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ON THE VANISHING OF LOCAL HOMOLOGY MODULES

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Abstract. Let R be a commutative Noetherian ring, $\mathfrak{a}$ is an ideal of R and M is an R-module. We intend to establish the dual of Grothendieck's Vanishing Theorem for local homology modules. We conjecture that $H_i^{\mathfrak{a}}(M) = 0$ for all $i > \operatorname{mag}_R M$. We prove this in several cases.

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1. Introduction. Throughout this paper, R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ is an ideal of R and M is an R-module.

The theory of local cohomology has developed much in six decades since its introduction by Grothendieck. But, its dual theory, the theory of local homology has not developed much. The theory of local homology was initiated by Matlis [**9**] in 1974. The study of this theory was continued by Simon in [**15**] and [**16**]. A new era in the study of local homology has started after Greenlees and May [**7**] and Alonso Tarrío et al. [**1**]; see for instance [**5**, **4**, **6**, **11**, **14**].

The most essential vanishing result for the local cohomology modules $H_{\mathfrak{a}}^i(M)$ is Grothendieck's Vanishing Theorem, which asserts that $H_{\mathfrak{a}}^i(M) = 0$ for all $i > \dim_R M$. There is no satisfactory dual of this result for local homology modules. To have such a dual, one should first have an appropriate dual notion of Krull dimension. There are two dual notions in literature: Noetherian dimension, $\operatorname{Ndim}_R M$, and magnitude, $\operatorname{mag}_R M$; see [**12**] and [**17**]. There are two partial duals of Grothendieck's Vanishing Theorem. If M is linearly compact with $\operatorname{Ndim}_R M = d$, then by [**5**, Theorem 4.8], $H_i^{\mathfrak{a}}(M) = 0$ for all $i > d$. Also, by [**11**, Proposition 4.2] if M is either finitely generated or Artinian, then $H_i^{\mathfrak{a}}(M) = 0$ for all $i > \sup\{\dim R/\mathfrak{p} | \mathfrak{p} \in \operatorname{Cosupp}_R M\}$.

When R is complete local, we show that $\operatorname{mag}_R M \leq \operatorname{Ndim}_R M$ with the equality if $M \neq 0$ and it is either N-critical or semi-discrete linearly compact. So for having a sharper upper bound for vanishing of the local homology modules $H_i^{\mathfrak{a}}(M)$, $\operatorname{mag}_R M$ could be a better candidate. In fact, we conjecture that $H_i^{\mathfrak{a}}(M) = 0$ for all $i > \operatorname{mag}_R M$. Our investigation on this conjecture is the core of this paper. We show this conjecture in several cases.

Namely, we prove that if
$\operatorname{Coass}_R M = \mathcal{A}tt_R M$, or

M is finitely generated, Artinian or Matlis reflexive, or

M is linearly compact, or

R is complete local and M has finitely many minimal co-associated prime ideals, or

R is complete local with the maximal ideal $\mathfrak{m}$ and $\mathfrak{m}^n M$ is minimax for some integer $n \geq 0$, then $H_i^{\mathfrak{a}}(M) = 0$ for all $i > \mathrm{mag}_R M$.

Zöschinger [19] has conjectured that any module over a local ring has finitely many minimal co-associated prime ideals. Thus, by fourth case, over a complete local ring R, Zöschinger's conjecture implies our conjecture.

2. The results. In what follows, we denote the faithful exact functor $\mathrm{Hom}_R(-, \underset{\mathfrak{m}\in \mathrm{Max}R}{\oplus} E(R/\mathfrak{m}))$ by $(-)^\vee$. Let M be an R-module. A prime ideal $\mathfrak{p}$ of R is said to be a co-associated prime ideal of M if there is an Artinian quotient L of M such that $\mathfrak{p} = (0 :_R L)$. The set of all co-associated prime ideals of M is denoted by $\mathrm{Coass}_R M$. Also, $\mathcal{A}tt_R M$ is defined by

$$\mathcal{A}tt_R M := \{\mathfrak{p} \in \mathrm{Spec} R |\ \mathfrak{p} = (0 :_R L) \text{ for some quotient } L \text{ of } M\}.$$

Clearly, $\mathrm{Coass}_R M \subseteq \mathcal{A}tt_R M$ and the equality holds if either R or M is Artinian. More generally, if M is representable, then it is easy to check that $\mathrm{Coass}_R M = \mathcal{A}tt_R M$. If $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ is an exact sequence of R-modules and R-homomorphisms, then it is easy to check that

$$\mathrm{Coass}_R L \subseteq \mathrm{Coass}_R N \subseteq \mathrm{Coass}_R L \cup \mathrm{Coass}_R M$$

and

$$\mathcal{A}tt_R L \subseteq \mathcal{A}tt_R N \subseteq \mathcal{A}tt_R L \cup \mathcal{A}tt_R M.$$

Also, if R is local, then one can see that $\mathrm{Coass}_R M = \mathrm{Ass}_R M^\vee$.

LEMMA 2.1. *Let M be an R-module. Then $\mathcal{A}tt_R M = \mathcal{A}tt_R M^{\vee\vee}$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R and X, Y are two R-modules. There are natural isomorphisms $(X/\mathfrak{p}X)^\vee \cong (0 :_{X^\vee} \mathfrak{p})$ and $(0 :_Y \mathfrak{p})^\vee \cong Y^\vee/\mathfrak{p}Y^\vee$. Hence, $(X/\mathfrak{p}X)^{\vee\vee} \cong X^{\vee\vee}/\mathfrak{p}X^{\vee\vee}$. Since $(-)^\vee$ is a faithfully exact functor, we deduce that

$$\mathrm{Ann_R}(\mathrm{X}/\mathfrak{p}\mathrm{X}) = \mathrm{Ann_R}((\mathrm{X}/\mathfrak{p}\mathrm{X})^{\vee\vee}) = \mathrm{Ann_R}(\mathrm{X}^{\vee\vee}/\mathfrak{p}\mathrm{X}^{\vee\vee}).$$

On the other hand, one can easily check that $\mathfrak{p} \in \mathcal{A}tt_R X$ if and only if $\mathfrak{p} = \mathrm{Ann_R}(\mathrm{X}/\mathfrak{p}\mathrm{X})$. This yields that $\mathcal{A}tt_R M = \mathcal{A}tt_R M^{\vee\vee}$, as required. □

Let $\mathfrak{a}$ be an ideal of R and $\mathcal{C}_0(R)$ denote the category of R-modules and R-homomorphisms. It is known that the $\mathfrak{a}$-adic completion functor

$$\Lambda_{\mathfrak{a}}(-) := \varprojlim_n (R/\mathfrak{a}^n \otimes_R -) : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$$

is not right exact in general. For any integer i, the ith local homology functor with respect to $\mathfrak{a}$ is defined as ith left derived functor of $\Lambda_{\mathfrak{a}}(-)$. For an R-module M, set $\mathrm{cd_{\mathfrak{a}}M} := \sup\{\mathrm{i} | \mathrm{H^i_{\mathfrak{a}}(M)} \neq 0\}$. By [7, Corollary 3.2], $H_i^{\mathfrak{a}}(M) = 0$ for all $i > \mathrm{cd_{\mathfrak{a}}R}$.

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LEMMA 2.2. *Let $\mathfrak{a}$ be an ideal of R and M an R-module. Then $H_i^{\mathfrak{a}}(M) = 0$ for all*

$$i > \sup\{\dim R/\mathfrak{p} | \mathfrak{p} \in \mathcal{A}tt_R M\}.$$

Proof. For any R-module N, let $d_N := \sup\{\dim R/\mathfrak{p}|\mathfrak{p} \in \mathcal{A}tt_R N\}$. If $d_N \geq \mathrm{cd}_{\mathfrak{a}}\mathrm{R}$, then [**7**, Corollary 3.2] implies that $H_i^{\mathfrak{a}}(N) = 0$ for all $i > d_N$. Hence, it is enough to show that for any R-module N with $d_N < \mathrm{cd}_{\mathfrak{a}}\mathrm{R}$, one has $H_i^{\mathfrak{a}}(N) = 0$ for all $d_N < i \leq \mathrm{cd}_{\mathfrak{a}}\mathrm{R} + 1$. We do decreasing induction on i. Clearly, the claim holds for $i = \mathrm{cd}_{\mathfrak{a}}\mathrm{R} + 1$. Now assume that $d_N < i < \mathrm{cd}_{\mathfrak{a}}\mathrm{R} + 1$ and that the claim holds for $i + 1$. We have to show that $H_i^{\mathfrak{a}}(N) = 0$. We have an exact sequence

$$0 \longrightarrow N \longrightarrow N^{\vee\vee} \longrightarrow C \longrightarrow 0,$$

which yields the long exact sequence

$$\cdots \longrightarrow H_{i+1}^{\mathfrak{a}}(N^{\vee\vee}) \longrightarrow H_{i+1}^{\mathfrak{a}}(C) \longrightarrow H_i^{\mathfrak{a}}(N) \longrightarrow H_i^{\mathfrak{a}}(N^{\vee\vee}) \longrightarrow \cdots.$$

By [**7**, Lemma 3.7] one has $H_j^{\mathfrak{a}}(N^{\vee\vee}) \cong H_{\mathfrak{a}}^j(N^{\vee})^{\vee}$ for all $j \geq 0$. It is easy to see that $\mathrm{Ass}_R N^{\vee} \subseteq \mathcal{A}tt_R N^{\vee\vee}$, and so by Lemma 2.1, $\dim_R N^{\vee} \leq d_N$. Thus, by Grothendieck's Vanishing Theorem, one has

$$H_{i+1}^{\mathfrak{a}}(N^{\vee\vee}) = 0 = H_i^{\mathfrak{a}}(N^{\vee\vee}),$$

and so $H_{i+1}^{\mathfrak{a}}(C) \cong H_i^{\mathfrak{a}}(N)$. From the above short exact sequence and Lemma 2.1, one has $\mathcal{A}tt_R C \subseteq \mathcal{A}tt_R N$. Hence, $d_C \leq d_N < i + 1$, and so by induction hypothesis,

$$H_i^{\mathfrak{a}}(N) \cong H_{i+1}^{\mathfrak{a}}(C) = 0.$$

Thus, the claim follows by induction. □

Next, we recall the definitions of $\mathrm{Ndim}_R M$ and $\mathrm{mag}_R M$.

DEFINITION 2.3. Let M be an R-module.

(i) (See [**12**]) The Noetherian dimension of M is defined inductively as follows: When $M = 0$, put $\mathrm{Ndim}_R M = -1$. Then by induction, for an integer $d \geq 0$, we put $\mathrm{Ndim}_R M = d$ if $\mathrm{Ndim}_R M < d$ is false and for every ascending sequence $M_0 \subseteq M_1 \subseteq \ldots$ of submodules of M, there exists n_0 such that $\mathrm{Ndim}_R M_{n+1}/M_n < d$ for all $n > n_0$.

(ii) (See [**17**]) The magnitude of M is defined by $\mathrm{mag}_R M := \sup\{\dim R/\mathfrak{p}|\mathfrak{p} \in \mathrm{Coass}_R M\}$. If $M = 0$, we put $\mathrm{mag}_R M = -\infty$.

(iii) (See [**11**]) The co-localization of M at a prime ideal $\mathfrak{p}$ of R is defined by

$${}^{\mathfrak{p}}M := \mathrm{Hom}_{R_{\mathfrak{p}}}((M^{\vee})_{\mathfrak{p}}, E_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})).$$

Then $\mathrm{Cosupp}_R M$ is defined by $\mathrm{Cosupp}_R M := \{\mathfrak{p} \in \mathrm{Spec} R | {}^{\mathfrak{p}}M \neq 0\}$.

(iv) (See [**3**]) M is said to be N-critical if $\mathrm{Ndim}_R N < \mathrm{Ndim}_R M$ for all proper submodules N of M.

It becomes clear from the definition that $\mathrm{Ndim}_R M = 0$ if and only if M is a non-zero Noetherian R-module. If $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is an exact sequence of R-modules and R-homomorphisms, then [**3**, Proposition 5] yields

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that $\mathrm{Ndim}_R Y = \max\{\mathrm{Ndim}_R X, \mathrm{Ndim}_R Z\}$. Also, it is easy to verify that $\mathrm{mag}_R Y = \max\{\mathrm{mag}_R X, \mathrm{mag}_R Z\}$.

Next, we compare $\mathrm{Ndim}_R M$ and $\mathrm{mag}_R M$. Recall that an R-module M is said to be *Matlis reflexive* if the natural homomorphism $M \longrightarrow M^{\vee\vee}$ is an isomorphism.

LEMMA 2.4. *Let M be an R-module.*

(i) *Suppose R is complete local. Then $\mathrm{mag}_R M \leq \mathrm{Ndim}_R M$ and equality holds if $M \neq 0$ and it is either N-critical or Matlis reflexive.*

(ii) $\mathrm{mag}_R M \leq \sup\{\dim R/\mathfrak{p} | \mathfrak{p} \in \mathrm{Cosupp}_R M\}$ *and equality holds if R is local.*

Proof. (i) Let $\mathfrak{p} \in \mathrm{Coass}_R M$. Then there is an Artinian $\mathfrak{p}$-secondary quotient M/N of M such that $\mathfrak{p} = \mathrm{Ann_R M/N}$. By [**3**, Proposition 5], we have $\mathrm{Ndim}_R M = \max\{\mathrm{Ndim}_R N, \mathrm{Ndim}_R M/N\}$. On the other hand, for any Artinian R-module A, [**17**, Theorem 2.10] asserts that $\mathrm{mag}_R A = \mathrm{Ndim}_R A$. (Note that the argument of [**17**, Theorem 2.10] is not correct without the completeness assumption on R.) Since $\mathfrak{p}$ is the only co-associated prime ideal of M/N, it turns out that

$$\dim R/\mathfrak{p} = \mathrm{mag}_R M/N = \mathrm{Ndim}_R M/N \leq \mathrm{Ndim}_R M.$$

Thus,

$$\mathrm{mag}_R M = \sup\{\dim R/\mathfrak{p} | \mathfrak{p} \in \mathrm{Coass}_R M\} \leq \mathrm{Ndim}_R M.$$

Now assume that $M \neq 0$. Let M be N-critical and let $\mathfrak{p}$ and N be as above. Then the $\mathrm{Ndim}_R N < \mathrm{Ndim}_R M$, and so

$$\begin{aligned} \mathrm{Ndim}_R M &= \max\{\mathrm{Ndim}_R N, \mathrm{Ndim}_R M/N\} \\ &= \mathrm{Ndim}_R M/N \\ &= \mathrm{mag}_R M/N \\ &\leq \mathrm{mag}_R M. \end{aligned}$$

Next, let M be Matlis reflexive. Then by [**2**, Theorem 12] there is a finitely generated submodule N of M such that M/N is Artinian. Clearly, we may suppose that $M \neq 0$. Then,

$$\mathrm{Ndim}_R M = \max\{0, \mathrm{Ndim}_R M/N\} = \max\{0, \mathrm{mag}_R M/N\} \leq \mathrm{mag}_R M.$$

(ii) It suffices to show that $\mathrm{Coass}_R M \subseteq \mathrm{Cosupp}_R M$. Let $\mathfrak{p} \in \mathrm{Coass}_R M$. There is an Artinian quotient L of M such that $\mathfrak{p} = \mathrm{Ann_R L}$. Set $E := \oplus_{\mathfrak{m} \in \mathrm{Max} R} E(R/\mathfrak{m})$. Since L is Artinian, we may assume that $L \subseteq E^r$ for an integer $r \geq 0$. Let $f : M \longrightarrow E^r$ denote the composition of the natural epimorphism $M \longrightarrow L$ and the natural monomorphism $L \longrightarrow E^r$. Then $\mathfrak{p} = (0 :_R f)(:= \{a \in R | af : M \longrightarrow E^r$ is the zero homomorphism$\})$, and so

$$\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Hom}_R(M, E^r)) = \mathrm{Ass}_R M^{\vee} \subseteq \mathrm{Supp}_R M^{\vee}.$$

Hence, $(M^{\vee})_{\mathfrak{p}} \neq 0$, and so ${}^{\mathfrak{p}}M \neq 0$. This means that $\mathfrak{p} \in \mathrm{Cosupp}_R M$.

Now assume that R is local. Since $\mathrm{Cosupp}_R M = \mathrm{Supp}_R M^{\vee}$, [**17**, Lemma 2.2 a)] implies that $\mathrm{mag}_R M = \sup\{\dim R/\mathfrak{p} | \mathfrak{p} \in \mathrm{Cosupp}_R M\}$. □

Recall that an R-module M is said to be *semi-Artinian* if every proper submodule of M contains a minimal submodule; see for example [**13**].

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EXAMPLE 2.5. Let M be an R-module.

(i) Suppose that M is semi-Artinian with finitely many associated prime ideals. Then by [**19**, Bemerkung after Satz 2.9], one has $\mathrm{Coass}_R M = \mathcal{A}tt_R M$. But, in general, the containment $\mathrm{Coass}_R M \subseteq \mathcal{A}tt_R M$ may be strict. To this end, let $\mathfrak{p}$ be a non-maximal prime ideal of R and set $M := R/\mathfrak{p}$. Then $\mathrm{Coass}_R M = \mathrm{V}(\mathfrak{p}) \cap \mathrm{MaxR}$ and $\mathcal{A}tt_R M = \mathrm{V}(\mathfrak{p})$, and so $\mathrm{Coass}_R M \subsetneq \mathcal{A}tt_R M$.

(ii) The inequality in Lemma 2.4 (i) may be strict. To see this, let $(R, \mathfrak{m})$ be a local ring and $M := \oplus_{i\in\mathbb{N}} R/\mathfrak{m}$. Then $\mathrm{Coass}_R M = \{\mathfrak{m}\}$, and so $\mathrm{mag}_R M = 0$. But, as M is not a Noetherian R-module, we have $\mathrm{Ndim}_R M > 0$.

(iii) In Lemma 2.4 (i), the completeness assumption on R cannot be skipped. To see this, let $(R, \mathfrak{m})$ be a two-dimensional local domain such that $\hat{R}$ possesses a one-dimensional embedded prime ideal $\mathfrak{q}$; see [**10**, Appendix, Example 2]. Let $A := (0 :_{E(R/\mathfrak{m})} \mathfrak{q})$. Then $\mathrm{mag}_R A = 2$ and

$$\mathrm{Ndim}_R A = \mathrm{Ndim}_{\hat{R}} A = \mathrm{mag}_{\hat{R}} A = 1.$$

LEMMA 2.6. *Let $(R, \mathfrak{m})$ be a complete local ring, $\mathfrak{a}$ is an ideal of R and M is an R-module. Then $H_i^{\mathfrak{a}}(M) = 0$ for all $i > \mathrm{mag}_R M^{\vee\vee}$.*

Proof. The proof is similar to the proof of Lemma 2.2. We use decreasing induction on i. For $i \geq \dim R + 1$, the claim holds by [**7**, Corollary 3.2]. Note that $\mathrm{mag}_R M^{\vee\vee} \leq \dim R$. Now assume that $\mathrm{mag}_R M^{\vee\vee} < i < \dim R + 1$ and that the claim holds for $i + 1$. We have an exact sequence

$$0 \longrightarrow M \longrightarrow M^{\vee\vee} \longrightarrow C \longrightarrow 0,$$

which yields the long exact sequence

$$\cdots \longrightarrow H_{i+1}^{\mathfrak{a}}(M^{\vee\vee}) \longrightarrow H_{i+1}^{\mathfrak{a}}(C) \longrightarrow H_i^{\mathfrak{a}}(M) \longrightarrow H_i^{\mathfrak{a}}(M^{\vee\vee}) \longrightarrow \cdots .$$

By [**7**, Lemma 3.7], one has $H_j^{\mathfrak{a}}(M^{\vee\vee}) \cong H_{\mathfrak{a}}^j(M^{\vee})^{\vee}$ for all $j \geq 0$. Since

$$i > \mathrm{mag}_R M^{\vee\vee} = \dim_R M^{\vee\vee\vee} \geq \dim_R M^{\vee},$$

Grothendieck's Vanishing Theorem implies that

$$H_{i+1}^{\mathfrak{a}}(M^{\vee\vee}) = 0 = H_i^{\mathfrak{a}}(M^{\vee\vee}).$$

Hence, $H_{i+1}^{\mathfrak{a}}(C) \cong H_i^{\mathfrak{a}}(M)$. Also, from the above short exact sequence, we deduce that

$$\mathrm{mag}_R(M^{\vee\vee\vee\vee}) = \max\{\mathrm{mag}_R M^{\vee\vee}, \mathrm{mag}_R C^{\vee\vee}\}.$$

On the other hand, [**18**, Lemma 2.9 and Folgerung 2.10] yields that $\mathrm{Coass}_R(M^{\vee\vee\vee\vee}) = \mathrm{Coass}_R(M^{\vee\vee})$, and so

$$\mathrm{mag}_R C^{\vee\vee} \leq \mathrm{mag}_R(M^{\vee\vee\vee\vee}) = \mathrm{mag}_R M^{\vee\vee}.$$

Now, since $\mathrm{mag}_R C^{\vee\vee} < i + 1$, by induction hypothesis, it turns out that

$$H_i^{\mathfrak{a}}(M) \cong H_{i+1}^{\mathfrak{a}}(C) = 0.$$

Thus, the claim follows by induction. □

LEMMA 2.7. *Let $(R, \mathfrak{m})$ be a complete local ring and M is an R-module. Assume that M has finitely many minimal co-associated prime ideals. Then* $\operatorname{mag}_R M^{\vee\vee} = \operatorname{mag}_R M$.

Proof. Let $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ be the set of all elements of $\operatorname{Coass}_R M$ which are minimal with respect to inclusion in $\operatorname{Coass}_R M$. By [**18**, Satz 2.6], for any prime ideal $\tilde{p}$ of R, $\tilde{p} \in \operatorname{Coass}_R M^{\vee\vee}$ if and only if $\tilde{p} = \bigcap_{\mathfrak{p}\in\Lambda} \mathfrak{p}$ for some subset Λ of $\operatorname{Coass}_R M$. In particular, one has $\operatorname{Coass}_R M \subseteq \operatorname{Coass}_R M^{\vee\vee}$, and so $\operatorname{mag}_R M \leq \operatorname{mag}_R M^{\vee\vee}$. Also, it follows that any prime ideal $\tilde{p} \in \operatorname{Coass}_R M^{\vee\vee}$ contains $\bigcap_{i=1}^{n} \mathfrak{p}_i$.

Let $\mathfrak{q} \in \operatorname{Coass}_R M^{\vee\vee}$ be such that $\dim R/\mathfrak{q} = \operatorname{mag}_R M^{\vee\vee}$. Then $\mathfrak{q}$ is minimal in $\operatorname{Coass}_R M^{\vee\vee}$. Since $\mathfrak{q} \supseteq \bigcap_{i=1}^{n} \mathfrak{p}_i$, there is $1 \leq j \leq n$ such that $\mathfrak{q} \supseteq \mathfrak{p}_j$. But $\mathfrak{p}_j \in \operatorname{Coass}_R M^{\vee\vee}$, and so $\mathfrak{q} = \mathfrak{p}_j$. Thus, $\mathfrak{q} \in \operatorname{Coass}_R M$, and so $\operatorname{mag}_R M^{\vee\vee} \leq \operatorname{mag}_R M$. □

At this point, we are ready to present our main result. But we first recall some definitions which are needed in its statement.

We begin by recalling the definition of linearly compact modules from [**8**]. Let M be a topological R-module. Then M is said to be *linearly topologized* if M has a base $\mathcal{M}$ consisting of submodules for the neighbourhoods of its zero element. A Hausdorff linearly topologized R-module M is said to be *linearly compact* if for any family $\mathcal{F}$ of cosets of closed submodules of M, which has the finite intersection property, the intersection of all cosets in $\mathcal{F}$ is non-empty. A Hausdorff linearly topologized R-module M is called *semi-discrete* if every submodule of M is closed. The class of semi-discrete linearly compact modules is very large, it contains many important classes of modules such as the class of Artinian modules, or the class of finitely generated modules over a complete local ring.

An R-module M is called *minimax* if it has a finitely generated submodule N such that M/N is Artinian. By [**18**, Lemma 1.1], over a complete local ring R, an R-module M is minimax if and only if M is semi-discrete linearly compact and if and only if M is Matlis reflexive.

THEOREM 2.8. *Let $\mathfrak{a}$ be an ideal of R and M an R-module. Assume that either*

(i) $\operatorname{Coass}_R M = \mathcal{A}tt_R M$,
(ii) *M is N-critical,*
(iii) *M is finitely generated, Artinian or Matlis reflexive,*
(iv) *M is linearly compact,*
(v) *R is complete local and M has finitely many minimal co-associated prime ideals; or*
(vi) *R is complete local with the maximal ideal $\mathfrak{m}$ and $\mathfrak{m}^n M$ is minimax for some integer $n \geq 0$.*

Then $H_i^{\mathfrak{a}}(M) = 0$ for all $i > \operatorname{mag}_R M$.

Proof. (i) Follows from Lemma 2.2.

(ii) By [**3**, Proposition 2], M is a secondary module. This implies that $\operatorname{Coass}_R M = \mathcal{A}tt_R M$, and so (ii) follows from (i).

(iii) For a finitely generated R-module N, one has $\operatorname{mag}_R N \leq 0$ and [**15** Proposition 3.2] yields that $H_i^{\mathfrak{a}}(N) = 0$ for all $i > 0$. When M is Artinian, the claim follows from (i). Now assume that M is Matlis reflexive. We may and do assume that M is non-zero.

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By [**2**, Theorem 12], there is a finitely generated submodule N of M such that M/N is Artinian. Then

$$\mathrm{mag}_R M = \max\{\mathrm{mag}_R N, \mathrm{mag}_R M/N\} = \max\{0, \mathrm{mag}_R M/N\}.$$

Thus, the claim follows by the following long exact sequence

$$\cdots \longrightarrow H^{\mathfrak{a}}_{i+1}(M/N) \longrightarrow H^{\mathfrak{a}}_i(N) \longrightarrow H^{\mathfrak{a}}_i(M) \longrightarrow H^{\mathfrak{a}}_i(M/N) \longrightarrow \cdots .$$

(iv) Let $\mathcal{M}$ be a base consisting of submodules for the neighbourhoods of the zero element of M. Then by [**8**, 3.11], $M \cong \varprojlim_{U\in\mathcal{M}} M/U$. By [**5**, Proposition 3.4], it turns out that $H^{\mathfrak{a}}_i(M) \cong \varprojlim_{U\in\mathcal{M}} H^{\mathfrak{a}}_i(M/U)$. Note that for each U, M/U is a semi-discrete linearly compact R-module with $\mathrm{mag}_R M/U \leq \mathrm{mag}_R M$. So we only need to prove the claim for the case M is a semi-discrete linearly compact R-module. Let M be a semi-discrete linearly compact R-module. Then [**20**] implies that M is Matlis reflexive. Thus (iii) completes the proof of this part.

(v) Follows by Lemmas 2.6 and 2.7.

(vi) By [**13**, Theorem 3.3] and [**20**, Lemma 2.2 (d)], there is a submodule N of M such that $\mathrm{Coass}_R N \subseteq \{\mathfrak{m}\}$ and the quotient module M/N is Artinian. Hence, $\mathrm{Coass}_R M$ is finite, and so (v) yields the conclusion. □

Let $(R, \mathfrak{m})$ be a local ring. Zöschinger [**19**] conjectured that any R-module M has finitely many minimal co-associated prime ideals. Clearly, this is equivalent to, say that for any R-module M, $\mathrm{Supp}_R M^{\vee}$ is a Zariski-closed subset of $\mathrm{Spec} R$.

Remark 2.9. Let M be an R-module and in the first three parts, suppose that R is local.

(i) By [**19**, Folgerung 1.5] if $\mathrm{Coass}_R M$ is countable, then M has finitely many minimal co-associated prime ideals. Hence, if R is countable, or $\mathrm{Spec} R$ is finite (e.g. $\dim R \leq 1$), or M is Matlis reflexive, or M is representable (e.g. injective), then M has finitely many minimal co-associated prime ideals.

(ii) Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules such that for each λ, M_λ has finitely many minimal co-associated prime ideals. Then both $\bigoplus_{\lambda\in\Lambda} M_\lambda$ and $\prod_{\lambda\in\Lambda} M_\lambda$ have finitely many minimal co-associated prime ideals; see [**19**, Satz 2.6] and [**18**, Satz 2.8 (b)].

(iii) If $\mathrm{Coass}_R M = \mathcal{A}tt_R M$, then by [**19**, Lemma 3.1] M has finitely many minimal co-associated prime ideals. Hence, for any infinite index set Λ and any R-module X, the R-modules $X^{(\Lambda)}$ and X^{Λ} have finitely many minimal co-associated prime ideals; see [**19**, Bemerkung after Satz 2.4].

(iv) In view of Lemma 2.4 (ii), clearly Theorem 2.8 (iii) extends [**11**, Proposition 4.2].

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