\log 1/t} )</td>
<td>( \frac{t}{\log 1/t} )</td>
<td>( \frac{t}{\log 1/t \log \log 1/t} )</td>
</tr>
</tbody>
</table>

More precisely, if

\[
\phi(t) = \int_0^t |\phi(u)| \, du = O(\chi(1/t)),
\]

then the series \( \sum_{n=1}^{\infty} \lambda_n A_n(t) \) is summable \( |R, P_n, 1| \), at \( t = x \).

In this paper we shall generalize Theorem A in the following form.

THEOREM. Let \( \{\lambda_n\} \) be a monotone decreasing sequence and \( \{p_n\} \) be a positive sequence such that \( \{p_n/p_n\} \) is decreasing.

If

\[
(2.1) \quad \int_0^T \frac{\phi(t)}{t^2} \, dt \int_0^t \frac{P(1/u)\lambda(1/u)}{u^2P(1/u)} \, du < \infty
\]

and

\[
(2.2) \quad \int_0^\infty \frac{\phi(t)\lambda(1/t)}{t^2} \, dt < \infty,
\]

then the series \( \sum_{n=1}^{\infty} \lambda_n A_n(t) \) is summable \( |R, P_n, 1| \), at \( t = x \).

The conditions (2.1) and (2.2) have been obtained by referring to the theorems on the absolute Nörlund summability of Fourier series, which are...
due to Izumi and Izumi [2], [3], Kolhekar [4] and Leindler [5].

In our theorem we put \( \lambda(t) = 1/t^\delta (\log t)^{1+\epsilon} \), \( \chi(t) = 1/t \log t \) and \( P(t) = \exp t^\delta \) \( (0 < \delta \leq 1) \). Then we have

\[
\int_0^\infty \frac{\phi(t)}{t^2} \, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du \leq A \int_0^\infty \frac{\log \log 1/u}{u(\log 1/u)^{1+\epsilon}} \, du < \infty
\]

and

\[
\int_0^\infty \frac{\phi(t)\lambda(1/t)}{t^2} \, dt \leq A \int_0^\infty \frac{dt}{t^{1-\delta}(\log 1/t)^{2+\epsilon}} < \infty.
\]

The other cases are similarly showed. Thus we see that our theorem is a generalization of Theorem A.

3.

We need the following lemma for the proof of our theorem.

**Lemma.** Under the same assumptions as those of the theorem, we have

\[
(i) \int_0^\infty \frac{\phi(t)p(1/t)\lambda(1/t)}{t^3P(1/t)} \, dt \leq A \int_0^\infty \frac{\phi(t)}{t^2} \, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du < \infty
\]

\[
(ii) \int_0^\infty \frac{\phi(t)}{t} \, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du \leq A \int_0^\infty \frac{\phi(t)}{t^2} \, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du < \infty, \text{ and}
\]

\[
(iii) \int_0^\infty \frac{\phi(t)\lambda(1/t)}{t} \, dt \leq A \int_0^\infty \frac{\phi(t)\lambda(1/t)}{t^2} \, dt < \infty.
\]

**Proof.** Let \( \epsilon > 0 \). Then we obtain by an integration by parts

\[
\int_\epsilon^\infty \frac{\phi(t)p(1/t)\lambda(1/t)}{t^3P(1/t)} \, dt = \frac{\phi(\pi)}{\pi} \int_0^\infty \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du - \frac{\phi(\epsilon)}{\epsilon} \int_0^\epsilon \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du
\]

\[
- \int_\epsilon^\infty \frac{\phi(t)}{t} \, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du + \int_\epsilon^\infty \frac{\phi(t)}{t^2} \, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du.
\]

Thus we have
Therefore we can obtain cases (i) and (ii) as \( \varepsilon \to 0 \).

Noting that \( \frac{d\lambda(t)}{dt} < 0 \), case (iii) is similarly proved.

4.

Proof of the theorem. Let \( \tilde{\tau}_n \) denote the Riesz means of the series \( \sum \lambda_n a_n \). Then we have, by (1.1),

\[
\tilde{\tau}_n - \tilde{\tau}_{n-1} = \frac{p_n}{P_n} \sum_{k=1}^{n-1} P_{k-1} \lambda_k a_k - \frac{p_n}{P_n} \sum_{k=1}^{n-2} (P_{k-1} \lambda_k - P_k \lambda_{k+1}) s_{k-1} - \frac{p_n}{P_n} \lambda_0 s_0.
\]

Let \( \tilde{\tau}_n(x) \) denote the Riesz means of the series \( \sum \lambda_n A_n(x) \). Then we obtain, by (4.1),

\[
\frac{\pi}{2} \sum_{n=1}^{\infty} \left| \tilde{\tau}_n(x) - \tilde{\tau}_{n-1}(x) \right| \leq \sum_{n=1}^{\infty} \frac{p_n}{P_n} \frac{P_{n-2} \lambda_{n-1}}{P_{n-1}} \int_0^{\pi} |\phi(t)| |D_{n-1}(t)| dt
\]

\[
+ \sum_{n=1}^{\infty} \frac{p_n}{P_n} \sum_{k=1}^{n-2} \left| P_{k-1} \lambda_k - P_k \lambda_{k+1} \right| \int_0^{\pi} |\phi(t)| |D_k(t)| dt
\]

\[
= I + J,
\]

say, where \( s_k(x) = \frac{2}{\pi} \int_0^\pi \phi(t) D_k(t) dt \).

Since \( D_n(t) = O(n) \) or \( O(1/t) \), we have

\[
I \leq A \sum_{n=1}^{\infty} \frac{np_n \lambda_{n-1}}{P_n} \int_0^{\pi/n} |\phi(t)| dt + A \sum_{n=1}^{\infty} \frac{p_n \lambda_{n-1}}{P_n} \int_0^{\pi/n} \left| \frac{\phi(t)}{t} \right| dt
\]

\[
= I_1 + I_2,
\]
say. Since \(\{\lambda_n\}\) and \(\{p_n/p_n\}\) are decreasing, we have, by the lemma,

\[
I_1 = A \sum_{n=1}^{\infty} \frac{np_n \lambda_{n-1}}{p_n} \sum_{k=n}^{\infty} \int_{\pi/k+1}^{\pi/k} |\phi(t)| \, dt
\]

\[
\leq A \sum_{k=1}^{\infty} \int_{\pi/k+1}^{\pi/k} |\phi(t)| \, dt \sum_{n=1}^{\infty} \frac{np_n \lambda_n}{p_n}
\]

\[
\leq A \sum_{k=1}^{\infty} \int_{\pi/k+1}^{\pi/k} |\phi(t)| \, dt \int_{t}^{\infty} \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du
\]

\[
= A \int_{0}^{\infty} |\phi(t)| \, dt \int_{t}^{\infty} \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du
\]

\[
= A \int_{0}^{\infty} \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du < \infty
\]

and

\[
I_2 = A \sum_{n=1}^{\infty} \frac{p_n \lambda_{n-1}}{p_n} \sum_{k=1}^{\infty} \int_{\pi/k+1}^{\pi/k} \frac{|\phi(t)|}{t} \, dt
\]

\[
\leq A \sum_{k=1}^{\infty} \int_{\pi/k+1}^{\pi/k} \frac{|\phi(t)|}{t} \, dt \sum_{n=k+1}^{\infty} \frac{p_n \lambda_{n-1}}{p_n}
\]

\[
\leq A \sum_{n=1}^{\infty} \int_{\pi/k+1}^{\pi/k} \frac{|\phi(t)|}{t} \, dt \sum_{n=k+1}^{\infty} \frac{p(n/n)\lambda(n/n)}{P(n/n)} \text{ for } n > \pi/(n-1)
\]

\[
\leq A \int_{0}^{\infty} \frac{|\phi(t)|}{t} \, dt \int_{0}^{\infty} \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du
\]

\[
= A \int_{0}^{\infty} \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du
\]

\[
\leq A \int_{0}^{\infty} \frac{|\phi(t)|}{t^2} \, dt \int_{0}^{\infty} \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} \, du < \infty.
\]

Thus, by \(I_1\) and \(I_2\), we see that \(I\) is finite.

Since \(p_{k-1} \lambda_k - p_k \lambda_{k+1} = -p_k \lambda_k + p_k \Delta \lambda_k\), we have
\[
J \leq \sum_{n=1}^{\infty} \frac{p_n}{n} \frac{1}{n-1} \sum_{k=1}^{n-2} p_k \lambda_k \int_{0}^{\pi} |\phi(t)||D_k(t)| \, dt \\
+ \sum_{n=1}^{\infty} \frac{p_n}{n} \frac{1}{n-1} \sum_{k=1}^{n-2} p_k \lambda_k \int_{0}^{\pi} |\phi(t)||D_k(t)| \, dt \\
= J_1 + J_2 ,
\]
say. The finiteness of \( J_1 \) is proved by the same estimation as \( I \) because we obtain
\[
J_1 \leq \sum_{k=1}^{\infty} \frac{p_k \lambda_k}{p_k} \int_{0}^{\pi} |\phi(t)||D_k(t)| \, dt .
\]
Next, we have,
\[
J_2 \leq \sum_{n=1}^{\infty} \frac{p_n}{n} \frac{1}{n-1} \sum_{k=1}^{n-2} p_k \lambda_k \int_{0}^{\pi/k} |\phi(t)||D_k(t)| \, dt \\
+ \sum_{n=1}^{\infty} \frac{p_n}{n} \frac{1}{n-1} \sum_{k=1}^{n-2} p_k \lambda_k \int_{\pi/k}^{\pi} |\phi(t)||D_k(t)| \, dt \\
= J_{21} + J_{22} ,
\]
say. Since \( \{ \lambda_k \} \) is decreasing, we have, by the lemma,
\[
J_{21} \leq A \sum_{n=1}^{\infty} \frac{p_n}{n} \frac{1}{n-1} \sum_{k=1}^{n-2} k p_k \lambda_k \int_{\pi/n}^{\pi/n+j} |\phi(t)| \, dt \\
= A \sum_{k=1}^{\infty} k p_k \lambda_k \int_{\pi/(n+k+1)}^{\pi/n} |\phi(t)| \, dt \\
\leq A \sum_{k=1}^{\infty} k \lambda_k \int_{\pi/(n+k+1)}^{\pi/n} |\phi(t)| \, dt \\
= A \sum_{j=1}^{\infty} \int_{\pi/n+j}^{\pi/n+j+1} |\phi(t)| \, dt \int_{k=1}^{j} \lambda_k \\
\leq A \sum_{j=1}^{\infty} \int_{\pi/n+j+1}^{\pi/n+j} |\phi(t)| \, dt \int_{t=1}^{\infty} \frac{\lambda(1/u)}{u^2} \, du \\
= A \int_{0}^{\pi} |\phi(t)| \, dt \int_{t=1}^{\infty} \frac{\lambda(1/u)}{u^2} \, du \\
= A \int_{0}^{\infty} \frac{\lambda(1/u)}{u^2} \, \phi(u) \, du < \infty
\]
Thus, by $J_1$ and $J_2$ we see that $J$ is finite. Therefore, by the above estimations, our theorem is completely proved.

5.

In this section we consider some applications of our theorem.

COROLLARY. Let $\{p_n\}$ be a positive sequence such that $\{p_n/n\}$ is decreasing. Suppose that $\chi(t)$ is a positive decreasing function satisfying the condition

\[(5.1)\quad p(t)^{-1}P(t)\chi(t) = O(1) \text{ for } t > 0 \]

and $\psi(t)$ is a positive decreasing function such that

\[(5.2)\quad \sum_{n=1}^{\infty} \frac{\alpha_n \psi_n}{\gamma_n} < \infty,\]

where $\{\alpha_n\}$ is a sequence defined by
\[ \alpha_n = \int_{1/n+1}^{t} \frac{x(1/t)}{t^2} \, dt. \]

If \( \Phi(t) = O(\chi(1/t)) \), then

\[ (5.3) \quad \sum_{n=1}^{\infty} \frac{P_n \mu_n}{n^p} A_n(t) \]

is summable \( |R, P_n, 1| \), at \( t = x \).

Proof. Putting \( \lambda(t) = P(t)\mu(t)/tp(t) \) in the theorem, we have, by (5.2),

\[
\int_{0}^{\pi} \frac{\Phi(t)}{t^2} \, dt \int_{0}^{t} \frac{p(1/u)\lambda(1/u)}{u^2p(1/u)} \, du
\]

\[
= \int_{0}^{\pi} \frac{p(1/u)\lambda(1/u)}{u^2p(1/u)} \, du \int_{0}^{\pi} \frac{\Phi(t)}{t^2} \, dt
\]

\[
\leq A + \int_{0}^{1} \frac{p(1/u)\lambda(1/u)}{u^2p(1/u)} \, du \int_{0}^{\pi} \frac{\Phi(t)}{t^2} \, dt
\]

\[
\leq A + A \sum_{n=1}^{\infty} \int_{1/n+1}^{1/n} \frac{p(1/u)\lambda(1/u)}{u^2p(1/u)} \, du \int_{1/n+1}^{\pi} \frac{x(1/t)}{t^2} \, dt
\]

\[
\leq A + A \sum_{n=1}^{\infty} \alpha_n \int_{1/n+1}^{1/n} \frac{\mu(1/u)}{u} \, du
\]

\[
\leq A + A \sum_{n=1}^{\infty} \frac{\alpha_n \mu_n}{n} < \infty.
\]

Similarly we have, by (5.1),

\[
\int_{0}^{\pi} \frac{\Phi(t)\lambda(1/t)}{t^2} \, dt \leq A + \int_{0}^{1} \frac{\Phi(t)\lambda(1/t)}{t^2} \, dt
\]

\[
\leq A + A \sum_{n=1}^{\infty} \int_{1/n+1}^{1/n} \frac{x(1/t)P(1/t)\mu(1/t)}{tp(1/t)} \, dt
\]

\[
\leq A + A \sum_{n=1}^{\infty} \int_{1/n+1}^{1/n} \frac{\mu(1/t)}{t} \, dt
\]

\[
\leq A + A \sum_{n=1}^{\infty} \frac{\mu_n}{n} < \infty.
\]
by virtue of the facts that $\{a_n\}$ is increasing and $\sum_{n=1}^{\infty} a_n u_n/n$ converges. Hence we prove the corollary.

Here we shall make a list of the interesting examples of $P_n$, $\lambda_n$ and $\chi(t)$ in the corollary. If we put $\lambda_n = P_n u_n n p_{-n}$ and $u_n = 1/L_s^e(n)$, then we have the following list:

<table>
<thead>
<tr>
<th></th>
<th>$P_n$</th>
<th>$\lambda_n$</th>
<th>$\chi(1/t)$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>$\exp n^{\delta}$</td>
<td>$\frac{1}{n^r L_s^e(n)}$</td>
<td>$\frac{t}{L_s^0(1/t)}$</td>
<td>$0 &lt; \delta \leq 1$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$n^{\delta}$</td>
<td>$\frac{1}{L_s^e(n)}$</td>
<td>$\frac{t}{L_s^0(1/t)}$</td>
<td>$0 &lt; \delta$</td>
</tr>
<tr>
<td>(iii)</td>
<td>$\exp L_s(n)^{\delta}$</td>
<td>$\frac{L_s^0(n)}{L_s(n) L_s^e(n)}$</td>
<td>$\frac{t}{L_s^0(1/t)}$</td>
<td>$0 &lt; \delta$</td>
</tr>
<tr>
<td>(iv)</td>
<td>$L_s(n)^{\delta}$</td>
<td>$\frac{L_s^0(n)}{L_s(n)^{\delta} L_s^{e+p}(n)}$</td>
<td>$\frac{t}{L_s^{e+p}(1/t)}$</td>
<td>$0 &lt; \delta$</td>
</tr>
<tr>
<td>(v)</td>
<td>$\exp \frac{n}{L_s(n)^{\delta}}$</td>
<td>$\frac{L_s^e(n)}{n L_s^{e+p}(n)}$</td>
<td>$\frac{t}{L_s^{e+p}(1/t)}$</td>
<td>$0 &lt; \delta$</td>
</tr>
<tr>
<td>(vi)</td>
<td>$\frac{n}{L_s(n)^{\delta}}$</td>
<td>$\frac{1}{L_s^{e+p}(n)}$</td>
<td>$\frac{t}{L_s^{e+p}(1/t)}$</td>
<td>$0 &lt; \delta$</td>
</tr>
</tbody>
</table>

where $s$ is a positive integer and $p$ is a non-negative integer.

By Mohanty's lemma [7], we see that cases (i), (iii) and (v) are also deduced from cases (ii), (iv) and (vi), respectively. The positive number $\varepsilon$ in $L_s^e(n)$ or $L_s^{e+p}(n)$ is indispensable from the theorems due to Matsumoto [6] and Dikshit [7].
Summability factor of Fourier series

References


Department of Mathematics,
Faculty of Engineering,
Shinshu University,
Nagano 380,
Japan.