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THE PSEUDO-ORBIT SHADOWING PROPERTY FOR MARKOV OPERATORS IN THE SPACE OF PROBABILITY DENSITY FUNCTIONS

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ABSTRACT. Let X be a space with two metrics d_1 and d_2. Let $S : (X, d_1) \longrightarrow (X, d_2)$ be continuous. We say S has the generalized pseudo-orbit shadowing property with respect to the metrics d_1 and d_2 if for every $\epsilon > 0\ \exists \delta > 0 \ni$ every δ-pseudo-orbit in d_1 can be ϵ-shadowed by a true orbit in d_2, i.e., if $\{x_0, x_1, \ldots\}$ satisfies $d_1(S(x_i), x_{i+1}) \leqq \delta$ for all $i \geqq 0$, then $\exists x \in X \ni d_2(S^i(x), x_i) \leqq \epsilon$ for all $i \geqq 0$. The main result of this note shows that certain Markov operators $P : L^1 \longrightarrow L^1$ have the generalized shadowing property on weakly compact subsets of the space of probability density functions, where d_1 is the metric of norm convergence and d_2 is the metric of weak convergence. An important class of such operators are the Frobenius-Perron operators induced by certain expanding and non-expanding maps on the interval. When there is exponential convergence of the iterates to the density, we can express δ in terms of ϵ. We also show that, unlike the situation in the space X itself, the generalized shadowing property is valid for all parameters in families of maps and that there is stability of the shadowing property.

1. Introduction. Let (X, d) be a compact metric space and let $S : X \longrightarrow X$ be a continuous map. The orbit of $x \in X$ is the sequence $\{x, S(x), S^2(x), \ldots\}$. Given a number $\delta > 0$, a δ-pseudo-orbit is a sequence $\{x_0, x_1, \ldots\}$ such that $d(S(x_i), x_{i+1}) \leqq \delta$ for all $i \geqq 0$. An important example of a δ-orbit is a computer orbit, where computation errors occur at each iteration. In such cases it is of interest to know that pseudo-orbits can be approximated by true orbits of the map S.

Definition 1. We say $S : X \longrightarrow X$ has the shadowing property (or the pseudo-orbit tracing property) if for every $\epsilon > 0\ \exists \delta > 0 \ni$ that every δ-pseudo-orbit can be ϵ-shadowed by a true orbit, i.e., if $\{x_0, x_1, \ldots\}$ satisfies $d(S(x_i), x_{i+1}) \leqq \delta$ for all $i \geqq 0$, then $\exists x \in X \ni d(S^i(x), x_i) \leqq \epsilon$ for all $i \geqq 0$.

The term shadowing was first introduced by Bowen [**1**] for Axiom A diffeomorphisms. In [**2**] it is shown that tent maps have the shadowing property for almost all parameter values, although there is an uncountable set of parameter values which is dense and for which the tent map does not have the shadowing

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property.

The map $S : X \longrightarrow X$ induces a continuous map $S_M : \mathcal{M}(X) \longrightarrow \mathcal{M}(X)$, defined by $S_M\mu(A) = \mu(S^{-1}A)$, where $\mathcal{M}(X)$ is the space of probability measures on X. The elements of $\mathcal{M}(X)$ can be viewed as statistical states, reflecting the fact that there is imperfect knowledge of the system. In [**3**] it is shown that many of the topological properties of S carry over to S_M. In [**5**] the pseudo-orbit tracing property of S is shown to imply the pseudo-orbit tracing property for S_M on certain closed subsets of measures with finite support.

In this note we shall find it useful to employ a generalized notion of shadowing. Since we shall be using only linear operators, we state this property for such mappings.

Definition 2. (Generalized Shadowing Property) Let X be a subset of a linear space with two metrics d_1 and d_2. Let $S : (X, d_1) \longrightarrow (X, d_2)$ be linear. We say S has the (δ, ϵ) generalized shadowing property (with respect to d_1 and d_2), or simply the generalized shadowing property, if for every $\epsilon > 0\ \exists \delta > 0 \ni$ every δ-pseudo-orbit (in d_1) can be ϵ-shadowed by a true orbit (in d_2), i.e., if $\{x_0, x_1, \ldots\}$ satisfies $d_1(S(x_i), x_{i+1}) \leqq \delta$ for all $i \geqq 0$, then $\exists x \in X \ni d_2(S^i(x), x_i) \leqq \epsilon$ for all $i \geqq 0$.

Let $(X, \mathcal{B}, \mu)$ be a finite measure space and let $L^1 \equiv L^1(X, \mathcal{B}, \mu)$ with the L^1-norm $\|\ \|_1$. Let D_1 denote the space of densities on X, i.e., the set of all normalized nonnegative elements of L^1:

$$D_1 = \{f \in L^1 : \|f\|_1 = 1, f \geqq 0\}.$$

A linear operator $P : L^1 \longrightarrow L^1$ is called **Markov** if $P(D_1) \subset D_1$. It follows that $\|Pf\|_1 \leqq \|f\|_1$. If $Pf = f$ for some $f \in L^1$, f is called a fixed point of P.

Definition 3. We say $P : L^1 \longrightarrow L^1$ is strongly (weakly) constrictive if there exists a strongly (weakly) compact set $A \subset L^1$ such that

$$\lim_{n\to\infty} \inf_{g \in A} \|P^n f - g\|_1 = 0$$

for $f \in D_1$. We shall refer to A as an attractor.

In [**8**] it is shown that if P is weakly constrictive, then P is strongly constrictive. This is useful since it is easier to check for weak compactness than for strong compactness. From now on, we will delete the adjectives strong and weak. Examples of constrictive Markov operators can be found in [**3**].

Since D_1 is a subset of L^1, (D_1, σ) is a metric space, where σ is the metric induced by the L^1 norm $\|\ \|_1$. We shall also consider D_1 with the topology of weak convergence, i.e., $f_n \longrightarrow f$ if and only if for every $h \in L^\infty = L^\infty(X, \mathcal{B}, \mu)$,

$$\int_X h(x) f_n(x) \mu(dx) \longrightarrow \int_X h(x) f(x) \mu(dx)$$

as $n \longrightarrow \infty$. If $h \in C(X)$, the space of real-valued continuous functions on X, we shall refer to this as vague convergence.

Let $\{\phi_n\}$ be a countable dense subset of $C(X)$ in the sup norm topology. Let

$$\beta_n = \sup_{x \in X} |\phi_n(x)| > 0$$

and let $\{\alpha_n\}$ be a sequence of positive real numbers such that

$$\sum_{n=1}^{\infty} \alpha_n \beta_n = \alpha \leqq 1.$$

Define the semi-norm $\| \ \|$ on L^1 by:

$$\|f\| = \sum_{n=1}^{\infty} \alpha_n \left| \int_X \phi_n(x) f(x) \mu(dx) \right|$$

Clearly $\|f\| \leqq \alpha \|f\|_1$, and $\| \ \|$ defines the topology of vague convergence on D_1. Let ρ be the metric induced by $\| \ \|$.

LEMMA 1. *Let $D \subset D_1$ be a weakly compact set. Then the weak topology of L_1, restricted to D, is defined by $\| \ \|$.*

Proof. Since D is weakly compact, it is vaguely compact. Hence, given any sequence $\{f_n\} \subset D$, there exists a subsequence, also labelled by n, such that for each $\phi \in C(X)$,

$$\int_X \phi(x) f_n(x) \mu(dx) \longrightarrow \int_X \phi(x) f(x) \mu(dx)$$

for some $f \in D$, i.e., $\|f_n - f\| \longrightarrow 0$ as $n \longrightarrow \infty$. We claim that $f_n \longrightarrow f$ weakly. Suppose that this is not the case. Then there exists a subsequence $\{f_{n_k}\}$ such that for some $\epsilon > 0$,

$$\left| \int_X h(x) f_{n_k}(x) \mu(dx) - \int_X h(x) f(x) \mu(dx) \right| > \epsilon$$

But $\{f_{n_k}\}$ is weakly compact. Thus there exists a further subsequence $\{f_{n_{k'}}\}$ such that $f_{n_{k'}} \longrightarrow f'$ weakly, which implies that for any $\phi \in C(X)$,

$$\int_X \phi(x) f_{n_{k'}}(x) \mu(dx) \longrightarrow \int_X \phi(x) f'(x) \mu(dx)$$

But

$$\int_X \phi(x) f_{n_{k'}}(x) \mu(dx) \longrightarrow \int_X \phi(x) f(x) \mu(dx)$$

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Hence,

$$\int_X \phi(x)f'(x)\mu(dx) = \int_X \phi(x)f(x)\mu(dx)$$

for all $f \in C(X)$. Hence $f = f'$ a.e., and we have a contradiction. Thus vague convergence in $\| \ \|$ implies weak convergence. Since weak convergence obviously implies vague convergence, we have the desired result. ∎

The main result of this note is that under certain conditions, the operator P, when viewed as a map from the metric space (D, σ) into the compact metric space (D, ρ) has the generalized shadowing property. An important example of such operators are the Frobenius-Perron operators induced by expanding and certain non-expanding maps of the interval.

Notice that although the map for $S : X \longrightarrow X$ may not have the shadowing property, the Frobenius-Perron operator corresponding to S, $P_S : L^1 \longrightarrow L^1$, may have the generalized shadowing property. For example, consider the tent map $S : [0, 2] \longrightarrow [0, 2]$, defined by:

$$S(x) = \begin{cases} \sqrt{2}x, & 0 \leqq x \leqq 1 \\ \sqrt{2}(2 - x), & 1 \leqq x \leqq 2 \end{cases}$$

It is shown in [**2**] that S does not have the shadowing property. However, we will see that P_S has the generalized shadowing property with respect to the metrics σ and ρ.

2. The shadowing property in the space of densities. Let (X, ρ) be a compact metric space. For $A \subset X$, closed, let

$$A_\epsilon = \{x \in X : \inf_{y \epsilon A} \rho(x, y) < \epsilon\}$$

be an ϵ-neighbourhood of A. In the sequel we shall require the following elementary stability result.

LEMMA 2. *Let $P : (X, \rho) \longrightarrow (X, \rho)$ be continuous. Assume there exists a closed set $A \subset X$ such that $PA = A$ and that*

$$\rho(P^i x,\ A) = \inf_{y \in A} \rho(P^i x,\ y) \longrightarrow 0 \tag{1}$$

as $i \longrightarrow \infty$ uniformly for all $x \in X$. Then for any $\epsilon > 0$ and any $\delta > 0$ $\exists$ a positive integer $N \ni P^n A_{\epsilon+\delta} \subset A_\epsilon$ for $n \geqq N$.

Proof. We know from [**6**] that there exists a neighbourhood U of A such that $P(U) \subset U$. We can assume $U \subset A_\epsilon$. Now it is enough to prove that for some positive integer N we have $P^N(A_{\epsilon+\delta}) \subset U$. Let $d = \inf\{\rho(x,y) : x \in A, y \notin U\}$. From (1), we obtain that there exists an N such that for any $n \geqq N$ and any $x \in X$ we have $\rho(P^n x, A) < d/2$. Hence $P^N(A_{\epsilon+\delta}) \subset A_{d/2} \subset U$, and the proof is complete. ■

We assume that $P : X \longrightarrow X$ is continuous in both metrics ρ and σ. We denote the modulii of continuity with respect to ρ and σ we denote by ω and η, respectively:

$$\omega(t) = \sup\{\rho(Px,\ Py) : x,\ y \in X,\ \rho(x,y) \leqq t\};$$

$$\eta(t) = \sup\{\sigma(Px,\ Py) : x,\ y \in X,\ \sigma(x,y) \leqq t\}.$$

For a modulus of continuity γ and $s, t > 0$, we define:

$$t_1 = s;$$

$$t_{k+1} = \gamma(t_k) + t \qquad \text{for } k = 1,\ 2, \ldots$$

We put $\Omega(\gamma, s, t, N) = \max\{t_1, t_2, \ldots, t_{N-1}\}$, where N is a positive integer. Note, that if $\gamma(t) \leqq t$, then $\Omega(\gamma, s, t, N) \leqq s + (N-1)t$.

LEMMA 3. *Let $P : X \longrightarrow X$ be continuous in both metrics ρ and σ. If P^N has the (δ, ϵ) generalized shadowing property, then P has the (δ_1, ϵ_1) generalized shadowing property, where δ_1 is chosen to satisfy $\Omega(\eta, \delta_1, \delta_1, N) < \delta$ and $\epsilon_1 = \Omega(\omega, \epsilon, \delta_1, N)$. Note, that if $\eta(t) \leqq t$ and $\omega(t) \leqq t$, then we can take $\delta_1 = \delta/N$ and $\epsilon_1 = \epsilon + \delta$. If P has the (δ, ϵ) generalized shadowing property, then P^N also has the (δ, ϵ) generalized shadowing property.*

Proof. We remark that $\delta_1 > 0$ satisfying $\Omega(\eta, \delta_1, \delta_1, N) < \delta$ can always be found because $\eta(t) \longrightarrow 0$ as $t \longrightarrow 0$. Now let $\{x_0, x_1, \ldots\}$ be a δ_1-pseudo-orbit for P. We will prove first that the sequence $\{x_0, x_N, x_{2N}, \ldots\}$ forms a δ-pseudo-orbit for P^N. Let us fix a nonnegative k and let $M = kN$. We have $\sigma(Px_M, x_{M+1}) < \delta_1$ so $\sigma(P^2 x_M, Px_{M+1}) < \eta(\delta_1)$. Since $\sigma(Px_{M+1}, x_{M+2}) < \delta_1$, we get

$$\sigma(P^2 x_M,\ x_{M+2}) < \eta(\delta_1) + \delta_1.$$

Repeating this reasoning $N-1$ times, we obtain

$$\sigma(P^N x_{kN}, x_{(k+1)N}) < \Omega(\eta, \delta_1, \delta_1, N).$$

Therefore $\{x_0, x_N, x_{2N}, \ldots\}$ is a δ-pseudo-orbit for P^N.

Since P^N has the (δ, ϵ) generalized shadowing property, there exists a point $y \in X$ such that for any positive integer k we have $\rho(P^{Nk} y, x_{kN}) < \epsilon$. We shall prove the existence of an ϵ_1 such that for any k and any $1 \leqq j \leqq N-1$,

$$\rho(P^{Nk+j} y, x_{kN+j}) < \epsilon_1. \tag{2}$$

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Now, we have $\rho(P^{Nk}y, x_{kN}) < \epsilon$. By the definition of ω, we obtain $\rho(P^{Nk+1}y, Px_{kN}) < \omega(\epsilon)$. Since $\{x_0, x_1, \ldots\}$ is a δ_1-pseudo-orbit for P, we obtain: $\rho(P^{Nk+1}y, x_{kN+1}) < \omega(\epsilon) + \delta_1$. We have therefore proved (2) for $j = 1$. Continuing in this way, we obtain (2) for all $j \leqq N - 1$.

To prove the second part of the lemma, we proceed as follows: If $\{x_0, x_1, x_2, \ldots\}$ is a δ-pseudo-orbit for P^N, then

$$\{x_0,\ Px_0,\ P^2x_0, \ldots, P^{N-1}x_0,\ x_1,\ Px_1, \ldots, P^{N-1}x_1,\ x_2, \ldots\} \tag{3}$$

is the δ-pseudo-orbit for P. There exists $y \in X$ such that the orbit $\{y, Py, P^2y, \ldots\}$ approximates the pseudo-orbit (3) within ϵ. It is obvious that the orbit $\{y, P^Ny, P^{2N}, \ldots\}$ approximates the orbit $\{x_0, x_1, x_2, \ldots\}$ within ϵ. ∎

LEMMA 4. *Let σ be the norm metric in L^1 and ρ the metric of weak convergence in D, defined above. Then, for all $f, g \in D$, we have $\rho(f, g) \leqq \sigma(f, g)$.*

Proof. The proof follows from the definition of the semi-norm $\|\ \|$ and the fact that $\alpha \leqq 1$. ∎

Let D be a compact set of (D_1, ρ). An example of such a set is $D_g = \{f \in D_1 : f(x) \leqq g(x)\}$, where g is an L^1 function, or any bounded set in L^p, $p > 1$.

THEOREM 1. *Let $P : L^1 \longrightarrow L^1$ be a constrictive Markov operator with the attractor A consisting of a single element f^* of a ρ-compact set $D \subset D_1$. Assume $PD \subset D$. If*

$$\lim_{n \to \infty} \|P^n f - A\|_1 = 0$$

uniformly for all $f \in D$, then $P : (D, \sigma) \longrightarrow (D, \rho)$ has the generalized shadowing property (with respect to the metrics σ and ρ).

Proof. Fix $\epsilon > 0$. By Lemma 2 there exists an integer $N_0 > 0$ such that $P^{N_0}(D) \subset A_\epsilon$. Let $\delta = \epsilon/N_0$. Let N be the smallest positive integer such that $P^N(A_{2\epsilon}) \subset A_\epsilon$. Let $\bar{P} = P^N$ and $k = [(N_0 - 1)/N] + 1$, where $[t]$ is the greatest integer $\leqq t$. We have $k \leqq N_0$, $kN \geqq N_0$ and $\bar{P}^k f_0 \in A_\epsilon$ for any $f_0 \in D$.

Let us consider any δ-pseudo-orbit, $\{f_0, f_1, f_2, \ldots\}$ for $\bar{P}$ starting from a point $f_0 \in D$, $\sigma(\bar{P}f_i, f_{i+1}) < \delta$, for all i. Lemma 4 implies that we also have:

$$\rho(\bar{P}f_i,\ f_{i+1}) < \delta, \qquad i = 0,\ 1,\ 2, \ldots \tag{4}$$

By induction it can be proved that

$$\rho(\bar{P}^j f_0,\ f_j) < j\delta, \qquad \text{for any } 1 \leqq j \leqq k,$$

since we have

$$\sigma(\bar{P}^{j+1} f_0,\ f_{j+1}) \leqq \sigma(\bar{P}(\bar{P}^j f_0),\ \bar{P}f_j) + \sigma(\bar{P}f_j,\ f_{j+1}) \leqq j\delta + \delta.$$

By Lemma 4, we get, for $j = 0, 1, \ldots, k$,

$$\rho(\bar{P}^j f_0, f_j) \leqq k\delta \leqq \epsilon. \tag{5}$$

So far we have shown that the δ-pseudo-orbit (in σ) $\{f_0, f_1, f_2, \ldots\}$ and the true orbit $\{\bar{P}^j f_0\}$ stay close to each other for the first k iterates. It remains to show that this is the case for all other iterates.

By the definition of N, $\bar{P}A_{2\epsilon} \subset A_\epsilon$, and by the definition of k, $\bar{P}^k f_0 \in A_\epsilon$. Now (5) implies $f_k \in A_{2\epsilon}$. Thus the definition of N once again implies that $\bar{P}f_k \in A_\epsilon$, and by (4), we have

$$f_{k+1} \in A_{\epsilon+\delta} \subset A_{2\epsilon}.$$

Therefore, since $\bar{P}^{k+1} f_0 \in A_\epsilon$, $f_{k+1} \in A_{\epsilon+\delta}$, and since A consists of a singleton,

$$\rho(\bar{P}^{k+1} f_0,\ f_{k+1}) < 2\epsilon + \delta < 3\epsilon.$$

We can repeat the reasoning inductively and combining this with (5), we get: $\sigma(\bar{P}f_j, f_{j+1}) < \delta$ for all $j \geqq 0$ implies that $\rho(\bar{P}^j f_0, f_j) < 3\epsilon$ for all $j \geqq 1$. Thus we have established the generalized shadowing property for P. By Lemma 3, we have it for P. ∎

Although it appears that the assumption $P(D) \subset D$ is restrictive, we shall show that there exists a natural family of weakly compact sets D with this property.

Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mathcal{B}$ a countably generated σ-algebra of measurable sets. If P is a Markov operator on $L^1 = L^1(X, \mathcal{B}, \mu)$ then there exists a transition function [**16**, section V.4], $P(\ ,\ )$, which is a measurable function in the first variable and a measure in the second variable, such that P is the unique operator satisfying:

$$\int (Pf)g\, d\mu = \int f(x) \left\{ \int P(x,\ dy) g(y) \right\} \mu(dx)$$

for all $f \in L^1$ and $g \in L^\infty$, i.e., P is the operator adjoint to the operator

$$Tg(x) = \int P(x,\ dy) g(y).$$

Now we assume that μ is a $P(\ ,\ )$ invariant measure, i.e.,

$$\mu(B) = \int P(x, B)\mu(dx), \qquad B \in \mathcal{B},$$

or, equivalently,

$$\int g(x)\mu(dx) = \int\int P(x, dy)g(y)\mu(dx), \qquad g \in L^\infty.$$

We shall prove that $P(D_M) \subset D_M$, where $D_M = \{f \in L^p : \|f\|_p \leqq M\}$ and $p > 1$. D_M is, of course, a weakly compact set in L^1.

For q such that $1/p + 1/q = 1$, we have:

$$\begin{aligned}\|Pf\|_p &= \sup\left\{\left|\int (Pf)g\, d\mu\right| : \|g\|_q \leqq 1\right\} \\ &= \sup\left\{\left|\int f(x)\left(\int P(x, dy)g(y)\right)\mu(dx)\right| : \|g\|_q \leqq 1\right\}\end{aligned}$$

so it is enough to prove that the operator T is a contraction in L^q. By Jensen's Inequality and the $P(\ ,\ )$ invariance of μ, we have:

$$\begin{aligned}\int\left|\int P(x, dy)g(y)\right|^q \mu(dx) &\leqq \int\int P(x, dy)|g(y)|^q\mu(dx) \\ &= \int |g(x)|^q\mu(dx).\end{aligned}$$

This ends the proof. ■

Using the above representation of the operator P, we can associate a special metric with P. Let $\{\phi_n\}$ be a countable dense subset of $C(X)$. We define the metric ρ as follows:

$$\rho(f, g) = \sum_{n=1}^{\infty}\alpha_n\sum_{k=0}^{\infty}c^k\left|\int (T^k\phi_n)(f-g)\, d\mu\right|,$$

where $0 < c < 1$ and

$$\sum_{n=1}^{\infty}\alpha_n\beta_n \leqq 1 - c,$$

where $\beta_n = \sup|\phi_n|$.

The metric ρ gives the weak L_1 topology on any weakly compact set in L_1. We shall now show that P is Lipschitz continuous with the constant $1/c$. Consider

$$\begin{aligned}\rho(Pf, Pg) &= \sum_{n=1}^{\infty}\alpha_n\sum_{k=0}^{\infty}c^k\left|\int (T^k\phi_n)(Pf-Pg)\, d\mu\right| \\ &= \sum_{n=1}^{\infty}\alpha_n\sum_{k=0}^{\infty}c^k\left|\int (T^{k+1}\phi_n)(f-g)\, d\mu\right| \\ &= 1/c\sum_{n=1}^{\infty}\alpha_n\sum_{k=1}^{\infty}c^k\left|\int (T^k\phi_n)(f-g)\, d\mu\right| \leqq 1/c\ \rho(f, g).\end{aligned}$$

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If we use the metric ρ associated with P, the number ϵ_1 obtained in Lemma 3 can be expressed more explicitly.

Remark 1. If ρ is the special metric associated with the operator P, then in Lemma 3 we can take

$$\epsilon_1 = \epsilon b^{N-1} + (\delta/N)(1 + b + \cdots + b^{N-1}) = \epsilon b^{N-1} + \frac{\delta(1 - b^N)}{N(1 - b)},$$

where $b = 1/c$.

In many practically important situations we have exponential convergence of the iterates to the invariant density. In such cases, we can express δ in terms of ϵ.

PROPOSITION 1. *Let us assume that there exist $H > 0$ and $0 < q < 1$ such that for any $f \in D$*

$$\|P^n f - f^*\|_1 < Hq^n.$$

If ρ is the metric associated with P, then P has the (δ, ϵ) generalized shadowing property with

$$\delta \simeq \text{const } (\eta)\epsilon^{(1+\eta)} \tag{6}$$

as $\epsilon \longrightarrow 0$, for any $\eta > 0$.

Proof. First, we obtain a bound on N_0 from the proof of Theorem 1. We want $\|P^{N_0} f - f^*\|_1 < \epsilon$ so it is enough to take $N_0 = [(\log_q(\epsilon/H)] + 1$. By Theorem 1, Lemma 3, and Remark 1, we know that P has the $(\tilde{\delta}, \tilde{\epsilon})$ generalized shadowing property with $\tilde{\delta} = 3\epsilon/NN_0$ and

$$\tilde{\epsilon} = 3\epsilon b^{N-1} + \frac{3\epsilon(b^N - 1)}{NN_0(b - 1)}$$

Since $N < N_0$, and b can be chosen as near to 1 as we like, we have

$$\tilde{\delta} \simeq \text{const } \epsilon(\log_q(\epsilon/H))^{-2} \simeq \text{const } \epsilon^{(1+\eta_1)}$$

and

$$\tilde{\epsilon} \simeq \text{const } \epsilon(\epsilon/H)^{\log_q b}(\log_q(\epsilon/H))^{-2} \simeq \text{const } \epsilon^{(1-\eta_2)},$$

where η_1, η_2 are positive and arbitraly small real numbers. Thus, for $\eta = (\eta_1 + \eta_2)/(1 - \eta_2)$ we obtain $\tilde{\delta} \simeq \text{const } \tilde{\epsilon}^{(1+\eta)}$. ∎

Remark 2. Since $\tilde{\delta}$ is the precision of the pseudo-orbit, it is of practical significance to know how it is related to $\tilde{\epsilon}$. If we desire a true orbit to be within $\tilde{\epsilon}$ of a $\tilde{\delta}$-pseudo-orbit, (6) tells us what $\tilde{\delta}$ must be.

3. Frobenius-Perron operator. Let (X, B, μ) be a measure space, $S : X \longrightarrow X$ a nonsingular transformation, i.e., $\mu(S^{-1}E) = 0$ for all $E \in B \ni \mu(E) = 0$. The unique operator $P_S : L^1 \longrightarrow L^1$, defined by

$$\int_E P_S f(x)\mu(dx) = \int_{S^{-1}E} f(x)\mu(dx)$$

for $E \in B$, is called the Frobenius-Perron operator corresponding to S. It is easy to show that P_S is a linear operator; $P_S f \geqq 0$ if $f \geqq 0$; $P_{S^n} = P_S^n$, where P_{S^n} is the Frobenius-Perron operator corresponding to S^n; and

$$\int_X P_S f(x)\mu(dx) = \int_X f(x)\mu(dx).$$

Clearly P_s is a Markov operator. The adjoint to the Frobenius-Perron operator is the Koopman operator:

$$U(g) = g \circ S,$$

where $g \in L^\infty$. In other words, we have

$$\int (Pf)(x)g(x)\mu(dx) = \int f(x)g(Sx)\mu(dx)$$

for any $f \in L^1$ and any $g \in L^\infty$.

For the Frobenius-Perron operator of S, the metric ρ associated with P_S is given by

$$\rho(f,\ g) = \sum_{n=1}^{\infty} \alpha_n \sum_{k=0}^{\infty} c^k \left| \int (\phi_n \circ S^k)(x)(f-g)(x)\mu(x) \right|.$$

Definition 4. Let (X, B, μ) be a probability space and $S : X \longrightarrow X$ a nonsingular, measure preserving transformation, i.e., $\mu(S^{-1}E) = \mu(E)$ for every $E \in B$. Assume $S(E) \in B$ for every $E \in B$. If

$$\lim_{n \to \infty} \mu(S^n E) = 1$$

for every $E \in B$, $\mu(E) > 0$, S is called μ-exact.

The connection between exactness of S and P_S is expressed in the following result [**3**, Theorem 4.4.1]:

PROPOSITION 2. *Let (X, B, μ) be a probability space and $S : X \longrightarrow X$ a non-singular transformation. Assume there exists a unique $f^* \in D \ni P_S f^* = f^*$, where P_s is the Frobenius-Perron operator corresponding to S. Then S is μ-exact, where μ is the measure whose density is f^*, if and only if for every $f \in D$,*

$$\lim_{n \to \infty} \|P^n f - f^*\|_1 = 0.$$

Since P_S is Markov, Proposition 1 states that P_S is a constrictive Markov operator if S is μ-exact. We can now apply the results of section 2, to obtain:

PROPOSITION 3. *Let (X, B, μ) be a probability space, and let $S : X \longrightarrow X$ be μ-exact, where μ is an absolutely continuous invariant measure with density f^*. Let D be a weakly compact subset of D_1, and assume that $P_S D \subset D$. Assume that $\|P_S^n f - f^*\|_1 \longrightarrow 0$ as $n \longrightarrow \infty$ uniformly with respect to $f \in D$. Then $P_S : (D, \sigma) \longrightarrow (D, \rho)$ has the generalized shadowing property.*

4. Examples

Example 1. (Expanding Maps of the Unit Interval)
Let $I = [0, 1]$. For a function $\tilde{f} : I \longrightarrow \mathbb{R}$, set

$$\text{var}(\tilde{f}) \equiv \sup_{a_0 < a_1 < \cdots < a_n \in I} \sum_{i=1}^{n} |\tilde{f}(a_i) - \tilde{f}(a_{i-1})|,$$

and for an equivalence class $f \in L^1$, define

$$V(f) \equiv \inf \{\text{var}(\tilde{f}) : \tilde{f} \in f\}$$

Let $BV \equiv \{f \in L^1 : V(f) < \infty\}$, and for $f \in BV$, define

$$\|f\|_V = V(f) + \|f\|_1$$

$\|f\|_V$ is a norm on BV, which makes $(BV, \| \ \|_V)$ into a Banach space. BV is a dense linear subspace of $(L^1, \| \ \|_1)$ and $\{f \in BV\colon \|f\|_V \leqq 1\}$ is a $\| \ \|_1$ compact subset of L^1. In the sequel we shall not distinguish between a function and its equivalence class.

Following [**9**], we denote by $\mathcal{S}$ the class of Markov operators $P : L^1 \longrightarrow L^1$ which satisfy the following condition: $P(BV) \subset BV$ and there exist constants $\lambda > 1$, $c > 0$, and a positive integer k such that $\|P\|_V < \infty$ and

$$\|P^k f\|_V \leqq \frac{1}{\lambda} \|f\|_V + C\|f\|_1$$

for $f \in BV$. The subclass of $\mathcal{S}$ satisfying the foregoing condition for a specific λ and C is called $\mathcal{S}(\lambda, C)$.

Let $\mathcal{E}$ denote the class of expanding, piecewise C^1 maps $S : I \longrightarrow I$ which satisfy the condition that $\frac{1}{S'_{|I_i}}$ is of bounded variation, where I_i is any interval of the defining partition. In [11] it is shown that P_S belongs to $\mathcal{S}$. From the ergodic theorem of Ionescu Tulcea and Marinescu [**12**], it follows that the operators in $\mathcal{S}$ are quasi-compact as operators on $(BV, \| \ \|_V)$. Thus, P_S has only finitely many eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ of modulus 1; the corresponding eigenspaces E_i are finite dimensional subspaces of BV and P_S has the following representation:

$$P_S = \sum_{i=1}^{p} \lambda_i \Phi_i + Q,$$

where the Φ_i are projections onto the E_i, $\|\Phi_i\|_1 \leqq 1$, $\Phi_i \circ \Phi_i = 0$ $(i \neq j)$, and where $Q : L^1 \longrightarrow L^1$ is a linear subspace with $\sup_n \|Q^n\|_1 \leqq p+1$, $Q(BV) \subset BV$,

$$\|Q^n\|_V \leqq Hq^n \tag{7}$$

for some $0 < q < 1$ and $H > 0$, and $Q \circ \Phi_i = \Phi_i \circ Q = 0$ for all i. If $\dim(E_1) = 1$, P_S is ergodic. P_S is mixing if P_S is ergodic and 1 is the only eigenvalue of P_S of modulos 1.

Let $S \in \mathcal{E}$. Then from the main result of [**7**], there exists a constant K, independent of f, such that

$$\limsup_{n \longrightarrow \infty} VP_S^n f \leqq K$$

for every $f \in D$ of bounded variation. Let $D = \{f \in D_1 : Vf \leqq K'\}$, where K' is any number greater than or equal to K. Then D is weakly compact. Let us assume that S admits a unique absolutely continuous invariant measure μ with respect to which it is exact (or even only mixing). Let f^* be the density of this measure. Then from (7) it follows that convergence to f^* is uniform with respect to all $f \in D$:

$$\|P_S^n f - f^*\|_V \leqq Hq^n$$

where $H > 0$ and $0 < q < 1$ are independent of $f \in D$. Hence

$$\|P_S^n f - f^*\|_1 \leqq Hq^n$$

and we have uniform convergence in the σ metric to f^*.

In view of Proposition 2, we have the generalized shadowing property for $P_S : (D, \sigma) \longrightarrow (D, \rho)$. In fact in this setting, it is easy to show that $P_S : (D, \| \ \|_1) \longrightarrow (D, \| \ \|_1)$ has the shadowing property [**10**].

A trivial example of a map $P_S \in \mathcal{S}$ is the case where S is any triangle map with slope > 1 in absolute value.

Example 2. (Random Maps of the Interval)

Let $S_1, S_2, \ldots, S_m$ be maps of I and define a "random map" S by $S(x) = S_i(x)$ with probability p_i. A measure μ is called S-invariant if

$$\mu(A) = \sum_{i=1}^{m} p_i \mu(S_i^{-1} A)$$

for each measurable set A. Assume each $S_i \in \mathcal{E}$. If for all $x \in I$,

$$\sum_{i=1}^{m} \frac{p_i}{|S_i'(x)|} \leqq \gamma < 1,$$

then it is shown in [**14**] that the Markov operator P_S defined by

$$P_S = \sum_{i=0}^{m} p_i P_{S_i}$$

satisfies, for all $f \in BV$,

$$VP_S f \leqq \alpha V f + K \|f\|_1 \tag{8}$$

for some $0 < \alpha < 1$ and $K > 0$, both independent of f. Hence

$$\begin{aligned} \|P_S f\|_V &= VP_S f + \|P_S f\|_1 \\ &\leqq \alpha V f + K\|f\|_1 + \|f\|_1 \\ &\leqq \alpha \|f\|_V + K' \|f\|_1 . \end{aligned}$$

Hence P_S satisfies (6) and is in $\mathcal{S}(\alpha, K')$.

If P_S is ergodic and mixing (see Cor. 7 of [**14**] and [**15**]), then we have the existence of a unique f^* such that

$$\|P_S^n f - f^*\|_1 \leqq H q^n$$

for all $f \in D = \{f \in D_1 : Vf \leqq K_1\}$, where $H > 0$ and $0 < q < 1$ are independent of f, and K_1 is any sufficiently large positive number. Hence

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Theorem 1 applies and the Markov operator P_S has the generalized shadowing property.

Example 3. (Markov Operators Defined by Kernels)
Consider the integral operator $P : L^1 \longrightarrow L^1$, defined by

$$Pu(x) = \int_0^1 b(x, y)u(y)\, dy,$$

where $b : (0,1)x(0,1) \longrightarrow [0,\infty)$ is measurable and stochastic, that is

$$\int_0^1 b(x, y)\, dx = 1$$

for $y \in (0,1)$. Assume that for some $B > 0$, $b(x,y) \leqq B$ for all x and $y \in (0,1)$. Then the operator P is constrictive [**11**]. Let $D = \{f \in D : f(x,y) \leqq B\}$. Then $PD \subset D$. Assume P has a unique fixed point $f^* \in D$. Then the convergence to f^* is uniform with respect to all $f \in D$ [**13**]. Hence Theorem 1 implies that P has the generalized shadowing property.

Let us moreover assume that there exist a positive integer N and $r > 0$ such that

$$r \leqq b^N(x,y)$$

for all $x \in X$ and y from some set of positive μ-measure, where

$$b^N(x, y) = \int \dots \int b(x, z_1)b(z_1, z_1)\dots b(z_{N-1}, y)\mu(dz_1)\dots\mu(dz_{N-1}).$$

Then the convergence to the invariant density f^* is exponential and all the results of section 2 apply.

5. Frobenius-Perron operators with respect to a non-invariant measure. In this section we treat a more general situation than in the previous sections. We will consider the Frobenius-Perron operators with respect to a non-invariant measure.

Let $(X, \mathcal{B}, m)$ be a measure space and $S : X \longrightarrow X$ a nonsingular transformation with respect to m, i.e., $m(B) = 0 \Rightarrow m(S^{-1}(B)) = 0$, for any $B \in \mathcal{B}$. Let P_S be the Frobenius-Perron operator with respect to the measure m induced by S.

In this case the assumption that $P_S(D) \subset D$ can be too restrictive, but we shall show that if D is a ball in some L^p space ($p > 1$), then we can find a smaller ball $\tilde{D}$ such that $P_S^n(\tilde{D}) \subset D$ for all $n \geqq 0$, and thus all our reasoning can be repeated.

5.1. Expanding maps of an interval. For Lasota-Yorke maps, piecewise convex maps [**3**, Theorem 6.3.1], and maps with $1/|S'|$ of bounded p-variation [**18**], the set $\{P_S^n(1) : n \geqq 1\}$ is bounded in L^∞. For any $f \in D_M = \{f \in L^p : \|f\|_p \leqq M\}$ and any $g \in L^q$ $(1/p + 1/q = 1)$, we have

$$\int (P_S^n f) g \, dm = \int f(g \circ S^n) \, dm$$

On the other hand,

$$\int |g \circ S^n|^q \, dm = \int |g|^q (P_S^n(1)) \, dm \leqq \sup_n \|P_S^n(1)\|_\infty \int |g|^q \, dm.$$

Thus there exists a constant K such that $\|P_S^n f\|_p \leqq KM$, for any positive integer n and any $f \in D_M$. This implies that $P_S^n(D_{M/2K}) \subset D_M$, for any n.

Hence all the results of sections 2 and 3 hold for the transformations considered here.

5.2. Non-expanding maps of Misiurewicz type. In this section we use the results of Szewc [**17**]. Let S be a Misiurewicz-type transformation of an interval I. Let C be the set of all "bad" points of S: singular points and endpoints of I. Let

$$B = \bigcup_{n \geqq 0} S^n(C)$$

and $B_0 = cl(B)$. By $\mathcal{J}$ we denote the partition of $I \setminus B_0$ into its connected components.

We define $C_{S,\epsilon}^{(0)+1}$ as a space of functions which are defined on $I \setminus B_0$ and are Lipschitz continuous on any compact subset of any $J \in \mathcal{J}$. The norm in $C_{S,\epsilon}^{(0)+1}$ is defined as follows:

$$\|f\|_{(0)+1} = \max \{\|f\|_\epsilon, |f|_{(0)+1}\},$$

where

$$\|f\|_\epsilon = \sup_{J \in \mathcal{J}} \sup_J \frac{|f|}{\phi_\epsilon}$$

$$|f|_{(0)+1} = \sup_{J \in \mathcal{J}} \operatorname{essup}_J \frac{|f'|}{\phi_1}$$

and ϕ_ϵ, ϕ_1 are appropriate weight functions.

$C_{S,\epsilon}^{(0)+1}$ is a Banach space, it is P_S invariant, and any ball in it is compact in L_1. By Theorem 6.3 of [**17**], there exist constants $\Gamma > 0$ and $0 < \gamma < 1$ such that for any $f \in C_{S,\epsilon}^{(0)+1}$:

$$\|\Phi^n(f)\|_1 \leqq \Gamma\gamma^n \|f\|_{(0)+1} \qquad \text{for } n = 1, 2, \ldots,$$

where $\Phi^n(f) = P_S^n f - P_S^n(\Pi f)$ and Π is the projection on the space of invariant densities. This space is spanned by the finite number of ergodic densities. Thus P_S^n is a constrictive operator. In case there exists only one absolutely continuous invariant measure (S is exact) all the results of sections 2 and 3 apply.

6. Stability of the shadowing property in families of maps. In [**2**] it is shown that the family of tent maps have the shadowing property for almost all parameter values, although they fail to have the shadowing property for an uncountable dense set of parameters. This implies that there is no continuity in the shadowing property; a small change in S may result in the loss of the shadowing property. Clearly this renders such a result unpractical for the analysis of experimental or computational systems. In the space of probability density functions, the situation is dramatically different. In this section, we shall prove that for many families of maps the shadowing property is preserved as the parameter varies over its range.

As in [**9**], we employ the following Skorokhod-like metric on $\mathcal{E}$:

$$r(S_1, S_2) = \inf\Big\{\epsilon > 0 : \exists E \subseteq I \exists \eta : I \longrightarrow I \ni m(E) > 1 - \epsilon, \ \eta \text{ is a diffeomorphism, } S_{1|_E} = S_2 \circ \eta_{|_E}, \text{ and for all } x \in X, \ |\eta(x) - x| < \epsilon, \ \left|\frac{1}{\eta'(x)} - 1\right| < \epsilon\Big\}$$

The following result is contained in Corollary 14 of [**9**].

LEMMA 5. *Let $\{S_n\} \subset \mathcal{E}$ and let $S \subset \mathcal{E}$. Let P_n be the Frobenius-Perron operator corresponding to S_n. Assume $\{P_n\} \subset \mathcal{S}(\lambda, C)$ for some $\lambda > 1$ and $C > 0$. If $r(S_n, S) \longrightarrow 0$ as $n \longrightarrow \infty$ and S is ergodic, then S_n is ergodic for n sufficiently large and the unique invariant densities of S_n converge in L^1 to that of S.*

Let $\{S_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathcal{E}$ be a family of maps ($\mathcal{A}$ is the parameter space) satisfying the following conditions:

(i) Let I be a fixed partition of I such that all S_α are piecewise C^2 with respect to this partition,

(ii) $|S'_\alpha(x)| \geqq \lambda > 1$ for all $\alpha \in \mathcal{A}$

(iii) There exists a real constant c such that

$$V\frac{1}{S'_{\alpha|_{I_i}}} \leqq c < \infty.$$

for all $\alpha \in \mathcal{A}$.

(iv) Let each S_α admit a unique absolutely continuous invariant measure μ_α on I.

We can now state the main result of this section.

THEOREM 2. (Stability of Shadowing Property). *Let* $\{S_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathcal{E}$ *satisfy conditions (i)–(iv). Assume that the map* $\alpha \longrightarrow S_\alpha$ *from* $(\mathcal{A}, |\ |) \longrightarrow (\mathcal{E}, r)$ *is continuous. Then for each* $\alpha_0 \in \mathcal{A} \exists$ *a neighbourhood* $\mathcal{N}$ *of* $\alpha_0 \ni$ *for each* $\epsilon > 0 \exists \delta > 0$ *and every* δ*-pseudo- orbit (in* σ*) can be* ϵ*-shadowed by a true orbit (in* ρ*) uniformly for all* $\alpha \in \mathcal{N}$ *i.e., if* $\{f_0, f_1, \ldots\}$ *satisfies* $\sigma(P_\alpha f_i, f_{i+1}) < \delta$ *for any* $\alpha \in \mathcal{N}$*, then* $\rho(P_\alpha^i f_0, f_i) < \epsilon$ *for all* $\alpha \in \mathcal{N}$*. (We refer to this property as the stability of shadowing property.)*

Proof. Condition (iv) implies that each S_α is μ_α-exact. Hence P_α is a constrictive Markov operator. The results of Example 1 show that the convergence is uniform for f in the weakly compact set $D = \{f \in D_1 : Vf \leqq K\}$, for K a large positive number.

From the proof of Theorem 1 of [**7**], it is easy to see that conditions (i) and (ii) imply the existence of $\lambda > 1$ and $C > 0$, both independent of $\alpha \ni$

$$\|P_\alpha^k f\|_V \leqq \frac{1}{\lambda}\|f\|_V + C\|f\|_1$$

Hence $\{P_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathcal{S}(\lambda, C)$. Let f_α denote the density of μ_α. Then Lemma 5 implies that the map $\alpha \longrightarrow f_\alpha$ from $(\mathcal{A}, |\ |) \longrightarrow (D, \sigma)$ is continuous, where $|\ |$ denote the absolute value norm. Fix $\alpha_0 \in \mathcal{A}$. Then given $\in > 0 \ \exists$ a ν-neighbourhood $\mathcal{N}_\nu \subset \mathcal{A}$ of α_0 such that $\alpha \in \mathcal{N}_\nu$ implies $\|f_\alpha - f_\alpha\|_1 < \delta$. Now, repeating the arguments of the proof of Theorem 1, we get the desired shadowing property of P_α uniformly for $\alpha \in \mathcal{N}_\nu$. ∎

Example 4. Consider the family of tent maps $S_\beta : I \longrightarrow I$, defined by

$$S_\beta(x) = \begin{cases} \beta x, & 0 \leqq x \leqq 1/2 \\ \beta(1-x), & 1/2 \leqq x \leqq 1 \end{cases}$$

where $\beta \in \mathcal{A} \equiv [1+\omega, 2]$, $\omega > 0$. Clearly $|S'_\beta(x)| \geqq 1 + \omega > 1$, and all the S_β have the same two intervals in their partition. Since each S_β has only one turning point in its partition, there is a unique absolutely continuous invariant measure. Furthermore, since S_β is piecewise linear, condition (iii) is satisfied for

$c = 0$. Therefore, conditions (i)–(iv) are fulfilled for this family. The continuity of $\beta \longrightarrow S_\beta$ from $(\mathcal{A}, |\ |) \longrightarrow (\mathcal{E}, r)$ is easy to prove. Thus, all the conditions of Theorem 2 are satisfied for the family of tent maps. Hence the generalized shadowing property is stable for this family.

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