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PROJECTIVELY TORSION-FREE MODULES

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1. Introduction

A right R-module A_R will be said to be *right projectively torsion-free* (A_R is PTF) if for every $a \in A$, there exist subsets $\{a_1, a_2, \cdots, a_n\} \subseteq A$ and $\{x_1, x_2, \cdots, x_n\} \subseteq R$ such that $a = \sum_{i=1}^{n} a_i x_i$ and for all $x \in R$, if $ax = 0$ then $x_i x = 0$ for all $1 \leqq i \leqq n$.

Let $\mathscr{S}_R$ denote the class of right PTF modules of the ring R. It is shown that $\mathscr{S}_R$ is the torsion-free class of a hereditary torsion theory if and only if R is a right non-singular ring and every right complement ideal of R is generated by an idempotent. These rings include the continuous regular rings as discussed by Y. Utumi (1961) and (1963).

Hattori (1960) defined a module A_R to be torsion-free if for all $a \in A$ and $x \in R$, $ax = 0$ implies there exist subsets $\{x_1, x_2, \cdots, x_n\} \subseteq R$, with $x_i x = 0$ for all $1 \leqq i \leqq n$, and $\{a_1, a_2, \cdots, a_n\} \subseteq A$ such that $a = \sum_{i=1}^{n} a_i x_i$. We will call such modules *H-torsion-free*. This property may be considered as a 'one-dimensional' flatness. Similarly we show that PTF may be regarded as a 'one dimensional' projectivity. Some of the results of this paper may be considered analogous to results on H-torsion-free R-modules by Hattori (1960) and the author (1973).

Baer rings, in the sense of Kaplansky (1969), are characterised by right PTF modules. In section three it is shown that $\mathscr{S}_R$ is closed under submodules if and only if every principal right ideal of R is projective. The property that $\mathscr{S}_R$ is closed under direct products is also considered in this section.

In the final section of this paper it is shown that for a commutative non-singular ring R, $\mathscr{S}_R$ is closed under forming injective hulls if and only if Min R, the space of minimal prime ideals of R, is compact and extremally disconnected.

Some portions of this paper appeared in the authors' Ph.D. thesis at Monash University.

2. Notation and Preliminaries

Throughout this paper R is an associative ring with identity and all modules are unitary. Let $\mathcal{M}_R$ denote the category of right R-modules. If $A \in \mathcal{M}_R$, $E_R(A)$ is the right injective hull of A (Lambek (1966 page 92)). The maximal right quotient ring of R is Q_R (Lambek (1966; page 94)). A ring R is said to be *right P.P.* if every principal right ideal of R is projective. By a *regular ring* we mean a von-Neumann regular ring. A ring R is regular if and only if every right R-module is flat (Lambek (1966; page 134)). For tring and homological noation the reader is referred to Cartan and Eilenberg (1956) and Lambek (1966).

Given a subset X of a right R-module A_R, we set $r_R(X) = \{y \in R : Xy = 0\}$. $l_R(X)$ is defined similarly when X is a subset of a left R-module.

A submodule B_R of a module A_R is said to be an *essential submodule* of A_R, written $B \subseteq'_R A$, if $B_R \cap C_R \neq 0$ for every non-zero submodule C_R of A_R. If $B_R \subseteq'_R A_R \subseteq D_R$ it is said that A_R is an essential extension of B_R in D_R. A right ideal of a ring R is said to be essential if $I_R \subseteq'_R R_R$. If $A \in \mathcal{M}_R$, let $Z_R(A_R) = \{a \in A : r_R(a) \text{ is an essential right ideal of } R\}$. When $Z_R(A) = 0$, A_R is said to be a *right non-singular* R-module. R is said to be a *right non-singular ring* if $Z_R(R_R) = 0$.

A submodule C_R of a right R-module, A_R is said to be a *right-complement submodule* of A_R if C_R has no proper essential extension in A_R. A right ideal I of R is a *right complement ideal* of R if I_R is a complement submodule of R_R. For further details of essential and complement submodules the reader is referred to Faith (1967).

A class $\mathscr{C}_R$ of right R-modules is said to be closed under group extensions if whenever B_R is a submodule of A_R with B_R and A/B in $\mathscr{C}$, then so is A_R. A class of right R-modules $\mathscr{C}_R$, is the *torsion-free class of a torsion theory* Stenström (1971; pages 5–6), if it is closed under isomorphic images, submodules, direct products and group extensions. $\mathscr{C}_R$ is the *torsion-free class of a hereditary torsion theory* if it is also closed under forming injective hulls. For details of torsion theories see Stenström (1971).

When $Z_R(R_R) = 0$, the right non-singular R-modules are the torsion-free R-modules of the torsion theory co-generated by $E_R(R_R)$, the right injective hull of R. This torsion theory was named the *Lambek torsion theory*, in Stenström (1971). The right complement ideals of R are closed ideals of the Lambek torsion theory when $Z_R(R_R) = 0$. Also when $Z_R(R_R) = 0$, Q_R coincides with $E_R(R_R)$ and Q_R is a right self-injective regular ring. For further details of this see page 106 of Lambek (1966).

The following lemmas will be used throughout the paper. Lemmas 2.1 and 2.2 explain the choice of the name 'projectively torsion-free'.

LEMMA 2.1 (i) $A_R \in \mathscr{S}_R$ *if and only if* $a \in A_R$ *and* $aX = 0$, *for a subset* $X \subseteq R$, *implies* $a \in Al_R(X)$.

(ii) *If* $A_R \in \mathscr{S}_R$ *it is H-torsion-free.*

(iii) *If* $Z_R(R_R) = 0$ *and* $A_R \in \mathscr{S}_R$, *then* $Z_R(A_R) = 0$.

(iv) *If R is a commutative integral domain,* $A \in \mathscr{S}_R$ *if and only if* A_R *is torsion-free in the classical sense.*

PROOF. (i) Assume $a \in A_R$ a right PTF module. Then there exists $\{a_1, a_2, \cdots, a_n\} \subseteq A$ and $\{y_1, y_2, \cdots, y_n\} \subseteq R$ such that, $a = \sum_{i=1}^n a_i y_i$ and $r_R(a) = \bigcap_{i=1}^n r_R(y_i)$. If $aX = 0$ for $X \subseteq R$, then $y_i X = 0$ for all $1 < i \leqq n$. Hence $y_i \in l_R(X)$ and $a \in Al_R(X)$.

Conversely assume $a \in A_R$. Then $ar_R(a) = 0$, which by the assumption implies $a \in Al_R r_R(a)$; i.e. there exists $a_i \in A$, $y_i \in l_R r_R(a)$, $1 \leqq i \leqq n$, such that $a = \sum_{i=1}^n a_i y_i$. Furthermore if $ax = 0$ for some $x \in R$, then $y_i x = 0$ for all $1 \leqq i \leqq n$, since every $y_i \in l_R r_R(a)$.

(ii) Let $a \in A$ and $ax = 0$ for some $x \in R$. Then by (i) $a \in Al_R(x)$ which of course implies A_R is H-torsion-free.

(iii) Let $a \in A_R$, a right PTF R-module. Then there exists $\{a_1, a_2, \cdots, a_n\} \subseteq A$ and $\{y_1, y_2, \cdots, y_n\} \subseteq R$ such that $a = \sum_{i=1}^n a_i y_i$ and $r_R(a) = \bigcap_{i=1}^n r_R(y_i)$. If $r_R(a)$ is an essential right ideal, then so is $r_R(y_i)$ for every $1 \leqq i \leqq n$. But this is impossible for we have assumed $Z_R(R_R) = 0$.

(iv) Assume R is a commutative integral domain and $A \in \mathscr{S}_R$. Then if $a \in A$ and $ad = 0$ for some $d \in R$ part (i) of this lemma gives that $a \in Al_R(d) = 0$. The converse of this result is just as straightforward and so will be omitted.

REMARK 1. It is not known in general when every H-torsion-free R-module is a PTF module. When R is a commutative ring for which every principal ideal is flat it may be shown that the two concepts coincide if and only if R is a finite direct sum of integral domains.

Further results concerning part (iii) of the lemma are presented in Section 4 of this paper.

LEMMA 2.2. (i) *Every projective right R-module is PTF.*

(ii) *If* A_R *is a cyclic R-module,* A_R *is projective if and only if* $A_R \in \mathscr{S}_R$.

PROOF. (i) Let $a \in A_R$ a projective R-module. Then by the dual basis lemma there exist homomorphisms $\{\phi_1, \phi_2, \cdots, \phi_n\} \subseteq \mathrm{Hom}_R(A, R)$ and elements $a_1, a_2, \cdots, a_n \subseteq A$ such that $a = \sum_{i=1}^n a_i \phi_i(a)$. If $ax = 0$ for some $x \in R$, then $\phi_i(a)x = \phi_i(ax) = 0$ for all $1 \leqq i \leqq n$. Hence $A_R \in \mathscr{S}_R$.

(ii) Let $aR \cong A_R$ be a cyclic PTF module. Then there exists elements $\{b_1, b_2, \cdots, b_n\} \subseteq P$ and $\{x_1, x_2, \cdots, x_n\} \subseteq R$ such that

$$(1) \qquad a = \sum_{i=1}^{n} (ab_i)x_i$$

and if $ay = 0$ for some $y \in R$, $x_i y = 0$ for all $1 \leqq i \leqq n$.

By (1) $a(1 - \sum_{i=1}^{n} b_i x_i) = 0$ and hence $x_i(1 - \sum_{i=1}^{n} b_i x_i) = 0$ for every $1 \leqq i \leqq n$. Thus we have that $\sum_{i=1}^{n} b_i x_i(1 - \sum_{i=1}^{n} b_i x_i) = 0$ and $(\sum_{i=1}^{n} b_i x_i)^2 = \sum_{i=1}^{n} b_i x_i$. Since $\sum_{i=1}^{n} b_i x_i$ is an idempotent of R we have $r_R(a) = r_R(\sum_{i=1}^{n} b_i x_i) = (1 - \sum_{i=1}^{n} b_i x_i)R$. Since $1 - \sum_{i=1}^{n} b_i x_i$ is also an idempotent of R $aR \cong R/r_R(a)$ is isomorphic to a direct summand of R_R and hence is projective.

Let $Pd(A)$ denote the right projective dimension of the right R-module A_R. The right global dimension of R, written $GD(R)$ is defined by $GD(R) = \sup\{Pd(A)\colon A \in \mathscr{M}_R\}$. Auslander has shown that $GD(R) = \sup\{Pd(R/I)\colon I$ is a right ideal of $R\}$. We will say a ring R is *semi-simple* if $GD(R) = 0$; i.e. if every R-module is projective.

COROLLARY 2.3. R is a semi-simple ring if and only if every right R-module is a PTF module.

PROOF. If R is a semi-simple ring every R-module is projective and hence PTF by Lemma 2.2 (i).

Conversely assume every right R-module is PTF. Then every cyclic right R-module is a PTF module and thus projective by Lemma 2.2 (ii). By the comments above we have $GD(R) = 0$.

REMARK 2. It may be suspected that every right flat R-module is PTF. But thus is not so. For example choose R to be a regular ring which is not semi-simple. Then every R-module is flat, but as R is not semi-simple there must exist an R-module which is not PTF.

LEMMA 2.4. *Suppose $\phi\colon R \to S$ to be a ring homomorphism. Inducing a right R-module structure on S. Then any PTF right S-module A_S remains PTF when considered as a right R-module under the induced action.*

PROOF. Let $a \in A$ and $X \subseteq R$ such that $aX = 0$ under the induced action; i.e. $a\phi(X) = 0$. Then there exists elements $\{a_1, a_2, \cdots, a_n\} \subseteq A$ and $\{s_1, s_2, \cdots, s_n\} \subseteq l_S(\phi(X))$ such that $a = \sum_{i=1}^{n} a_i s_i$. But the assumption on S implies that $l_S(\phi(X)) \subseteq S\phi(l_R(X))$. Hence for each s_i, $1 \leqq i \leqq n$ there exists element $\{t_1, t_2, \cdots, t_m\} \subseteq S$ and $\{y_1, y_2, \cdots, y_m\} \subseteq \phi(l_R(X))$ such that $s_i = \sum_{j=1}^{m} t_j m_j$. Thus $a = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} t_j y_j$ which is an element of $Al_R(X)$. Lemma 2.1 gives that $A \in \mathscr{S}_R$.

3. Conditions on $\mathscr{S}_R$

LEMMA 3.1. *For any ring* R:

(i) $\mathscr{S}_R$ *is closed under group extensions.*

(ii) $\mathscr{S}_R$ *is closed under isomorphic images.*

PROOF. (i) Let $B_R \subseteq A_R$ and assume B and A/B are PTF modules. Let $a \in A$ and $aX = 0$ for some $X \subseteq R$. Then $(a+B)X = 0 + B$ in A/B. Hence there exists $\{a_1, a_2, \cdots, a_n\} \subseteq A$ and $\{y_1, y_2, \cdots, y_n\} \subseteq l_R(X)$ such that $a + B = (\sum_{i=1}^n a_i + B)y_i = \sum_{i=1}^n a_i y_i + B$. Hence a $- \sum_{i=1}^n a_i y_i \in B$. By the assumption $aX = 0$ and as $y_i \in l_R(X)$, $(a - \sum_{i=1}^n a_i y_i)X = 0$. Since $B_R \in \mathscr{S}_R$ there exists $\{b_1, b_2, \cdots, b_m\} \subseteq B$ and $\{z_1, z_2, \cdots, z_m\} \subseteq l_R(X)$ such that $a - \sum_{i=1}^n a_i y_i = \sum_{j=1}^n b_j z_j$. Thus $a = \sum_{i=1}^n a_i y_i + \sum_{j=1}^m b_j y_j \in Al_R(X)$. Lemma 2.1 gives that $A_R \in \mathscr{S}_R$.

(ii) This is clear.

These are the only two of the necessary and sufficient conditions for $\mathscr{S}_R$ to be the torsion-free class of a hereditary torsion theory which hold for an arbitrary ring. In the following results of this section we investigate when the remainder of these conditions hold.

The following Lemma's proof is taken from page 96 of Blythe and Janowitz (1972).

LEMMA 3.2. *Let R be a right P.P. ring. Then if $\{x_1, x_2, \cdots, x_n\} \subseteq R$ there exists an idempotent $h \in R$, such that $\cap_{i=1}^n r_R(x_i) = hR$.*

PROOF. We give a proof for $n = 2$. Let $\{x_1, x_2\} \subseteq R$. Then as R is a right P.P. ring there exists idempotents $\{e, f\} \subseteq R$ such that $r_R(x_1) = eR$ and $r_R(x_2) = fR$. Let $g^2 = g \in R$ be such that $gR = r_R(f - ef)$. Then $(f - ef)fg = (f - ef)g = 0$. Hence $fg \in gR$ and $fg = gfg$. Thus (fg) is an idempotent of R.

We show $r_R(x_1) \cap r_R(x_2) = (fg)R$. Clearly $(fg)R \subseteq fR$. Also, $(1-e)fg = (f - ef)g = 0$. Hence $fgR \subseteq r_R(1-e) = eR$ and thus $fgR \subseteq fR \cap eR$.

Conversely assume $x \in eR \cap fR$. Then $(f - ef)x = (1-e)fx = (1-e)x = (1-e)x = x - ex = 0$ and so $x \in gR$. Then $x = gx$, $x = fx$ and $x = fgx$ and we have $eR \cap fR = fgR$.

The following result is a generalisation of Theorem 3.2 of Evans (1972).

THEOREM 3.3. *If R is a ring, the following are equivalent*:

(i) *R is a right P.P. ring.*

(ii) *$\mathscr{S}_R$ is closed under submodules.*

(iii) *Every submodule of a projective R-module is PTF.*

(iv) *Every submodule of a free R-module is PTF.*

(v) *Every right ideal of R is a right PTF module.*

Proof. (i) ⇒ (ii) Let $a \in A_R$, a right PTF module. Then $a = \sum_{i=1}^n a_i x_i$ with $\{a_1, a_2, \cdots, a_n\} \subseteq A$ and $\{x_1, x_2, \cdots, x_n\} \subseteq R$ such that $r_R(a) = \cap_{i=1}^n r_R(x_i)$. By the Lemma 3.2 there is an idempotent $h \in R$ such that $r_R(a) = hR$. Thus $aR \cong R/r_R(a) \cong (1-h)R$ is projective as an R-module. Thus every cyclic submodule of A_R is PTF by Lemma 2.2. Clearly this implies every submodule of A_R is PTF.

(ii) ⇒ (iii) Lemma 2.2 gives that every right projective R-module is PTF.

(iii) ⇒ (iv) Every free R-module is PTF.

(iv) ⇒ (v) R_R is free.

(v) ⇒ (i) Let $x \in R$. Then xR is a PTF module and thus projective by Lemma 2.2.

It is shown by Hattori (1960), that every principal right ideal of a ring R is flat if and only if every submodule of right H-torsion-free R-module is H-torsion-free.

A direct sum of right PTF modules is always a PTF module. The following result is analogous to Proposition 8 of Hattori (1960).

Theorem 3.4. *If R is a ring the following are equivalent:*

(i) *$\mathscr{S}_R$ is closed under direct products.*

(ii) *Every direct product of copies of R_R is a right PTF module.*

(iii) *Every left annihilator ideal of R is finitely generated.*

Proof. (i) ⇒ (ii) R_R is a right PTF module.

(ii) ⇒ (iii). Let $X \subseteq R$ and $A_X = \Pi_{z \in l_R(X)} R_z$ be a direct product of isomorphic copies of R indexed over $l_R(X)$. Let $a \in A_X$, be the element of A_X having z in the zth place for all $z \in l_R(X)$. Since $A_X \in \mathscr{S}_R$ and $aX = 0$. Lemma 2.1 gives that there exists $\{y_1, y_2, \cdots, y_n\} \subseteq l_R(X)$ and $\{a_1, a_2, \cdots, a_n\} \subseteq A$ such that $a = \sum_{i=1}^n a_i y_i$.

Write $a(z)$ as the zth component of a and $a_j(z)$ as the zth component of a_j where $a_j \in \{a_1, a_2, \cdots, a_n\}$. Then $z = a(z) = \sum_{i=1}^n a_i(z) y_i$ for all $z \in l_R(X)$ and thus the set $\{y_1, y_2, \cdots, y_n\}$ is a generating set for $l_R(X)$.

(iii) ⇒ (i) Let $\{A_\alpha\}_{\alpha \in I}$ be a family of right PTF modules and $A = \Pi A_\alpha$ its direct product. For $a = (a_\alpha) \in A$ and $X \subseteq R$, $aX = 0$ implies $a_\alpha X = 0$ for each α. Assume $l_R(X)$ is generated by a finite number of elements, say $\{y_1, y_2, \cdots, y_n\}$. Then for each α there exists elements $a_{i\alpha} \in A_\alpha$ such that $a_\alpha = \sum_{i=1}^n a_{i\alpha} y_i$. Put $a_i = (a_{i\alpha})$; then $a = \sum_{i=1}^n a_i y_i$ and hence $a \in Al_R(X)$. Lemma 2.1 gives that A is PTF.

For the question of when $\mathscr{S}_R$ is closed under injective hulls we have been able to obtain results for the case $Z_R(R_R) = 0$.

Proposition 3.5. *If $Z_R(R_R) = 0$, $\mathscr{S}_R$ is closed under forming injective hulls if and only if Q_R the maximal right quotient ring of R is a right PTF module.*

PROOF. Assume $Q_R \in \mathscr{S}_R$ and let $A \in \mathscr{S}_R$. Then by Lemma 2.1, $Z_R(A_R) = 0$. Lemma 8.3 of Stenström (1971) gives that $E_R(A)$ is a Q-module and clearly $Z_Q(E_R(A)) = 0$. Now since $Z_R(R_R) = 0$, Q is a right self injective regular ring. Theorem 4, page 71 of Faith (1967) gives that every right complement ideal of Q is generated by an idempotent. Let $a \in E_R(A)$. Then $r_Q(a)$ is a right complement ideal of Q and thus aQ is a projective Q-submodule of $E_R(A)$. Hence by Lemma 2.2 (i), $E(A) \in \mathscr{S}_Q$. Lemma 2.4 now gives that $E(A) \in \mathscr{S}_R$.

Cateforis (1969) gave the following definitions.

DEFINITION 3.6. *A right R-module, A_R is essentially finitely generated (A_R is EFG) if there exists $\{a_1, a_2, \cdots, a_n\} \subseteq A$ such that $\sum_{i=1}^n a_i R \subseteq'\, {}_R A$.*

DEFINITION 3.7. *A right R-module, B_R, is essentially finitely related (A_R is EFR) if there exists an exact sequence $0 \to A \to F \to B \to 0$ with F_R finitely generated and free with A_R EFG.*

THEOREM 3.8. *If $Z_R(R_R) = 0$, Q_R, the maximal right quotient ring of R, is left PTF as an R-module if and only if for every subset X of R, $r_R(X)$ is EFG.*

PROOF. Assume Q_R is a left PTF module. Then by the dual result to Lemma 2.1 we have that for each subset X of R $r_R(X)Q = r_Q(X)$. Now since Q is a right self-injective regular ring there exists $e^2 = e \in Q$ such that $r_Q(X) = eQ$. Thus there exists elements $\{x_1, x_2, \cdots, x_n\} \subseteq r_R(X)$ and $\{q_1, q_2, \cdots, q_n\} \subseteq Q$ such that $\sum_{i=1}^n x_i q_i = e$.

Now if $y \in r_R(X) \subseteq r_Q(X)$, $y = ey = (\sum_{i=1}^n x_i q_i) y = \sum_{i=1}^n x_i (q_i y)$. Since Q is a right essential extension of R, there exists $z \in R$ such that $0 \neq yz = \sum_{i=1}^n x_i(q_i yz) \in \sum_{i=1}^n x_i R$. Hence $\sum_{i=1}^n x_i R \subseteq'\, {}_R r_R(X)$ and $r_R(X)$ is EFG.

Conversely assume $r_R(X)$ is EFG for every $X \subseteq R$. It is sufficient to show that if $X \subseteq R$, $r_R(X)Q = r_Q(X)$. By the assumption there exists $\{x_1, x_2, \cdots, x_n\} \subseteq R$ such that $\sum_{i=1}^n x_i R \subseteq'\, {}_R r_R(X)$. Thus $\sum_{i=1}^n x_i Q \subseteq'\, {}_Q r_Q(X)$ as Q is a right essential extension of R. Since Q is a regular ring there exists an idempotent $e \in Q$ such that $\sum_{i=1}^n x_i Q = eQ$. But eQ is a right complement ideal of R and thus has no proper essential extension in Q. Hence $r_Q(X) = eQ \subseteq r_R(X)Q$. Lemma 2.1 now gives the result.

In Evans (1973) it is shown that if $Z_R(R_R) = 0$, $r_R(x)$ is EFG for every $x \in R$ if and only if Q_R is a left H-torsion-free R-module. In the commutative case it is shown that this is equivalent to Q_R being flat.

From Proposition 3.5 and Theorem 3.8 we have the following Corollary.

COROLLARY 3.9. If R is commutative and $Z_R(R_R) = 0$ the following are equivalent:

(i) $\mathscr{S}_R$ is closed under injective hulls.

(ii) Q_R is a PTF module.

(iii) For every subset X of R, $r_R(X)$ is EFG.

4. $\mathscr{S}_R$ froms a torsion-free class

A ring R is said to be a *Baer ring* if for each $X \subseteq R$, there exists an idem potent $e^2 = e \in R$ such that $r_R(X) = eR$. These rings are discussed by Kaplansky (1969). If $X \subseteq R$, where R is a Baer ring, there exists $e^2 = e \in R$ such that $l_R(X) = l_R r_R l_R(X) = l_R(eR) = R(1-e)$. It is easily seen that R is a Baer ring if and only if each left annihilator ideal of R is generated by an idempotent.

A module A_R is said to be *torsionless* if A_R can be imbedded in a direct product of copies of R_R. This is equivalent to: for every $0 \neq a \in A$ there exists $f \in \mathrm{Hom}_R(A, R)$ with $f(a) \neq 0$.

THEOREM 4.1. *If R is a ring the following are equivalent:*

(i) *R is a Baer ring.*

(ii) *$\mathscr{S}_R$ is closed under direct products and submodules.*

(iii) *$\mathscr{S}_R$ is the torsion-free class of a torsion theory.*

(iv) *Every right torsionless R-module is a PTF module.*

PROOF. (i) $\Rightarrow$ (ii). Since R is a Baer ring, $x \in R$ implies $r_R(x) = eR$ for some $e^2 = e \in R$. Hence xR is projective and R is a right P.P. ring. Thus by Theorem 3.3 $\mathscr{S}_R$ is closed under the operation of taking submodules. Theorem 3.4 gives that $\mathscr{S}_R$ is closed under taking direct products.

(ii) $\Rightarrow$ (iii). Lemma 3.1 gives that $\mathscr{S}_R$ is also closed under group extensions.

(iii) $\Rightarrow$ (iv). If A_R is a right torsionless R-module, A_R can be imbedded in a direct product of copies of R_R. Since $\mathscr{S}_R$ is closed under isomorphisms, direct products and submodules the result follows.

(iv) $\Rightarrow$ (i). If $x \in R$, xR is a right torsionless R-module and is thus PTF by the assumption. Lemma 2.2 (ii) gives that xR is projective and thus we have R is a right P.P. ring.

If $X \subseteq R$, then $r_R(X) = r_R l_R r_R(X)$. By Theorem 3.3 $l_R r_R(X)$ is a finitely generated left ideal of R. Let $\{y_1, y_2, \cdots, y_n\}$ be a generating set for $l_R r_R(X)$. Then $r_R l_R r_R(X) = \bigcap_{i=1}^n r_R(y_i)$ Lemma 3.2 gives that R is a Baer ring.

REMARKS 3 (a). Hattori (1960) shows that the left H-torsion-free R-modules form the torsion-free class of a torsion theory if and only if R is a right P.P. ring.

(b) Let ${}_R\mathscr{S}$ denote the class of left PTF modules of R. From the comments preceding the statement of Theorem 4.1 it may be seen that the three other equivalent conditions may be added. These are:

(ii)′ ${}_R\mathscr{S}$ is closed under direct products and submodules.

(iii)′ ${}_R\mathscr{S}$ is the torsion-free class of a torsion theory.

(iv)′ Every left torsionless R-module is a left PTF module.

We have another characterisation of Baer rings in terms of PTF modules.

PROPOSITION 4.2. *R is a left P.P. ring, $Z_R(R_R) = 0$ and Q_R, the maximal right quotient ring of R, is left PTF as an R-module if and only if R is a Baer ring.*

PROOF. First assume R is a left P.P. ring and $Q_R \in {}_R\mathscr{S}$. Let $X \subseteq R$ and $y \in r_R(X)$. Then $l_R r_R(X) \subseteq l_R(y)$. Since R is a left P.P. ring, $l_R(y) = \text{Re}$ for some $e^2 = e \in R$. Hence $r_R(X) = r_R l_R r_R(X) \supseteq r_R l_R(y) = (1-e)R$. Thus we have $1 - e \in r_R(X)$ and $y = (1-e)y$. It follows that for each $z \in r_R(X)$ there exists $c \in r_R(X)$ such that $z = cz$. Thus if J is a left ideal of R, $J \cap r_R(X) = r_R(X)J$. Hence by Proposition 3 page 133 of Lambek (1966), $R/r_R(X)$ is a flat R-module.

Now since $Q_R \in {}_R\mathscr{S}$, Theorem 3.8 gives that $r_R(X)$ is EFG and thus by the exact sequence

$$0 \to r_R(X) \to R \to R/r_R(X) \to 0 \tag{2}$$

we see that $R/r_R(X)$ is EFR. By Theorem 1.7 of Cateforis (1969), $R/r_R(X) \otimes {}_R Q$ is projective as an R-module.

Theorem 3.1 of Jϕndrup (1970) gives that $R/r_R(X)$ is a projective R-module. Hence the exact sequence (2) splits and we see that $r_R(X)$ is a direct summand of R.

The converse is immediate as any Baer ring, R is a left P.P. ring, $Z(R_R) = 0$ and Theorem 3.8 gives thas $Q_R \in {}_R\mathscr{S}$.

REMARK 4. The analogous result for H-torsion-free R-modules appeared in Evans (1973). Q_R is regular, left H-torsion-free as an R-module and every principal left ideal of R is flat if and only if R is a right P.P. ring.

As noted in Section 2, Q_R the maximal right quotient ring of R, a right non-singular ring, is a right self-injective regular ring. The right complement ideals of Q_R are simply the principal right ideals of Q. The lattice of right complement ideals of R, $C(R)$, is isomorphic to the lattice of right complement ideals of Q. The isomorphism being determined by the map α: $C(Q) \to C(R)$ defined by $\alpha(J) = J \cap R$. This material appears on pages 61 and 71 of Faith (1967).

THEOREM 4.3. *Let R be a ring. Then the following are equivalent:*

(i) *$\mathscr{S}_R$ is the torsion-free class of a hereditary torsion theory.*
(ii) *R is a right P.P. ring and $Q_R \in \mathscr{S}_R$.*
(iii) *R is a right P.P. ring and $\mathscr{S}_R$ is closed under injective hulls.*
(iv) *Every right complement ideal of R is generated by an idempotent.*
(v) *For all $A \in \mathscr{M}_R$, $Z_R(A_R) = 0$ if and only if $A_R \in \mathscr{S}_R$.*

PROOF. (i) $\Rightarrow$ (ii). By Theorem 4.1 R is a Baer ring. Hence $Z_R(R) = 0$ and Q_R is the right injective hull of R. Thus $Q_R \in \mathscr{S}_R$.

(ii) $\Leftrightarrow$ (iii). This is Proposition 3.5.

(ii) $\Rightarrow$ (iv). Let J be a right complement ideal of R. Then by the comments preceding the theorem there exists an idempotent e of Q such that $J = eQ \cap R$. Now $eQ = r_Q(1-e)$ and thus $J = r_Q(1-e) \cap R = r_R(1-e)$.

By the assumption R is a right P.P. ring and thus by Theorem 3.3 every submodule of Q_R is also a PTF module. In particular $(1-e)R \in \mathscr{S}_R$ and is thus projective by Lemma 2.2 (ii). Hence the exact sequence

$$0 \to J \to R \to (1-e)R \to 0$$

splits and J is a direct summand of R.

(iv) $\Rightarrow$ (v). Let $a \in A_R$, where $Z_R(A_R) = 0$. Then $Z(aR) = 0$ and $r_R(a)$ is a right complement ideal of R. Thus $aR \cong R/r_R(a)$ is isomorphic to a direct summand of R and is thus projective. Hence $aR \in \mathscr{S}_R$ and so $A_R \in \mathscr{S}_R$.

(v) $\Rightarrow$ (i). $R_R \in \mathscr{S}_R$ and thus $Z(R_R) = 0$. When $Z(R_R) = 0$ the non-singular R-modules are the torsion-free class of a hereditary torsion theory.

We will call the rings defined by the equivalent conditions of Theorem 4.3 *right strongly Baer rings*. Left strongly Baer is defined similarly. A ring R, is said to be *strongly Baer* if it is both left and right strongly Baer. An important example of strongly Baer rings are the continuous regular rings as defined by J. V n Neumann.

DEFINITION 4.3. (Utumi (1960)). *A regular ring R is right (left) continuous if its principal right (left) ideals form a complete lattice L and for any directed set $\{A_i\}_{i \in I}$ of elements of L and $A \in L$.*

$$A \wedge \Big(\bigvee_{i \in I} A_i \Big) = \bigvee_{i \in I} (A \wedge A_i)$$

A regular ring is *continuous* if and only if it is both left and right continuous. Utumi (1961) has given the following characterisation of right continuous regular rings.

LEMMA 4.4. *A regular ring R is right continous if and only if every right ideal of R is an essential submodule of a direct summand of R.*

We will need the following lemma.

LEMMA 4.5. *Let $Z_R(R_R) = 0$. Then the following are equivalent:*

(i) *Every right complement ideal of R is EFG.*
(ii) *For all right ideals I of Q, $(I \cap R)Q = I$.*

PROOF. (i) $\Rightarrow$ (ii). This is Lemma 2.4 of Cateforis (1969a).

(ii) $\Rightarrow$ (i). If J is a right complement ideal of R, then there exists an idempotent $e \in Q$ such that $eQ \cap R = J$.

Hence by the assumption $JQ = (eQ \cap R)Q = eQ$. Thus there exists elements $\{x_1, x_2, \cdots, x_n\} \subseteq J$ and $\{q_1, q_2, \cdots, q_n\} \subseteq Q$ such that $\sum_{i=1}^n x_i q_i = e$. Let $y \in J \subseteq eQ$. Then $y = ey$ and hence $y = (\sum_{i=1}^n x_i q_i)y$. Since Q is an essential right extension of R, there exists $z \in R$ such that $0 \neq yz = \sum_{i=1}^n x_i(q_i y)z \in \sum_{i=1}^n x_i R$. Hence J is EFG.

Corollary 4.6. If $R \subseteq S \subseteq Q$, S a subring of Q and R satisfies the equivalent conditions of Lemma 4.5, then I being a right ideal of Q implies $(I \cap S)Q = I$.

Proof. Let I be a right ideal of Q. Then $(I \cap S)Q \subseteq I$ and as $I = (I \cap R)Q \subseteq (I \cap S)Q$, $I = (I \cap S)Q$.

Proposition 4.7. *If R is a regular ring, the following are equivalent:*

(i) *R is a right continuous regular ring.*

(ii) *Every right complement ideal of R is EFG.*

(iii) *For all right ideals I of Q, $(I \cap R)Q = I$.*

(iv) *R is a right strongly Baer ring.*

Proof. (i) $\Rightarrow$ (ii). Let J be a right complement ideal of R. Then by definition J has no proper essential extension. Thus J is a direct summand of R by Lemma 4.4

(ii) $\Leftrightarrow$ (iii). This is Lemma 4.5.

(ii) $\Rightarrow$ (iv). By the assumption there exists elements $\{x_1, x_2, \cdots, x_n\} \subseteq J$ such that $\sum_{i=1}^n x_i R \subseteq' J$. Since R is regular there exists $e^2 = e \in R$ such that $\sum_{i=1}^n x_i R = eR$. But eR is a right complement ideal of R and has no proper essential extension.

(iv) $\Rightarrow$ (i). Let J be a right ideal of R. Then by Proposition 7, page 61 of Faith (1967) there exists a right complement ideal K of R such that $J \subseteq'_R K$. Since K is generated by an idempotent Lemma 4.4 now gives the result.

In Lemma 1 of, Utumi (1960) it is shown that if L, the lattice of principal right ideals of a regular ring R, is complete then R is a Baer ring. In Utumi (1963) the following result is given

'A regular ring for which L is complete is continuous if and only if Q_R is also the maximal left quotient ring of R'.

We have the following generalisation of this.

Proposition 4.8. *A Baer ring R is strongly Baer if and only if Q_R is also the maximal left quotient ring of R.*

Proof. This follows from Theorems 2.2 and 3.3 of Utumi (1963).

Corollary 4.9. *If R is a commutative ring, then R is a Baer ring if and only if R is strongly Baer.*

5. Min R compact and extremally disconnected

A characterisation of commutative non-singular rings for which $\mathscr{S}_R$ is closed under forming injective hulls was given in Corollary 3.9. In this section it is shown that this condition may also be characterised by a topological condition on the minimal prime ideals of the ring.

For a commutative ring R, let Spec R denote the set of prime ideals of R with the hull kernel topology and Min R the set of minimal prime ideals of R with this topology. If I is an ideal of R, let $D(I) = \{P \in \operatorname{Min} R \mid I \nsubseteq P\}$. A set $U \subseteq \operatorname{Min} R$ is open if and only if $U = D(I)$ for some ideal I of R. A topological space Z is said to be *extremally disconnected* if the closure of every open set of Z is open.

When R is a commutative ring, R is non-singular, if and only if R contains no non-zero nilpotent elements. In this case the complement ideals of R are simply the annihilator ideals.

Before we proceed to the main result of this section we need two lemmas.

LEMMA 5.1. *Let X be a subset of R, where R is a commutative non-singular ring. Then the following are equivalent:*

(i) *$r_R(X)$ is EFG, with $\sum_{i=1}^n z_iR \subseteq'_R r_R(X)$, $z_i \in$ for $1 \leqq i \leqq n$.*

(ii) *There exists elements $\{z_1, z_2, \cdots, z_n\} \subseteq R$ such that $r_Rr_R(X) = \bigcap_{i=1}^n r_R(z_i)$.*

PROOF. (i) $\Rightarrow$ (ii). Since $\sum_{i=1}^n z_iR \subseteq' r_R(X)$ we have

$$\sum_{i=1}^n z_iR \subseteq r_Rr_R\left(\sum_{i=1}^n r_Rr_R(z_i)\right) \subseteq'_R r_R(X).$$

But $r_Rr_R(\sum_{i=1}^n r_Rr_R(z_i))$ is a complement ideal of R and thus has no proper essential extension. Hence $r_Rr_R(\sum_{i=1}^n r_Rr_R(z_i)) = r_R(X)$. Thus $r_Rr_R(X) = r_R(r_Rr_R(\sum_{i=1}^n r_Rr_R(z_i))) = \bigcap_{i=1}^n r_R(z_i)$.

(ii) $\Rightarrow$ (i). Suppose $r_Rr_R(X) = \bigcap_{i=1}^n r_R(z_i)$. Then if $y \in r_R(X)$ and $yR \cap \sum_{i=1}^n z_iR = 0$, $yz_i = 0$ for every $1 \leqq i \leqq n$. But this implies that $y \in r_Rr_R(X) \cap r_R(X) = 0$. Hence $\sum_{i=1}^n z_iR \subseteq' r_R(X)$.

LEMMA 5.2. *If R is a commutative ring, $M(R)$ the flat epimorphic hull of R is PTF when considered as an R-module.*

PROOF. Let $X \subseteq R$. Then $r_M(X) = (r_M(X) \cap R)M = r_R(X)M$ by Proposition 14.3 of Stenström (1971). By Lemma 2.1, $M \in \mathscr{S}_R$.

For details of $M(R)$, the left flat epimorphic hull of a ring R, see Stenström (1971).

THEOREM 5.3. *If R is a commutative non-singular ring the following are equivalent:*

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sidered condition (v) and have given an example which shows that condition (v) does not imply Q_R is the flat epimorphic hull of R.

(b) It is shown in Evans (1973) that if R is a commutative non-singular ring, Min R is compact if and only if Q_R is H-torsion-free or equivalently if the injective hull of every H-torsion-free R-module is H-torsion-free.

COROLLARY 5.4. *Let R be a commutative ring. Then R is a Baer ring if and only if R is a P.P. ring and satisfies the equivalent conditions of Theorem* 5.3.

PROOF. This is immediate from Theorem 4.3 and Corollary 4.9.

EXAMPLE *of a commutative ring which satisfies the equivalent conditions of Theorem* 5.3 *but which is ot Baer.*

Let $R' = 2Z \oplus 2Z$, where 2Z is the ring of even integers. Let R be the ring obtained from R' by adjoining a unity in the usual way (see for example, page 111, Henriksen and Jerison (1965)). Then R has only finitely many minimal prime ideals and is thus compact. Hence Proposition 1 of Quentel (1971) gives that $M(R)$ is regular and $M(R)$ will have only a finite number of minimal prime ideals. This implies $M(R)$ is isomorphic to a finite direct sum of fields and hence, in particular, isomorphic to a continuous regular ring.

R is not Baer as it has only two idempotents but many zero divisors.

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