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Abstract

In this paper we prove that a smooth family of canonically polarized manifolds parametrized by a special (in the sense of Campana) quasi-projective variety is isotrivial.

1. Introduction

In 1962 Shafarevich conjectured that any smooth family of curves of genus $g \geqslant 2$ over non-hyperbolic algebraic curves, namely $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{P}^1$ and elliptic curve E, is isotrivial. More generally, it was conjectured that any smooth family of *canonically polarized manifolds* over these curves has no (algebraic) variation. This gives rise to a natural question: is there a large class of higher-dimensional bases over which every such family is isotrivial? Campana has introduced *special* varieties as higher-dimensional analogues of non-quasi-hyperbolic curves (the curves listed above) and conjectured that they serve as natural candidates for such bases.

Conjecture 1.1 (The isotriviality conjecture of Campana). Let Y° be a smooth quasi-projective variety parametrizing a smooth family of canonically polarized manifolds. If Y° is special (see the definition below), then the family is isotrivial.

Definition 1.2 (Special logarithmic pairs). Let (Y, D) be a pair consisting of a smooth projective variety Y and a simple normal-crossing reduced boundary divisor D. We call (Y, D) special if, for every invertible subsheaf $\mathscr{L} \subseteq \Omega^p_Y \log(D)$ and $p > 0$, we have $\kappa(\mathscr{L}) < p$. Moreover, we shall call a smooth quasi-projective variety Y° special if (Y, D) is special as a logarithmic pair, where Y is a smooth compactification with a simple normal-crossing (snc) boundary divisor D.

So, by definition, $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{P}^1$ and E is the list of all special quasi-projective curves. Other important examples of special varieties include rationally-connected varieties, varieties with zero Kodaira dimension [Cam04, Theorem 5.1] and those with nef anti-canonical divisor [Lu02, Theorem 11.1]. These examples, however, are very particular instances of special varieties and it is important to recall that in every dimension n, there are special quasi-projective manifolds of all possible log-Kodaira dimensions ($< n$).

Conjecture 1.1 is generalization of the following celebrated conjecture of Viehweg.

Conjecture 1.3 (Viehweg's hyperbolicity conjecture). Let $f^\circ : X^\circ \to Y^\circ$ be a smooth family of canonically polarized varieties over a quasi-projective variety Y°. Assume that Y is a smooth compactification of Y° with snc boundary divisor $D \cong Y \backslash Y^\circ$. If $\mathrm{Var}(f^\circ)$ is maximal, then (Y, D) is of log-general type (see [Vie83, Introduction] for the definition of $\mathrm{Var}(f^\circ)$).

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Conjecture 1.3 has recently been established in [CP15] by using, among many other things, an important generalization of Miyaoka's generic semi-positivity (see Theorem 3.1) and the following remarkable result of Viehweg and Zuo.

THEOREM 1.4 (Existence of pluri-logarithmic forms in the base; cf. [VZ02, Theorem 1.4]). *In the same notation as in Conjecture 1.3, if f° is not isotrivial, then for a positive integer $N \in \mathbb{N}^+$, there exists an invertible subsheaf $\mathscr{L} \subseteq \mathrm{Sym}^N(\Omega_Y \log(D))$ such that $\kappa(\mathscr{L}) \geqslant \mathrm{Var}(f^\circ)$.*

Clearly, Conjecture 1.3 in the case of $\dim Y^\circ = 1$ is an immediate corollary of Theorem 1.4. Conjecture 1.3 was already known in $\dim(Y^\circ) \leqslant 3$ by Kebekus and Kovács [KK10, Theorem 1.1]. The stronger conjecture of Campana (Conjecture 1.1) has also been established when $\dim(Y^\circ) \leqslant 3$, thanks to Kebekus and Jabbusch [JK11b, Theorem 1.5]. In § 5, and after following Campana and Păun's proof of Viehweg's conjecture very closely, we give a proof of Conjecture 1.1. The proof heavily depends on a recent generic semi-positivity result of Campana and Păun, existence of log-minimal models for Kawamata log terminal pairs with big boundary divisors established by [BCHM10, Theorem 1.1], and an important refinement of Theorem 1.4 given by [JK11a, Theorem 1.4].

THEOREM 1.5 (Isotriviality of smooth families of canonically polarized manifolds). *Conjecture 1.1 holds in all dimensions.*

Let $\mathfrak{M}$ be the quasi-projective scheme [Vie95] equipped with transformations

$$\Psi : \mathcal{M} \to \mathrm{Hom}(\cdot, \mathfrak{M}),$$

such that $\mathfrak{M}$ is the coarse moduli scheme of the moduli functor $\mathcal{M}$ of smooth families of canonically polarized manifolds. According to Campana's reduction theory, for every projective variety Y there exists an almost holomorphic map $C_Y : Y \dashrightarrow Z$, called the *core*, whose general fibre is special and contracts almost all special subvarieties of Y. As a result of Theorem 1.5 it follows that the moduli maps associated to smooth families of canonically polarized manifolds factors through the (logarithmic) core.

COROLLARY 1.6 (Factorization of the moduli map through the core). *Let Y° be a smooth quasi-projective variety admitting a morphism $\mu : Y^\circ \to \mathfrak{M}$, where $\mu = \Psi(\mathcal{M}(Y^\circ))$. Let $\widetilde{\mu}$ be the induced morphism between smooth compactifications Y, $\overline{\mathfrak{M}}$ of Y and $\mathfrak{M}$, respectively. Then $\widetilde{\mu}$ factors through the core $C_{(Y,D)} : (Y, D) \dashrightarrow Z$ associated to a smooth compactification (Y, D) of Y°.*

Notice that Corollary 1.6 immediately implies that Viehweg's hyperbolicity conjecture (already settled in [CP15]) holds: let $f^\circ : X^\circ \to Y^\circ$ and (Y, D) be as in the set-up of Conjecture 1.3. If (Y, D) is not of log-general type, then $C_Y : Y \dashrightarrow Z$ has positive-dimensional general fibres. On the other hand, by Corollary 1.6, the moduli map $\mu : Y \to \overline{\mathfrak{M}}$ factors through C_Y. But by the assumption μ is generically finite, a contradiction.

The proof of Theorem 1.5 essentially consists of the following two steps. First we use Viehweg and Zuo's factorization result (Theorem 1.4), together with its refinement by [JK11a], to reduce the problem to the following (see Theorem 4.3 for details): given a smooth pair (X, D), existence of an invertible subsheaf $\mathscr{L} \subseteq (\Omega_X \log(D))^{\otimes_{\mathcal{C}} N}$ (see Definition 2.11), for some $N \in \mathbb{N}^+$, with maximal $\mathcal{C}$-Kodaira dimension (this is defined in Definition 2.12) implies that (X, D) is of log-general type. The second step (§ 5) is to prove this statement using the positivity result of [CP15] and results of [BCHM10] (Theorem 5.2).

2. Preliminaries

To approach the isotriviality conjecture (Conjecture 1.1), it is essential to work with pairs (or the orbifold pairs in the sense of Campana) instead of just logarithmic ones. We refer the reader to [Cam11, JK11b] for an in-depth discussion of the definitions and background. In the present section we give a brief overview of the key ingredients of this theory to the extent that is necessary for our arguments in the rest of the paper.

Definition 2.1 (Smooth pairs). Let X be an n-dimensional normal (quasi-)projective variety and $D = \sum d_i D_i$, where $d_i \in \mathbb{Q} \cap [0, 1]$, a $\mathbb{Q}$-Weil divisor in X. We shall call the pair (X, D) a *smooth pair* if X is smooth and $\operatorname{supp}(D)$ is snc.

Definition 2.2 ($\mathcal{C}$-multiplicity). Let (X, D) be a smooth pair as in Definition 2.1. When $d_i \neq 1$, let a_i and b_i be the positive integers for which the equality $1 - b_i/a_i = d_i$ holds. For every i, we define the $\mathcal{C}$-multiplicity of the irreducible component D_i of D by

$$m_D(D_i) := \begin{cases} \dfrac{1}{1-d_i} = \dfrac{a_i}{b_i} & \text{if } d_i \neq 1, \\ \infty & \text{if } d_i = 1. \end{cases}$$

A classical result of Kawamata (see [Laz04, Proposition 4.1.12]) proves that, given a collection of smooth prime divisors $\{D_1, \ldots, D_l\}$ and positive integers $\{c_1, \ldots, c_l\}$, one can always construct a *smooth* variety Y together with a finite, flat morphism $\gamma : Y \to X$ such that

$$\gamma^*(D_i) = c_i \sum D_{ij},$$

where $\sum D_{ij}$ is an snc divisor in Y. In particular, given a smooth pair (X, D), we may take the coefficients c_i to be equal to a_i (a_i being the numerator of $m_D(D_i)$, as in Definition 2.2), so that the resulting Kawamata cover $\gamma : Y \to X$ is, in a sense, *adapted* to the structure of the pair (X, D).

Definition 2.3 (Adapted covers). Let (X, D) be a smooth pair, Y a smooth variety, and $\gamma : Y \to X$ a finite, flat, Galois cover with Galois group G such that if $m_D(D_i) = a_i/b_i < \infty$, then every prime divisor in Y that appears in $\gamma^*(D_i)$ has multiplicity exactly equal to a_i. We call γ an adapted cover for the pair (X, D) if it additionally satisfies the following properties.

(2.3.1) The branch locus is given by

$$\operatorname{supp}\Big(H + \bigcup_{m_D(D_i) \neq \infty} D_i\Big),$$

where H is a general member of a linear system $|L|$ of a very ample divisor L in X.

(2.3.2) γ is totally branched over H.

(2.3.3) γ is not branched at the general point of $\operatorname{supp}(\lfloor D \rfloor)$.

Notation 2.4. Let $\gamma : Y \to X$ be an adapted cover of a smooth pair (X, D), where $D = \sum d_i D_i$, $d_i = 1 - b_i/a_i$ as in Definition 2.2. For every prime component D_i of D with $m_D(D_i) \neq \infty$, let $\{D_{ij}\}_{j(i)}$ be the collection of prime divisors that appear in $\gamma^{-1}(D_i)$. We define new divisors in Y by:

(2.4.1) $D_Y^{i,j} := b_i D_{ij}$, $m_D(D_i) \neq \infty$;
(2.4.2) $D_\gamma := \gamma^*(\lfloor D \rfloor)$.

DEFINITION 2.5 ($\mathcal{C}$-cotangent sheaf). Given a smooth pair (X, D) with an adapted cover $\gamma : Y \to X$, define the $\mathcal{C}$-cotangent sheaf Ω_{Y^∂} to be the unique maximal locally free subsheaf of $\Omega_Y \log(D_\gamma)$ for which the sequence

$$0 \longrightarrow \Omega_{Y^\partial}|_{(Y\setminus D_\gamma)} \longrightarrow \gamma^*(\Omega_X \log(\ulcorner D \urcorner))|_{(Y\setminus D_\gamma)} \xrightarrow{\rho} \bigoplus_{i,j(i)} \mathscr{O}_{D_Y^{i,j}} \longrightarrow 0,$$

induced by the natural residue map, is exact.

Remark 2.6. The $\mathcal{C}$-cotangent sheaf of Definition 2.5 coincides with Campana and Păun's notion [CP15, § 1.1] of the coherent sheaf on Y which they denote by $\gamma^*\Omega^1(X, D)$. It is also identical to the sheaf defined in [Lu02, Lemma 4.2]. See also [JK11b, Definition 2.13] for an equivalent definition in the classical setting, when the $\mathcal{C}$-multiplicities are all integral.

Remark 2.7 (Determinant of $\mathcal{C}$-cotangent sheaf). Given a smooth pair (X, D), let $\gamma : Y \to X$ be an adapted cover of degree d. There exists a natural isomorphism between the two invertible sheaves $\det(\Omega_{Y^\partial})$ and $\mathscr{O}_Y(\gamma^*(K_X + D))$,

$$\det(\Omega_{Y^\partial}) \cong \mathscr{O}_Y(\gamma^*(K_X + D)). \tag{2.7.1}$$

This follows from the ramification formula for the adapted cover γ.

DEFINITION 2.8 (Symmetric $\mathcal{C}$-differential forms, cf. [Cam11, §§ 2.6–7]). Let (X, D) be a smooth pair, $D = \sum d_i D_i$, and V_x an open neighbourhood of a given point $x \in X$ equipped with a coordinate system $z_1, \ldots, z_n$ such that $\mathrm{supp}(D) \cap V_x = \{z_1 \cdot \cdots \cdot z_l = 0\}$, for a positive integer $1 \leqslant l \leqslant n$. For every $N \in \mathbb{N}^+$, define the sheaf of symmetric $\mathcal{C}$-differential forms $\mathrm{Sym}^N_{\mathcal{C}}(\Omega_X \log(D))$ by the locally free subsheaf of $\mathrm{Sym}^N(\Omega_X \log(\ulcorner D \urcorner))$ that is locally generated, as an $\mathscr{O}_{V_x}$-module, by the elements

$$\frac{dz_1^{k_1}}{z_1^{\lfloor d_1 \cdot k_1 \rfloor}} \cdot \cdots \cdot \frac{dz_l^{k_l}}{z_l^{\lfloor d_l \cdot k_l \rfloor}} \cdot dz_{l+1}^{k_{l+1}} \cdot \cdots \cdot dz_n^{k_n},$$

where $\sum k_i = N$.

Remark 2.9 (An equivalent definition). There is an alternative definition for the sheaf of $\mathcal{C}$-differential forms. Let V_x be an open neighbourhood of $x \in X$ as in Definition 2.8 and take $\gamma : W \to V_x$ to be an adapted cover for $(V_x, D|_{V_x})$. Let $\sigma \in \Gamma(V_x, \mathrm{Sym}^N(\Omega_X(*\ulcorner D \urcorner)))$, that is, σ is a local rational section of $\mathrm{Sym}^N(\Omega_X)$ with poles along $\ulcorner D \urcorner$. Then

$$\sigma \in \Gamma(V_x, \mathrm{Sym}^N_{\mathcal{C}}(\Omega_X \log(D))) \iff \gamma^*(\sigma) \in \Gamma(W, \mathrm{Sym}^N(\Omega_{W^\partial})), \tag{2.9.1}$$

so that, in particular, $\gamma^*(\sigma)$ has at worst logarithmic poles *only* along those prime divisors in W that dominate $(\lfloor D \rfloor \cap V_x)$, and is regular otherwise.

Explanation 2.10. Assume that $\sigma \in \Gamma(V_x, \mathrm{Sym}^N_{\mathcal{C}}(\Omega_X \log(D)))$ is a local $\mathcal{C}$-differential form in the sense of (2.9.1). By the classical result of Iitaka [Iit82, ch. 11], it follows that $\sigma \in \Gamma(V_x, \mathrm{Sym}^N(\Omega_X \log(\ulcorner D \urcorner)))$. In particular, we find that along the reduced component of D the equivalence between the two definitions trivially holds. So assume, without loss of generality, that $m_D(D_i) \neq \infty$, for all irreducible components D_i of D. Furthermore, let us assume, for simplicity, that

$$\sigma = f \cdot \frac{dz_1^{k_1}}{z_1^{e_1}} \cdot \dots \cdot \frac{dz_l^{k_l}}{z_l^{e_l}} \cdot dz_{l+1}^{k_l+1} \cdot \dots \cdot dz_n^{k_n} \in \Gamma(V_x, \mathrm{Sym}^N(\Omega_X \log(\ulcorner D \urcorner))),$$

where $f \in \mathscr{O}_{V(x)}$, with no zeros along the D_i, is the local explicit description of σ. Since $\gamma^*(\sigma) \in \mathrm{Sym}^N(\Omega_{W^\partial})$, the inequality

$$k_i \cdot (a_i - 1) - a_i \cdot e_i \geqslant k_i(b_i - 1)$$

holds for $1 \leqslant i \leqslant l$, where $d_i = 1 - (b_i/a_i)$, that is,

$$e_i \leqslant k_i.d_i, \quad \text{for all } 1 \leqslant i \leqslant l.$$

In particular, σ is a symmetric $\mathcal{C}$-differential form on V_x in the sense of Definition 2.8.

Remark 2.11 (Tensorial $\mathcal{C}$-differential forms). Similarly to Definition 2.8 and (2.9.1), we can define the sheaf of tensorial $\mathcal{C}$-differential forms $(\Omega_X \log(D))^{\otimes_{\mathcal{C}} N}$ as the maximal subsheaf of $(\Omega_X \log(\ulcorner D \urcorner))^{\otimes N}$ such that

$$\gamma^*((\Omega_X \log(D))^{\otimes_{\mathcal{C}} N}) \subseteq (\Omega_{Y^\partial})^{\otimes N}.$$

Using the notation in Remark 2.9, pluri-$\mathcal{C}$-differential forms are locally defined as follows:

$$\sigma \in \Gamma(V_x, (\Omega_X \log(D))^{\otimes_{\mathcal{C}} N}) \iff \gamma^*(\sigma) \in \Gamma(W, (\Omega_{W^\partial}^{\otimes N})), \tag{2.11.1}$$

As we shall see in § 4, the Viehweg–Zuo subsheaves generically come from the coarse moduli space, as long as we extend the sheaf of symmetric differential forms to that of $\mathcal{C}$-differential forms associated to the naturally imposed $\mathcal{C}$-structures or orbifold structures (see Definition 2.14 below or [Cam11, § 3]) that appear over the moduli variety. But, as the usual Kodaira dimension of subsheaves of symmetric $\mathcal{C}$-differential forms is not sensitive to the fractional positivity of the non-reduced components of the boundary divisor (see Remark 2.13 below), a new birational notion is needed to measure the positivity of the Viehweg–Zuo subsheaves in the moduli.

Definition 2.12 ($\mathcal{C}$-Kodaira dimension; cf. [Cam11, § 2.7]). Let (X, D) be a smooth pair and $\mathscr{L} \subseteq (\Omega_X \log(D))^{\otimes_{\mathcal{C}} r}$ a saturated coherent subsheaf of rank one. Define the $\mathcal{C}$-product $\mathscr{L}^{\otimes_{\mathcal{C}} m}$ of $\mathscr{L}$, to the order of m, to be the saturation of the image of $\mathscr{L}^{\otimes m}$ inside $(\Omega_X \log(D))^{\otimes_{\mathcal{C}} (m.r)}$ and define the $\mathcal{C}$-Kodaira dimension of $\mathscr{L}$ by

$$\kappa_{\mathcal{C}}(X, \mathscr{L}) := \max \left\{ k \;\middle|\; \limsup_{m \to \infty} \frac{h^0(X, \mathscr{L}^{\otimes_{\mathcal{C}} m})}{m^k} \neq 0 \right\}.$$

If $h^0(X, \mathscr{L}^{\otimes_{\mathcal{C}} m}) = 0$ for all $m \in \mathbb{N}^+$, then, by convention, we define $\kappa_{\mathcal{C}}(X, \mathscr{L}) = -\infty$.

Remark 2.13 (Comparing Kodaira dimensions). When $D = 0$ or when D is reduced the sheaf of pluri-$\mathcal{C}$-differential forms $(\Omega_X \log(D))^{\otimes_{\mathcal{C}} r}$ is equal to $(\Omega_X)^{\otimes r}$ and $(\Omega_X \log(D))^{\otimes r}$, respectively, so that the $\mathcal{C}$-Kodaira dimension $\kappa_{\mathcal{C}}(X, \mathscr{L})$ of a rank-one coherent subsheaf $\mathscr{L}$ of $(\Omega_X \log(D))^{\otimes_{\mathcal{C}} r}$ coincides with the usual Kodaira dimension $\kappa(X, \mathscr{L})$ of $\mathscr{L}$.

Table 1. Notation.

Ω_{Y^∂}	$\mathcal{C}$-cotangent sheaf (Definition 2.5)
$\mathrm{Sym}^N_{\mathcal{C}}(\Omega_X \log(D))$	Symmetric $\mathcal{C}$-differential forms (Definition 2.8)
$(\Omega_X \log(D))^{\otimes_{\mathcal{C}} N}$	Tensorial $\mathcal{C}$-differential forms (Remark 2.11)
$\mathscr{L}^{\otimes_{\mathcal{C}} N}$	$\mathcal{C}$-product (Definition 2.12)
$\kappa_{\mathcal{C}}(X, \mathscr{L})$	$\mathcal{C}$-Kodaira dimension (Definition 2.12)

Let (Y, D) be a smooth pair, Z a smooth variety, and $f : Y \to Z$ a fibration with connected fibres. Assume that every f-exceptional prime divisor F, that is, $\mathrm{codim}_Z\big(f(F)\big) \geqslant 2$, is a reduced component of D. Then simple local calculations show that there exists a maximal (in the sense of multiplicities of the irreducible components) divisorial structure Δ on Z, whose support coincides with the codimension-one closed subset of the log-discriminant locus B of $f : (Y, D) \to Z$ and that the natural pull-back map

$$(df)^m : f^*(\mathrm{Sym}^m_{\mathcal{C}}(\Omega_Z \log(\Delta))) \to \mathrm{Sym}^m_{\mathcal{C}}(\Omega_Y \log(D))$$

is well defined. Recall that the log-discriminant locus B is the smallest closed subset of Z such that f is smooth over its complement, and that for every point $z \in Z \backslash \Delta$, the set-theoretic fibre $f^{-1}(z)$ is not contained in D, and that the scheme-theoretic intersection of the fibre Y_z with D is an snc divisor in Y_z. We call Δ the $\mathcal{C}$-*base* (or the orbifold base) of the fibration $f : (Y, D) \to Z$.

Definition 2.14 ($\mathcal{C}$-base of a fibration). Given a smooth pair (Y, D), let $f : Y \to Z$ be a fibration with connected fibres onto a smooth variety Z. Let $\{\Delta_i\}_i$ be the set of the irreducible components of the divisorial part of the log-discriminant locus of f. For every i, define $\{\Delta_{ij}\}_j$ to be the collection of prime divisors in $f^{-1}(\Delta_i)$ that are *not* f-exceptional. To each divisor Δ_i, assign a positive rational number $m_\Delta(\Delta_i)$ defined by

$$m_\Delta(\Delta_i) := \min_j\{d_j \cdot m_\Delta(\Delta_{ij})\},$$

d_j being the positive integer satisfying the equality

$$f^*(\Delta_i) = \sum_j d_j \Delta_{ij} + E.$$

We define the $\mathcal{C}$-base of the fibration $f : (Y, D) \to Z$ by the divisor

$$\Delta := \sum_i \left(1 - \frac{1}{m_\Delta(\Delta_i)}\right) \Delta_i.$$

Table 1 gathers together the notation we have introduced in this section.

3. The orbifold generic semi-positivity

Miyaoka's generic semi-positivity result (see [Miy87a, Miy87b]) establishes a correspondence between abundance of rational curves (uniruledness) on a smooth projective variety and generic (semi-)positivity of the cotangent sheaf Ω_X. On the other hand, the results of [BDPP13] prove that uniruledness of X is characterized by the pseudo-effectivity of K_X. This suggests (as was

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originally formulated by Campana) that a generalization of Miyaoka's result in the logarithmic context should read as follows: pseudo-effectivity of $K_X + D$ implies the generic semi-positivity of $\Omega_X \log(D)$. But the positivity result of Miyaoka was achieved via certain characteristic p-arguments which cannot be adapted to the context of pairs. Nevertheless, in [CP15], Campana and Păun overcome this obstacle by using the Bogomolov–McQuillan criterion for the algebraicity of foliations induced by positive subsheaves of the tangent sheaf.

Theorem 3.1 (Generic semi-positivity of $\mathcal{C}$-cotangent sheaf [CP15, Theorem 2.1]). *Let (X, D) be a smooth pair with an adapted cover $\gamma : Y \to X$, whose Galois group we denote by G. If $K_X + D$ is pseudo-effective, then every torsion-free, coherent, $\mathscr{O}_Y$-module quotient $\mathscr{F}$ of $(\Omega_{Y^\partial})^{\otimes N}$ satisfies the inequality*

$$\mathrm{c}_1(\mathscr{F}) \cdot \gamma^*(H_1) \cdot \dots \cdot \gamma^*(H_{n-1}) \geqslant 0, \tag{3.1.1}$$

for all $(n-1)$-tuples of ample divisors $(H_1, \dots, H_{n-1})$ in X.

As an immediate corollary we get an inequality involving the intersection of $K_X + D$ and any invertible subsheaf $\mathscr{L} \subseteq (\Omega^1_X \log(D))^{\otimes_{\mathcal{C}} N}$ with nef divisors.

Corollary 3.2. *Let (X, D) be a smooth pair of dimension n. Let $\mathscr{L} \subseteq (\Omega^1_X \log(D))^{\otimes_{\mathcal{C}} N}$ be an invertible subsheaf and L a divisor in X such that $\mathscr{O}_X(L) \cong \mathscr{L}$. If $K_X + D$ is pseudo-effective, then for every collection of $(n-1)$ $\mathbb{Q}$-Cartier nef divisors $P_1, \dots, P_{n-1}$ the following inequality holds:*

$$(N \cdot (n^N)^{N-1} \cdot (K_X + D) - L) \cdot P_1 \cdot \dots \cdot P_{n-1} \geqslant 0.$$

4. Viehweg–Zuo subsheaves in the parametrizing space

The result of Jabbusch and Kebekus [JK11b] shows that the $\mathcal{C}$-differential forms are the correct framework to study the positivity of subsheaves of forms in the coarse moduli space of canonically polarized manifolds. In this section we give a brief explanation of how one can then reduce the isotriviality conjecture (Conjecture 1.1) to the problem of showing that existence of rank-one subsheaves of the sheaf pluri-$\mathcal{C}$-differential forms, attached to a smooth pair, with maximal $\mathcal{C}$-Kodaira dimension implies that the given pair is of log-general type (see Theorem 4.3 below). To prepare the correct setting for this reduction, we introduce a notion that, as far as the author is aware, is originally due to Campana (see [Cam11, § 1.1]).

Definition 4.1 (Neat model of a pair). Let (Y, D) be a normal logarithmic pair (Y is normal and the Weil divisor D is reduced) and $h : Y \to Z$ a fibration with connected fibres onto an algebraic base Z. We call a smooth pair (Y_h, D_h) a neat model for (Y, D) and h if there exists a fibration $\widetilde{h} : Y_h \to Z_h$ that is birationally equivalent to h, that is, there are birational morphisms $\mu : Y_h \to Y$ and $\alpha : Z_h \to Z$ such that the diagram

$$\begin{tikzcd} Y \arrow[d, "h"'] & Y_h \arrow[l, "\mu"'] \arrow[d, "\widetilde{h}"] \\ Z & Z_h \arrow[l, "\alpha"'] \end{tikzcd}$$

commutes, for which the following conditions are satisfied.

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(4.1.1) D_h is the extension of the μ-birational transform $\widetilde{D}$ of D by some reduced μ-exceptional divisor, i.e. $D_h = \widetilde{D} + E'$, where E' is μ-exceptional.

(4.1.2) (Z_h, Δ_h) is a smooth pair, Δ_h being the $\mathcal{C}$-base (see Definition 2.14) of the fibration $\widetilde{h} : (Y_h, D_h) \to Z_h$.

(4.1.3) Every $\widetilde{h}$-exceptional prime divisor P in Y_h (P satisfies the inequality $\operatorname{codim}_{Z_h}(\widetilde{h}(P)) \geqslant 2$) is contained in $\operatorname{supp}(D_h)$.

The interest in the neat models of pairs (that are equipped with fibrations), is twofold. First, conditions (4.1.2) and (4.1.3) ensure that $(\widetilde{h})^*$ defines a well-defined pull-back map from symmetric $\mathcal{C}$-differential forms $(\Omega_{Z_h} \log(\Delta_h))^{\otimes_{\mathcal{C}} N}$ attached to (Z_h, Δ_h) to the sheaf of tensorial logarithmic forms $(\Omega_{Y_h} \log(D_h))^{\otimes N}$ (see the discussion before the Definition 2.14). Secondly, according to property (4.1.1), the neat model (Y_h, D_h) inherits the birational properties of the original pair (Y, D). For example if (Y, D) special, then so is (Y_h, D_h). These attributes will be crucial to the proof of the main result (Theorem 4.3) of this section.

PROPOSITION 4.2 (Construction of neat models; cf. [JK11b, § 10]). *Every normal logarithmic pair (Y, D) and a surjective morphism with connected fibres $h : Y \to Z$, where Z is a projective variety, admits a neat model.*

Proof. Let $\alpha_1 : Z_1 \to Z$ be a suitable modification of the base of the fibration h such that the normalization of the induced fibre product $Y \times_Z Z_1$, which we denote by Y_1, gives rise to an equidimensional fibration $h_1 : Y_1 \to Z_1$, that is, a *flattening* of h, and a birational map $\mu_1 : Y_1 \to Y$ (see the diagram below). Define D_1 to be the maximal reduced divisor contained in the $\operatorname{supp}(\mu_1^{-1} D)$ and let

$$D_1 = D_1^{\text{ver}} + D_1^{\text{hor}}$$

be the decomposition of D_1 into sum of its vertical D_1^{ver} and horizontal D_1^{hor} components. Introduce a closed subset in Z_1 by $D_{Z_1} := h_1(D_1^{\text{ver}})$. Let $\Delta_1 \subset Z_1$ denote the log-discriminant locus defined by the fibration h_1 and the divisor D_1. Now, let $\alpha_2 : Z_h \to Z_1$ be a desingularization of Z_1 such that the maximal reduced divisor in the $\operatorname{supp}(\alpha_2^{-1} \Delta_1 \cup \alpha_2^{-1} D_{Z_1})$ is snc. Set Y_2 to be the normalization of the fibre product $Y_1 \times_{Z_1} Z_2$, and μ_2 the naturally induced birational morphism. Define D_2 in Y_2 by the maximal reduced divisor contained in the $\operatorname{supp}(\mu_2^{-1} D_1)$. Finally, let $\mu_3 : Y_h \to Y_2$ be a log-resolution of (Y_2, D_2) and take $\widetilde{h} : Y_h \to Z_h$ to be the induced fibration.

$$\begin{tikzcd}
Y & Y_1 \arrow[l, "\mu_1"'] \arrow[d, "h_1"] & Y_2 \arrow[l, "\mu_2"'] \arrow[d, "h_2"] & Y_h \arrow[l, "\mu_3"'] \arrow[dl, "\widetilde{h}"] \\
Z \arrow[from=u, "h"] & Z_1 \arrow[l, "\alpha_1"'] & Z_h \arrow[l, "\alpha_2"'] &
\end{tikzcd}$$

Now set $\widetilde{D}_2$ to be the maximal reduced divisor in $\operatorname{supp}(\mu_3^{-1})$. Note that h_1 remains equidimensional under the base change of α_2, that is, h_2 is also equidimensional. This implies that when we desingularize Y_2 by μ_3, every $\widetilde{h}$-exceptional divisor is μ_3-exceptional. Let E_3 be the sum of all $\widetilde{h}$-exceptional prime divisors in Y_h and define $D_h := \widetilde{D}_2 + E_3$ to be the extension of $\widetilde{D}_2$ by E_3. We finish by defining the birational morphisms μ and α in Definition 4.1 by $\mu_3 \circ \mu_2 \circ \mu_1$ and $\alpha_2 \circ \alpha_1$, respectively. Now, by construction, the $\mathcal{C}$-structure Δ_h on Z_h induced by D_h and $\widetilde{h}$ defines a smooth pair (Z_h, Δ_h), as required. □

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Theorem 4.3 (Reduction of the isotriviality conjecture). *Conjecture 1.1 holds if the following assertion is true.*

(4.3.1) *Let (T, B) be a smooth pair. If $(\Omega_T \log(B))^{\otimes_{\mathcal{C}} N}$ admits an invertible subsheaf $\mathscr{L}$ with $\kappa_{\mathcal{C}}(T, \mathscr{L}) = \dim T$, then (T, B) is of log-general type.*

Proof. Let $f^\circ : X^\circ \to Y^\circ$ be a smooth family of canonically polarized manifolds, where Y° is a special quasi-projective variety. We assume that $\dim(Y^\circ) > 0$ (otherwise there is nothing to prove). Let Y be a smooth compactification with boundary divisor D such that $D \cong Y \backslash Y^\circ$ and that the induced map $\widetilde{\mu} : Y \to \overline{\mathfrak{M}}$ to a compactification of $\mathfrak{M}$ is a morphism. Aiming for a contradiction, assume that the family $f^\circ : X^\circ \to Y^\circ$ is *not* isotrivial, that is, $\dim(\mathrm{Im}(\widetilde{\mu})) > 0$. Now if $\widetilde{\mu}$ is generically finite, then, thanks to Campana and Păun's solution to Viehweg's conjecture (Conjecture 1.3), we find that $K_Y + D$ is big, contradicting the assumption that (Y, D) is special. Therefore to prove the theorem, we only need to treat the case where $\widetilde{\mu} : Y \to \overline{\mathfrak{M}}$ is *not* generically finite (so that $\widetilde{\mu}$ has positive-dimensional general fibres). In this case, by the Stein factorization, we can find a projective variety Z such that the morphism $\widetilde{\mu}$ factors through a fibration with connected fibres $h : Y \to Z$ and a finite morphism $Z \to \overline{\mathfrak{M}}$. According to Proposition 4.2, we can find a neat model (Y_h, D_h) of the pair (Y, D) and the fibration $h : Y \to Z$.

$$\begin{tikzcd}
 & Y \arrow[dl, "\widetilde{\mu}"'] \arrow[d, "h"] & Y_h \arrow[l, "\mu"'] \arrow[d, "\widetilde{h}"] \\
\overline{\mathfrak{M}} & Z \arrow[l, "\text{finite}"'] & Z_h \arrow[l, "\alpha"']
\end{tikzcd}$$

We observe that since $Y_h \backslash D_h$ is isomorphic to an open subset of Y°, it also parametrizes a smooth family of canonically polarized manifolds. Thus by [VZ02, Theorem 1.4], for some positive integer N, we can find a line subbundle $\mathscr{L} \subseteq (\Omega_{Y_h} \log(D_h))^{\otimes N}$ such that $\kappa(Y_h, \mathscr{L}) \geqslant \dim Z_h$. Moreover, by [JK11a, Theorem 1.4], we know that the Viehweg–Zuo subsheaf $\mathscr{L}$ generically comes from the coarse moduli space. More precisely, there exists an inclusion $\mathscr{L} \subseteq \mathscr{B}^{\otimes N}$, where $\mathscr{B}$ is the saturation of the image of

$$d\widetilde{h} : (\widetilde{h})^*(\Omega_{Z_h}) \to \Omega_{Y_h} \log(D_h).$$

Let us now collect the various properties of the pairs (Y_h, D_h) and (Z_h, Δ_h), and the fibration $\widetilde{h} : Y_h \to Z_h$ (recall that, by definition, the divisor Δ_h is the $\mathcal{C}$-base of the fibration $\widetilde{h} : (Y_h, D_h) \to Z_h$), that we have found so far.

- (Y_h, D_h) and (Z_h, Δ_h) are both smooth pairs (property (4.1.2)).
- D_h contains all $\widetilde{h}$-exceptional prime divisors (property (4.1.3)).
- There exists a saturated rank-one subsheaf $\mathscr{L} \subseteq \mathscr{B}^{\otimes N}$, for some positive integer N, such that $\kappa(Y_h, \mathscr{L}) \geqslant \dim Z_h$.

With these conditions, we can apply [JK11a, Corollary 5.8] to find a saturated rank-one subsheaf $\mathscr{L}_{Z_h} \subseteq (\Omega_{Z_h} \log(\Delta_h))^{\otimes_{\mathcal{C}} N}$ such that

$$\kappa_{\mathcal{C}}(Z_h, \mathscr{L}_{Z_h}) = \kappa(Y_h, \mathscr{L}) \geqslant \dim(Z_h). \tag{4.3.2}$$

Finally, if the statement (4.3.1) holds, then (Z_h, Δ_h) is of log-general type. On the other hand, by property (4.1.1), for every $1 \leqslant p \leqslant n$, we can push forward invertible subsheaves of $\Omega^p_{Y_h} \log(D_h)$ to those of $\Omega^p_Y \log(D)$. In particular, since (Y, D) is special, then so is (Y_h, D_h). But this is a contradiction to our previous finding that $K_{Z_h} + \Delta_h$ is big (recall that for a neat model $\widetilde{h} : Y_h \to Z_h$, and for sufficiently divisible positive integer m, we always have

$$h^0(Y_h, \mathscr{G}^{\otimes m}) = h^0(Z_h, \mathscr{O}_{Z_h}(K_{Z_h} + \Delta_h)^{\otimes m}), \tag{4.3.3}$$

where $\mathscr{G}$ denotes the saturation of the pull-back bundle $(\widetilde{h})^*(\mathscr{O}_{Z_h}(K_{Z_h}))$ inside $\Omega_{Y_h}^{\dim(Z_h)} \log(D_h)$). □

5. The isotriviality conjecture: the approach of Campana and Păun

In this section we prove statement (4.3.1) from the previous section. The isotriviality conjecture will then follow from Theorem 4.3. The proof is completely based on the solution of [CP15, § 4] to Viehweg's hyperbolicity conjecture (Conjecture 1.3). In particular, Theorem 5.2 should be taken as the generalization of [CP15, Theorem 4.1] from the category of purely logarithmic smooth pairs (the boundary divisor is reduced) to that of smooth pairs in general.

For ease of notation we have replaced the pair (T, B) in the reduction statement (4.3.1) by (X, D), with the warning that D should not be confused with the boundary divisor of the compactification of Y° that was introduced in the previous sections.

PROPOSITION 5.1. *Let (X, D) be a smooth pair of dimension n and $\mathscr{L} \subseteq (\Omega_X \log(D))^{\otimes cN}$ a saturated rank-one subsheaf with $\kappa_{\mathcal{C}}(X, \mathscr{L}) = \dim X$. For every ample divisor A in X, there exists a rational number $c = c(A, \mathscr{L}) \in \mathbb{Q}^+$, depending on A and $\mathscr{L}$, such that the inequality*

$$\mathrm{vol}(K_X + D + G) \geqslant c \cdot \mathrm{vol}(A) \tag{5.1.1}$$

holds for every $\mathbb{Q}$-Cartier divisor G satisfying the following properties.

(5.1.2) *$(D + G) \sim_{\mathbb{Q}} P$, for some big $\mathbb{Q}$-Cartier divisor P such that $\lfloor P \rfloor = 0$.*

(5.1.3) *$(X, D + G)$ and (X, P) are both smooth pairs.*

(5.1.4) *The $\mathbb{Q}$-Cartier divisor $(K_X + D + G)$ is pseudo-effective.*

Proof. First, let us fix an ample divisor A. We notice that, by an argument similar to that of Kodaira's lemma [Laz04, Proposition 2.2.6], we can always find a (sufficiently large) positive integer m such that

$$H^0(X, (\mathscr{L})^{\otimes cN} \otimes \mathscr{O}_X(-A)) \neq 0.$$

Let the invertible subsheaf $\mathscr{L}' \subseteq (\Omega_X \log(D))^{\otimes c(m.N)}$ denote the line bundle $\mathscr{L}^{\otimes cm}$, so that the inequality

$$A \leqslant L' \tag{5.1.5}$$

holds between Cartier divisors L' and A, L' being the divisor satisfying the isomorphism $\mathscr{O}_X(L') \cong \mathscr{L}'$. We shall prove the proposition in two steps. First, we run the log-minimal model program (LMMP) for the smooth pair (X, P). We notice that since P is big and has no reduced components (assumption (5.1.2)), according to [BCHM10, Theorem 1.1], after a finite number of divisorial contractions and log-flips, the program terminates in a log-minimal model (X', P'), that is, $K_{X'} + P'$ is nef. Here, at the minimal level, we shall find a lower-bound for $\mathrm{vol}(K_{X'} + P')$ in terms of $\mathrm{vol}(A)$ and *independent of* G. The second step of the proof is standard; we will just use the negativity lemma in the minimal model theory and replace $\mathrm{vol}(K_{X'} + P')$ by $\mathrm{vol}(K_X + P)$ to establish the required inequality (5.1.1).

Step 1: log-minimal model of (X, P) *and the volume of its log-canonical divisor.* Let $\pi : (X, P) \dashrightarrow (X', P')$ be the birational map defined by the LMMP. Take $\mu : \widetilde{X} \to X$ to be a modification of X resolving the indeterminacy of π, with resulting morphism $\widetilde{\pi} : \widetilde{X} \to X'$, and such that $\operatorname{supp}(\operatorname{Exc}(\mu) \cup \widetilde{D} \cup \widetilde{G})$, where $\widetilde{D}$, $\widetilde{G}$ are the μ-birational transforms of D and G, respectively, is snc in $\widetilde{X}$:

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "\gamma", "\text{adapted cover}"'] & \widetilde{X} \arrow[d, "\mu"'] \arrow[dr, "\widetilde{\pi}\text{, birational}"] & \\
& (X, P) \arrow[r, dotted, "\pi", "LMMP"'] & (X', P')
\end{tikzcd}$$

Let $\gamma : \widetilde{Y} \to \widetilde{X}$ be an adapted cover for the pair $(\widetilde{X}, \widetilde{D} + \widetilde{G} + E)$, where E is the maximal reduced divisor contained in $\operatorname{Exc}(\mu)$. We notice that, as $\mathscr{L}'$ is a subsheaf of $(\Omega_X \log(D))^{\otimes_{\mathcal{C}}(m.N)}$ $(\subseteq (\Omega_X \log(\ulcorner D \urcorner))^{\otimes(m.N)})$, the inclusion

$$\mu^*(\mathscr{L}') \subseteq (\Omega_{\widetilde{X}} \log(\widetilde{D} + \widetilde{G} + E))^{\otimes_{\mathcal{C}}(m.N)}$$

follows from the definition. Now, in order for us to use the generic semi-positivity result (Corollary 3.2), we need $K_{\widetilde{X}} + \widetilde{D} + \widetilde{G} + E$ to be pseudo-effective. This is indeed the case: from the ramification formula for μ we have $K_{\widetilde{X}} + \widetilde{D} + \widetilde{G} = \mu^*(K_X + D + G) + \widetilde{E}$, $\widetilde{E}$ being an effective exceptional divisor (the effectivity follows from our assumption that $(X, D + G)$ is a smooth pair (5.1.2)). So, from the pseudo-effectivity of $(K_X + D + G)$ (assumption (5.1.4)) it follows that $K_{\widetilde{X}} + \widetilde{D} + \widetilde{G}$ is pseudo-effective, and thus so is $K_{\widetilde{X}} + \widetilde{D} + \widetilde{G} + E$, as required. Therefore Corollary 3.2 applies and the inequality

$$\mu^*(L') \cdot P^{n-1} \leqslant u(K_{\widetilde{X}} + \widetilde{D} + \widetilde{G} + E) \cdot P^{n-1}$$

holds, where $u := (mN)(n^{mN})^{(mN-1)}$, for any nef divisor P in $\widetilde{X}$. In particular, for any fixed ample divisor H' in X' and positive integer r, we have

$$\mu^*(L') \cdot \widetilde{\pi}^* \left(K_{X'} + P' + \frac{1}{r} H' \right)^{n-1} \leqslant u.(K_{\widetilde{X}} + \widetilde{D} + \widetilde{G} + E) \cdot \widetilde{\pi}^* \left(K_{X'} + P' + \frac{1}{r} H' \right)^{n-1}. \tag{5.1.6}$$

Now let U be a Zariski open subset of X' such that $\operatorname{codim}_{X'}(X' \backslash U) \geqslant 2$ where $\pi^{-1}|_U$ and $\widetilde{\pi}^{-1}|_U$ are both isomorphisms. For every $r \in \mathbb{N}^+$, define d_r to be a sufficiently large positive integer such that the linear system $|d_r(K_{X'} + P' + (1/r)H')|$ is basepoint-free and that the irreducible curve $C_r := B_r^1 \cap \cdots \cap B_r^{n-1}$, cut out by general members $B_r^i \in |d_r(K_{X'} + P' + (1/r)H')|$, is a subset of U. We notice that as $C_r \subset U$, and because of our assumption (5.1.2), the right-hand side of inequality (5.1.6) is equal to $(1/d_r)^{n-1} u.(K_{X'} + P') \cdot (K_{X'} + P' + (1/r))^{n-1}$. Therefore, we may write inequality (5.1.6) as

$$(d_r)^{n-1} \mu^*(L') \cdot \widetilde{\pi}^* \left(K_{X'} + P' + \frac{1}{r} H' \right)^{n-1} \leqslant u. \left(K_{X'} + P' + \frac{1}{r} H' \right)^n,$$

so that

$$\mu^*(L') \cdot \widetilde{\pi}^* \left(K_{X'} + P' + \frac{1}{r} H' \right)^{n-1} \leqslant u. \operatorname{vol} \left(K_{X'} + P' + \frac{1}{r} H' \right). \tag{5.1.7}$$

Next, we notice that, as $L' - A \geqslant 0$ (inequality (5.1.5)), the pull-back $\mu^*(L' - A)$ is also effective. Therefore, and again by using the fact that the nef cone in the Néron–Severi

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space $\mathrm{N}^1(\widetilde{X})_{\mathbb{R}}$ is equal to the closure of the ample one, we have $\mu^*(L'-A)\cdot\widetilde{\pi}^*(K_{X'}+P'+(1/r)H')^{n-1}\geqslant 0$. Hence, we can rewrite inequality (5.1.7) as

$$\mu^*(A)\cdot\widetilde{\pi}^*\left(K_{X'}+P'+\frac{1}{r}H'\right)^{n-1}\leqslant u.\operatorname{vol}\left(K_{X'}+P'+\frac{1}{r}H'\right). \tag{5.1.8}$$

Now, by applying Teissier's inequality [Laz04, Theorem 1.6.1] (to the left-hand side of inequality (5.1.8)), we have

$$\operatorname{vol}(A)^{1/n}\cdot\operatorname{vol}\left(K_{X'}+P'+\frac{1}{r}H'\right)^{(n-1)/n}\leqslant u.\operatorname{vol}\left(K_{X'}+P'+\frac{1}{r}H'\right),$$

that is,

$$\operatorname{vol}(A)^{1/n}\leqslant u.\operatorname{vol}\left(K_{X'}+P'+\frac{1}{r}H'\right)^{1/n}. \tag{5.1.9}$$

Finally, thanks to the continuity of vol(.), by taking $r\to\infty$ in inequality (5.1.9) we have

$$\frac{1}{u^n}\cdot\operatorname{vol}(A)\leqslant\operatorname{vol}(K_{X'}+P'), \tag{5.1.10}$$

that is, inequality (5.1.1) holds for the log-minimal model (X',P') if we take $c:=1/u^n$.

Step 2: lower bound for the volume of K_X+P. By the negativity lemma in the minimal model theory, we know that $H^0(X,m(K_X+P))\cong H^0(X',m(K_{X'}+P'))$, for all $m\in\mathbb{N}^+$. In particular, the equality $\operatorname{vol}(X,K_X+P)=\operatorname{vol}(X',K_{X'}+P')$ holds. The required inequality (5.1.1) now follows from inequality (5.1.10) in the previous step and assumption (5.1.2). □

Theorem 5.2. *Let (X,D) be a smooth pair and $\mathscr{L}\subseteq(\Omega_X\log(D))^{\otimes cN}$ an invertible subsheaf. If $\kappa_{\mathcal{C}}(X,\mathscr{L})=\dim X$, then K_X+D is big.*

Proof. Let H be a very ample divisor such that $H-D$ is ample, and let r be a (fixed) sufficiently large positive integer for which the divisor $r(H-D)$ is very ample. Define the hyperplane section B_D to be a general member of the linear system $|r(H-D)|$. By construction it follows that, for every integer $M>r$, the $\mathbb{Q}$-divisor $D+(1/M)B_D$ is $\mathbb{Q}$-linearly equivalent to an snc divisor, which we denote by P_M, with no reduced components:

$$\begin{aligned} D+\frac{1}{M}B_D &\sim_{\mathbb{Q}} D+\frac{1}{M}(r(H-D))\\ &= \left(1-\frac{r}{M}\right)D+\frac{r}{M}H=:P_M. \end{aligned}$$

Claim 5.2.1. *The divisor K_X+P_M is pseudo-effective, for all integers M satisfying the inequality $M>r$.*

Let us for the moment assume that the claim holds. Define the $\mathbb{Q}$-Cartier divisor G in Proposition 5.1 by $G:=(1/M)B_D$. As conditions (5.1.2), (5.1.3) and (5.1.4) in Proposition 5.1 are all satisfied, it follows from inequality (5.1.1) that for any fixed ample divisor A, there exists a constant c such that

$$\operatorname{vol}\left(K_X+D+\frac{1}{M}B_D\right)\geqslant c\cdot\operatorname{vol}(A),\quad\forall M\in\mathbb{N}\text{ such that }M>r. \tag{5.2.2}$$

Therefore, by taking $M\to\infty$, the continuity property of vol(.) and the fact that *the constant c in Proposition 5.1 is independent of M*, it follows that the divisor K_X+D is big.

It now remains to prove Claim 5.2.1.

Proof of Claim 5.2.1. Aiming to extract a contradiction, suppose that $K_X + P_M$ is not pseudo-effective for some positive integer $M > r$. Let H' be a suitably-chosen very ample divisor such that the effective log-threshold given by

$$\epsilon := \min\{t \in \mathbb{R}^+ : K_X + P_M + tH' \text{ is pseudo-effective}\}$$

is smaller than 1. According to [BCHM10, Corollary 1.1.7], ϵ is rational. Now by applying Proposition 5.1 to the pair (X, D) with $G := (1/M)B_D + \epsilon H'$, we find that $K_X + P_M + \epsilon H'$ is big. But as the big cone forms the interior of the cone of pseudo-effective $\mathbb{Q}$-Cartier classes, for sufficiently small δ, $K_X + D_M + (\epsilon - \delta)H'$ is also pseudo-effective, contradicting the minimality assumption on ϵ. $\square$

The isotriviality conjecture (Conjecture 1.1) now follows from Theorem 5.2 together with Theorem 4.3 in the previous section.

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