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Combinatorially Factorizable Cryptic Inverse Semigroups

Mario Petrich

Abstract. An inverse semigroup S is combinatorially factorizable if $S = TG$ where T is a combinatorial (*i.e.,* $\mathcal{H}$ is the equality relation) inverse subsemigroup of S and G is a subgroup of S. This concept was introduced and studied by Mills, especially in the case when S is cryptic (*i.e.,* $\mathcal{H}$ is a congruence on S). Her approach is mainly analytical considering subsemigroups of a cryptic inverse semigroup.

We start with a combinatorial inverse monoid and a factorizable Clifford monoid and from an action of the former on the latter construct the semigroups in the title. As a special case, we consider semigroups that are direct products of a combinatorial inverse monoid and a group.

1 Generalities

An inverse semigroup S is *factorizable* if $S = TG$ where T is a semilattice and G is a group. There is a modest literature concerning the structure of these semigroups, see [1]. This concept was generalized to include the case when T is a combinatorial inverse monoid by Mills [1] under the label of *combinatorially factorizable* inverse semigroups. In [1], she successfully analyzed such inverse semigroups as subsemigroups of cryptic inverse semigroups and some related cases. In her study appear two types of semigroups as essential ingredients: combinatorial inverse monoids and factorizable inverse monoids. The structure of the former is still an enigma, while for the latter we have the following.

Proposition 1.1 ([1, Proposition 3.1]) *A cryptic inverse monoid S is factorizable if and only if S is a Clifford monoid in which all structure homomorphisms are surjective.*

Hence we can say that, within cryptic inverse monoids, factorizable ones are completely determined. Analyzing subsemigroups of cryptic inverse monoids S that are susceptible to being of the form TG, where T is a combinatorial inverse monoid and G is a group, in [1] Mills arrived at a factorizable Clifford submonoid A of S having its semilattice of idempotents in common with T. In terms of structure, she arrived at a submonoid of a semidirect product of A and T.

We denote by $E(S)$ the set of all idempotents of any semigroup S. Taking a hint from the cited paper, and starting with a combinatorial inverse monoid T and a factorizable Clifford monoid A with group of units G, we construct a subsemigroup of a semidirect product of A and T which is a monoid of the form $S = TG$. To start with, we need not have $E(T) = E(A)$, but if this is not the case, we still must have

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$E(T) \cong E(A)$. In such a case, we can replace either T or A by an isomorphic copy to satisfy this condition, and thus we will assume that $E(T) = E(A)$, which will simplify the notation. This leads to Theorem 2.2 from which we derive Theorem 6.2 giving conditions for the direct product $T \times G$.

For terminology and notation we follow [2], with the exception of $E(S)$ for any semigroup S. We also write $\mathcal{E}(S)$ for the semigroup of endomorphisms of S written as operators on the left. If S is a monoid, we denote by $U(S)$ its groups of units, and its identity usually by 1.

We first clarify the difference between the semigroup and the monoid case.

Proposition 1.2 *Let $S = TG$ where T is an inverse semigroup and G is a group. If* 1 *is the identity of either S or T, then* 1 *is the identity of S, T, and G.*

Proof Denote by e the identity of G.

Let 1 be the identity of S. Then $1 = tg$ for some $t \in T$ and $g \in G$. Hence $e = 1e = tge = tg = 1$. It follows that $e = tg$, whence $g^{-1} = te = t1 = t$ and thus

$$1 = tgtg = tgg^{-1}g = tg = tt^{-1} \in T.$$

Now let 1 be the identity of T. First $e = tg$ for some $t \in T$ and $g \in G$ whence $e = 1e$. Also $1 = t'g'$ for some $t' \in T$ and $g' \in G$ so that $1e = 1$ and thus $e = 1$. For any $s = tg$, we get $s = 1s = se = s1$, and 1 is then identity of S. ■

We will need the following result.

Proposition 1.3 *Let S be an inverse semigroup and $H = \bigcup_{e\in E(S)} H_e$. Then the following conditions on S are equivalent.*

(i) *S is cryptic.*
(ii) *H is contained in the centralizer of $E(S)$.*
(iii) *H is closed under multiplication.*

Proof See [4, Theorem 4]. ■

If $S = TG$ where T is a semilattice and G is a group, neither S nor T need be monoids. Just let $S = T$ be a 3-element nonmonoid semilattice and $G = \{1\}$. Then $S = TG$ but neither S nor T is a monoid.

2 The Main Result

The following is our basic device.

Construction Let A be a factorizable Clifford monoid and T be a combinatorial inverse monoid such that

(A) $E(A) = E(T)\ (= E)$.

Let T act on A by endomorphisms on the left satisfying:

(B) ${}^{t}e = tet^{-1}$ $(t \in T, e \in E)$;
(C) ${}^{e}a = ea$ $(e \in E, a \in A)$;
(D) if $a \in A$, $t \in T$, and $aa^{-1} = tt^{-1}$, then $a = aa^{-1} \cdot {}^{t}g$ for some $g \in U(A)$.

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Set

$$S = \{(a,t) \in A \times T \mid aa^{-1} = tt^{-1}\}$$

with multiplication

$$(a,t)(b,u) = (a \cdot {}^{t}b, tu).$$

The first property of this construction is as follows.

Lemma 2.1 *S is a subsemigroup of the semidirect product of A and T determined by the given action.*

Proof Let $(a,t),(b,u) \in S$. Then $aa^{-1} = tt^{-1}$, $bb^{-1} = uu^{-1}$, and since $tuu^{-1}t^{-1} \in E(A)$, by Proposition 1.3, we get

$$\begin{aligned}(a \cdot {}^{t}b)(a \cdot {}^{t}b)^{-1} &= (a \cdot {}^{t}b)({}^{t}b)^{-1}a^{-1} = a \cdot {}^{t}b \cdot {}^{t}(b^{-1})a^{-1} \\ &= a \cdot {}^{t}(bb^{-1})a^{-1} = atbb^{-1}t^{-1}a^{-1} = atuu^{-1}t^{-1}u^{-1} \\ &= aa^{-1}tuu^{-1}t^{-1} = tt^{-1}tuu^{-1}t^{-1} = (tu)(tu)^{-1}.\end{aligned}$$

Hence S is closed under the above multiplication. Since the semidirect product is associative, S is a semigroup. ∎

The action of T on A is equivalent to the existence of a homomorphism $\zeta\colon T \to \mathcal{E}(A)$. Hence we denote the above semigroup S by

$$[A, \zeta, T].$$

We are now ready for the main result of the paper.

Theorem 2.2 *The semigroup $S = [A, \zeta, T]$ is a cryptic inverse monoid satisfying $S = T'G = GT'$ where $T' \cong T$ and $G \cong U(A)$. Conversely, every combinatorially factorizable cryptic inverse monoid is isomorphic to one so constructed.*

3 Proof of Theorem 2.2: Direct Part

This will follow from the next four lemmas, in which we include further information about $[A, \zeta, T]$ to clarify its structure. In them, $S = [A, \zeta, T]$, where we freely use conditions (A)–(D). The first lemma concerns general properties of this construction.

Lemma 3.1 *Denote by 1 the identity of T. Then S is an inverse monoid with identity $(1,1)$ and*

(3.1) $$E(S) = \{(e,e) \mid e \in E\} \cong E.$$

For $s = (a,t) \in S$, we have

(3.2) $$s^{-1} = ({}^{t^{-1}}a^{-1}, t^{-1}), \quad ss^{-1} = (tt^{-1}, tt^{-1}), \quad s^{-1}s = (t^{-1}t, t^{-1}t).$$

Proof Simple verification will show that $(1,1)$ is the identity of S and that if $e \in E$, then $(e,e) \in E(S)$. Conversely, let $(a,t) \in E(S)$. Then

$$(a,t) = (a,t)^2 = (a \cdot {}^{t}a, t^2)$$

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so that $t = t^2 = tt^{-1}$ and thus

$$a = a \cdot {}^{t}a = ata = att^{-1}a = aaa^{-1}a = a^2 = aa^{-1} = tt^{-1} = t \in E,$$

proving the equality in (3.1). For $e, f \in E$, we have

$$(e,e)(f,f) = (e \cdot {}^{e}f, ef) = (ef, ef)$$

which implies the isomorphism in (3.1).

For $s = (a,t) \in S$, we have

$$({}^{t^{-1}}a^{-1})({}^{t^{-1}}a^{-1})^{-1} = ({}^{t^{-1}}a^{-1})({}^{t^{-1}}a) = {}^{t^{-1}}(a^{-1}a) = t^{-1}a^{-1}at = t^{-1}tt^{-1}t = t^{-1}t$$

and thus $({}^{t^{-1}}a^{-1}, t^{-1}) \in S$. Straightforward verification will show that

$$s({}^{t^{-1}}a^{-1})s = s, \quad ({}^{t^{-1}}a^{-1})s({}^{t^{-1}}a^{-1}) = {}^{t^{-1}}a^{-1}. \tag{3.3}$$

It follows that S is a regular semigroup. By (3.1), $E(S)$ is a semilattice so that S is an inverse semigroup. We have mentioned above that S is a monoid. Now (3.3) implies the first formula in (3.2); the second and the third formulas in (3.2) are verified by direct multiplication. ∎

The second lemma handles the relationship of A and S.

Lemma 3.2 *The mapping*

$$\varphi\colon a \longmapsto (a, aa^{-1}), \quad a \in A,$$

embeds A into S and maps E onto $E(S)$. With $A' = A\varphi$, we have $A' = \bigcup_{e \in E} H_{(e,e)}$.

Proof The first assertion is readily verified; the second follows from Lemma 3.1.

Let $s = (a,t) \in A'$. Then $t = aa^{-1} = tt^{-1} = t^{-1}t$ and by Lemma 3.1, we get $ss^{-1} = s^{-1}s = (a, aa^{-1})$ and thus $s \in H_{(aa^{-1},aa^{-1})}$. Conversely, let $s = (a,t) \in H_{(e,e)}$, where $e \in E$. Then $ss^{-1} = s^{-1}s = (e,e)$ and so, by Lemma 3.1, we get $tt^{-1} = t^{-1}t = e$. Hence $t = e$ since T is combinatorial and thus $t = aa^{-1}$. Therefore $s \in A'$. ∎

The third lemma relates maximal subgroups.

Lemma 3.3 *The group of units of S equals*

$$G = \{(a,1) \mid a \in U(A)\}. \tag{3.4}$$

For every $e \in E$, we have $H_{(e,e)} = (e,e)G$.

Proof For $e \in E$ and $s = (a,t) \in S$, by Lemma 3.1, we obtain

$$s\mathcal{H}e \Longleftrightarrow ss^{-1} = s^{-1}s = (e,e) \Longleftrightarrow tt^{-1} = t^{-1}t = e \Longleftrightarrow t = e$$

since T is combinatorial, and thus

$$H_{(e,e)} = \{(a,e) \in S \mid aa^{-1} = e\}. \tag{3.5}$$

If $e = 1$, we get (3.4) which by (3.5) implies $H_{(e,e)} = (e,e)G$. ∎

The fourth lemma treats the relationship of T and S. Recall that a subsemigroup T of an inverse semigroup S is a *transversal* of $\mathcal{H}$ if each $\mathcal{H}$-class of S contains exactly one element of T.

Lemma 3.4 *The mapping*

$$\psi\colon t \longmapsto (tt^{-1}, t), \quad t \in T,$$

embeds T into S. For $(a,t),(b,u) \in S$, we have

$$(a,t)\mathcal{H}(b,u) \Longleftrightarrow t = u. \tag{3.6}$$

Moreover, $T' = T\psi$ is a transversal of $\mathcal{H}$, $\mathcal{H}$ is a congruence on S, and $S/\mathcal{H} \cong T$.

Proof The first assertion requires simple verification. Let $s = (a,t), q = (b,u) \in S$. Using Lemma 3.1, we get

$$s\mathcal{H}q \Leftrightarrow ss^{-1} = qq^{-1}, \quad s^{-1}s = q^{-1}q \Leftrightarrow tt^{-1} = uu^{-1}, \quad t^{-1}t = q^{-1}q \Leftrightarrow t = u$$

since T is combinatorial. This proves (3.6) and implies that $\mathcal{H}$ is a congruence on S. Further, $(tt^{-1}, t) \in T'$ and thus $(a,t)\mathcal{H}(tt^{-1}, t)$. Hence every $\mathcal{H}$-class of S contains at least one element of T'; by (3.6), it contains at most one element of T'. Therefore T' is a transversal of $\mathcal{H}$. It follows that $T \cong T' \cong S/\mathcal{H}$. ∎

We can now complete the proof of the direct part of Theorem 2.2. By Lemma 3.1, S is an inverse monoid and by Lemma 3.4, S is cryptic. Let $(a,t) \in S$. By hypothesis, there exists $g \in U(A)$ satisfying $a = aa^{-1} \cdot {}^{t}g$. Hence

$$(tt^{-1}, t)(g, 1) = (tt^{-1} \cdot {}^{t}g, t) = (aa^{-1} \cdot {}^{t}g, t) = (a,t),$$

where $(tt^{-1}, t) \in T'$ and $(g,1) \in G$, which shows that $S \subseteq T'G$, and equality follows.

4 Proof of Theorem 2.2: Converse Part

Let S be a cryptic inverse monoid such that $S = TG$ where T is a combinatorial inverse monoid and G is a group. Denote by 1 the identity of S. By Proposition 1.2, the element 1 is the identity of both T and G. If $e \in E(S)$, then $e = tg$ for some $t \in T$ and $g \in G$ so that $e = (tg)(tg)^{-1} = tgg^{-1}t^{-1} = tt^{-1}$ and thus $e \in T$. Let

$$E = E(S) = E(T).$$

We break the argument into seven lemmas, starting with the group units of S.

Lemma 4.1 $G = U(S)$.

Proof By Proposition 1.2, the element 1 is the identity of G. If $g \in G$, then $gg^{-1} = g^{-1}g = 1$ so that $g \in U(S)$, and thus $G \subseteq U(S)$. Conversely, let $a \in U(S)$. Then $aa^{-1} = a^{-1}a = 1$, and also $a = tg$ for some $t \in T$ and $g \in G$. Then

$$1 = aa^{-1} = tgg^{-1}t^{-1} = tt^{-1},$$

and using Proposition 1.3, we get

$$1 = a^{-1}a = g^{-1}t^{-1}tg = t^{-1}t,$$

whence $1\mathcal{H}t$ and thus $1 = t$ since T is combinatorial. Therefore $a = g \in G$ whence $U(S) \subseteq G$ and equality prevails. ∎

Next we characterize idempotent $\mathcal{H}$-classes of S.

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Lemma 4.2 *For every $e \in E$, we have $H_e = eG$.*

Proof Let $a \in H_e$. Then $a = tg$ for some $t \in T$ and $g \in G$. Then

$$e = aa^{-1} = tg(tg)^{-1} = tgg^{-1}t^{-1} = tt^{-1},$$

and using Proposition 1.3, we obtain

$$e = a^{-1}a = (tg)^{-1}tg = g^{-1}t^{-1}tg = t^{-1}t$$

and thus $e\mathcal{H}t$. Since T is combinatorial, we get $e = t$. Hence $a = eg \in eG$ and thus $H_e \subseteq eG$. Conversely, let $a = eg$ where $g \in G$. Then $eg \in eS$,

$$eg = a = ae \in Se, \quad e = e1 = egg^{-1} \in egS, \quad e = 1e = g^{-1}ge = g^{-1}eg \in Seg,$$

and thus $a = eg\mathcal{H}e$. Therefore $eG \subseteq H_e$ and equality prevails. ∎

Next we introduce a Clifford semigroup.

Lemma 4.3 *Let $A = \bigcup_{e \in E} H_e$. Then A is a factorizable Clifford monoid contained in the centralizer of E, with $G = U(A)$ and $E(A) = E$.*

Proof By Proposition 1.3, A is a Clifford semigroup contained in the centralizer of E; it is a monoid by Proposition 1.2; by Lemma 4.1, $G = U(S) = U(A)$; by Lemma 4.2, A is factorizable. Obviously $E(A) = E(S)$ so that $E(A) = E$. ∎

In view of Lemma 4.3, for $a \in A$ and $e \in E$, we have $ae = ea$ since A is a Clifford semigroup and we do not have to invoke Proposition 1.3.

The next result is [1, Proposition 1.3(ii)] which is stated there without proof. We will use it in the continuation of the proof of the converse of Theorem 2.2.

Lemma 4.4 *$S = GT$.*

Proof Let $g \in G$ and $t \in T$. We consider the element $s = tg$. Hence $s = tgt^{-1}t$ since g commutes with idempotents. Next,

$$tgt^{-1}(tgt^{-1})^{-1} = tgt^{-1}tg^{-1}t^{-1} = tgg^{-1}t^{-1} = tt^{-1},$$
$$(tgt^{-1})^{-1}tgt^{-1} = tg^{-1}t^{-1}tgt^{-1} = tg^{-1}gt^{-1} = t^{-1}t,$$

which shows that $tgt^{-1}\mathcal{H}tt^{-1}$ and thus $tgt^{-1} \in A$. By Lemma 4.3, A is a factorizable Clifford monoid. Hence there exist $e \in E(A)$ and $h \in U(A) = G$ such that $tgt^{-1} = eh$. It follows that $s = (tgt^{-1})t = eht = h(et)$ where $et \in T$. Therefore $S \subseteq GT$ and thus $S = GT$. ∎

The next lemma concerns the monoid T.

Lemma 4.5 *For every $t \in T$, define*

$$\zeta_t \colon a \longmapsto tat^{-1}, \quad a \in A,$$

and then define

$$\zeta \colon t \longmapsto \zeta_t, \quad t \in T.$$

Then $\zeta \colon T \to \mathcal{E}(A)$ is a homomorphism.

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Proof Let $t \in T$ and $a \in A$. Then

$$(tat^{-1})(tat^{-1})^{-1} = tat^{-1}ta^{-1}t^{-1} = taa^{-1}t^{-1},$$
$$(tat^{-1})^{-1}(tat^{-1}) = ta^{-1}t^{-1}tat^{-1} = ta^{-1}at^{-1},$$

and tat^{-1} commutes with its inverse which implies that tat^{-1} is an idempotent $\mathcal{H}$-class, and thus $tat^{-1} \in A$. It follows that $\zeta_t \colon A \to A$. For $a, b \in A$ and $t \in T$, we obtain

$$(\zeta_t a)(\zeta_t b) = (tat^{-1})(tbt^{-1}) = tabt^{-1} = \zeta_t(ab)$$

and $\zeta_t \in \mathcal{E}(A)$. For $t, u \in T$ and $a \in A$, we get

$$\zeta_t\zeta_u a = \zeta_t(uau^{-1}) = t(uau^{-1})t^{-1} = (tu)a(tu)^{-1} = \zeta_{tu}a,$$

whence $\zeta_t\zeta_u = \zeta_{tu}$ and $\zeta \colon T \to \mathcal{E}(A)$ is a homomorphism. ∎

Next we arrive at $[A, \zeta, T]$.

Lemma 4.6 *Conditions* (A)–(D) *in the Construction at the beginning of Section* 2 *are fulfilled.*

Proof Condition (A) is satisfied in view of the preamble to the proof of the converse of Lemma 4.3. Lemma 4.5 implies the validity of conditions (B) and (C).

Let $a \in A$ and $t \in T$ be such that $aa^{-1} = tt^{-1}$. By Lemma 4.3, A is factorizable so that $a = he$ for some $h \in G$ and $e \in E$. By hypothesis, $ht = ug$ for some $u \in T$ and $g \in G$. Hence

$$ht(ht)^{-1} = htt^{-1}h^{-1} = hh^{-1}tt^{-1} = tt^{-1},$$
$$(ht)^{-1}ht = t^{-1}h^{-1}ht = t^{-1}t,$$

and thus $ht\mathcal{H}t$. Similarly $ug\mathcal{H}u$ so that $ht = ug$ implies that $t\mathcal{H}u$. But then $t\mathcal{H}u$ within T which is combinatorial and thus $t = u$. Hence $ht = tg$. Also $a = he$ implies

$$att^{-1} = heaa^{-1} = haa^{-1}e = ha^{-1}(ae) = ha^{-1}a = haa^{-1} = htt^{-1}.$$

Consequently,

$$aa^{-1} \cdot {}^t g = tt^{-1}(tgt^{-1}) = tgt^{-1} = htt^{-1} = att^{-1} = aa^{-1}a = a,$$

which proves condition (D). ∎

This lemma implies that $[A, \zeta, T]$ is defined. We have finally arrived at the desired isomorphism.

Lemma 4.7 *The mapping*

$$\chi \colon gt \longmapsto (gtt^{-1}, t), \quad g \in G, t \in T,$$

is an isomorphism of S onto $[A, \zeta, T]$.

Proof We show first that χ is single valued. Let $g, h \in G$ and $t, u \in T$ satisfy $gt = hu$. As in the preceding proof, we get $t\mathcal{H}gt = hu\mathcal{H}u$ and thus $t = u$. It follows that $gtt^{-1} = huu^{-1}$ and $t = u$, and thus χ is single valued. Since $gtt^{-1}(gtt^{-1})^{-1} = tt^{-1}$, χ maps S into $[A, \zeta, T]$.

Without the hypothesis $gt = hu$, we obtain

$$(4.1)\qquad (gt)\chi(hu)\chi = (gtt^{-1}, t)(huu^{-1}, u) = (gtt^{-1} \cdot {}^{t}(huu^{-1}), tu) \\ = (gthuu^{-1}t^{-1}, tu) = (gthu(tu)^{-1}, tu).$$

By Lemma 4.4, $th = kv$ for some $k \in G$ and $v \in T$. As above we get $t\mathcal{H}v$ and hence $t = v$ since T is combinatorial. Now

$$(4.2)\qquad (gthu)\chi = (gktu)\chi = \big(gktu(tu)^{-1}, tu\big)$$

and $gthu = gktu$ implies the equality of (4.1) and (4.2). Therefore χ is a homomorphism.

If $(gtt^{-1}, t) = (huu^{-1}, u)$, then $t = u$ and $gt = hu$, which shows that χ is injective.

Let $(a, t) \in [A, \zeta, T]$ so that $aa^{-1} = tt^{-1}$. By Lemma 4.3, A is factorizable and thus $a = ge$ for some $g \in G$ and $e \in E$. Hence $e = g^{-1}a$ and thus

$$gtt^{-1} = gaa^{-1} = g^2(g^{-1}a)a^{-1} = g^2ea^{-1} = g(ge)ea^{-1} = (ge)(ge)a^{-1} = a^2a^{-1} = a.$$

It follows that $(gt)\chi = (a, t)$. By Lemma 4.4, we have $S = GT$, and thus χ is also surjective. ■

We can now complete the proof of the converse part of Theorem 2.2. By Lemma 4.4, $GT = TG$, and thus χ is an isomorphism of S onto $[A, \zeta, T]$. The equality $GT = TG$ also implies the equality $GT' = T'G$ in the direct part of the theorem.

5 Comments

We could have used some of the results in [1] to design a shorter proof of Theorem 2.2, but a direct proof is by far clearer.

Corollary 5.1 *A combinatorially factorizable cryptic inverse monoid is embeddable into a semidirect product of a factorizable Clifford monoid and a combinatorial inverse monoid.* ■

Let S be a cryptic inverse monoid with $S = TG$ where T is a combinatorial inverse monoid and G is a group. We have seen in Lemma 3.4 that $S/\mathcal{H} \cong T$ and in Lemma 4.1 that $G \cong U(S)$. It then follows that, in the obvious notation, $TG \cong T'G'$ implies that $T \cong T'$ and $G \cong G'$.

In the construction of $[A, \zeta, T]$ a basic role is played by the homomorphism $\zeta \colon T \to \mathcal{E}(A)$. We can represent A as $[Y; G_\alpha, \chi_{\alpha,\beta}]$. Homomorphisms between two such semigroups were constructed in [2, Proposition II.2.8] and can be specialized to represent endomorphisms.

With the above notation, in the lattice of inverse submonoids of S containing the identity of S, we have $T \vee G = S$ and $T \cap G = \{1\}$, that is, T and G are complements of each other. Can one say more?

6 Direct Products

We now approach the second problem of this note. Given a combinatorial inverse monoid T and a group G, so far we have provided a construction of all cryptic inverse monoids $S = T'G'$ where $T' \cong T$ and $G' \cong G$. Now we consider the direct product of T and G. This is a special case of $T'G'$ evoked above, since clearly $(t,1)(1,g) = (t,g)$ and thus for $T' = T \times \{1\}$ and $G' = \{1\} \times G$, we have $T'G' = T \times G$. We can take advantage of the observation that $T \times G \cong [A, \zeta, T]$, where ζ must satisfy conditions stronger than (A)–(D).

We start with a lemma that may not be new.

Lemma 6.1 *Let E be a semilattice and G be a group. Let $\xi \in \mathcal{E}(E)$, $\eta \in \mathcal{E}(G)$, and define φ by*

$$\varphi\colon (e,g) \longmapsto (\xi e, \eta g), \quad e \in E, g \in G.$$

Then $\varphi \in \mathcal{E}(E \times G)$. Conversely, every endomorphism of $E \times G$ can be so represented.

Proof The direct part is obvious. Conversely, let $\varphi \in \mathcal{E}(E \times G)$ and define functions ξ and η by

$$\varphi(e,g) = \big(\xi(e,g), \eta(e,g)\big), \quad e \in E, g \in G.$$

Clearly ξ is a homomorphism of $E \times G$ into E and η is a homomorphism of $E \times G$ into G. Then

$$\xi(e,g) = \xi\big((e,g)(e,1)\big) = \xi(e,g)\xi(e,1),$$
$$\xi(e,1) = \xi\big((e,g)(e,g^{-1})\big) = \xi(e,g)\xi(e,g^{-1})$$

where the first line implies that $\xi(e,g) \le \xi(e,1)$, and the second implies that $\xi(e,g) \ge \xi(e,1)$ so that $\xi(e,g) = \xi(e,1)$. It follows that $\xi(e,g)$ is independent of g. Next, for $e \le f$, we get

$$\eta(e,g) = \eta\big((f,g)(e,1)\big) = \eta(f,g)\eta(e,1) = \eta(f,g)$$

whence it follows that $\eta(e,g)$ is independent of e. We can write $\xi e = \xi(e,g)$ and $\eta g = \eta(e,g)$ thereby obtaining $\xi \in \mathcal{E}(E)$ and $\eta \in \mathcal{E}(G)$. Therefore φ is of the form in the statement of the lemma. ∎

Theorem 6.2 *Let $S = [A, \zeta, T]$ and assume*

(E) *$A \cong E \times G$ where $E = E(T)$ and $G = U(A)$,*

(F) *${}^t g = g$, ($g \in U(A)$, $t \in T$).*

Then S is a direct product of T and G. Conversely, the direct product of T and G is isomorphic to a semigroup so constructed.

Proof *Direct*. We may set $A = E \times G$. In view of Lemma 6.1, we can write the action of T on A in the form ${}^t(e,g) = ({}^t e, {}^t g)$ where now T acts on both E and G. For $(a,t) \in [A, \zeta, T]$, the requirement is $aa^{-1} = tt^{-1}$. Letting $a = (e,g)$, this requirement becomes $(e,g)(e,g)^{-1} = (e,1) = (tt^{-1}, tt^{-1})$ and we get $e = tt^{-1}$. Condition (F) implies that ${}^t(e,g) = ({}^t e, g)$. The mapping

$$\varphi\colon \big((tt^{-1}, g), t\big) \longmapsto (g,t)$$

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is thus an isomorphism of $[A, \zeta, T]$ onto $G \times T$.

Converse. We may set $S = T \times G$. Then $S \cong T' \times G'$ where $T' = \{1\} \times T$ and $G' = G \times \{1\}$ so that $T' \cong T$ and $G' \cong G$. By Theorem 2.2, we may set $S = [A, \zeta, T]$. Clearly $A \cong E \times G$ so condition (E) holds, and since the product in $T \times G$ is by components, condition (F) holds as well. ∎

We could have used [1, Theorem 5.1], but the above proof is shorter and more transparent. Theorem 6.2 exhibits the difference between the product TG and the direct product $T \times G$. We can specialize this result to the case of a direct product of a semilattice and a group, or can proceed as follows.

Let $A = [Y; G_\alpha, \chi_{\alpha,\beta}]$ be a factorizable Clifford monoid. Hence Y is a monoid, say with identity ε. Condition [1, Theorem 5.1(iii)] is equivalent to saying that each $\chi_{\varepsilon,\alpha}$ is injective. Proposition 1.1 asserts that all $\chi_{\alpha,\beta}$ are surjective. Hence all $\chi_{\varepsilon,\alpha}$ are isomorphisms. For $\alpha \geq \beta$, we have $\chi_{\varepsilon,\alpha}\chi_{\alpha,\beta} = \chi_{\varepsilon,\beta}$. Since both $\chi_{\varepsilon,\alpha}$ and $\chi_{\varepsilon,\beta}$ are isomorphisms, so is $\chi_{\alpha,\beta}$. We need the following.

Lemma 6.3 *Let $S = [Y; G_\alpha, \chi_{\alpha,\beta}]$ where every G_α is a group. Then every $\chi_{\alpha,\beta}$ is an isomorphism if and only if $S \cong Y \times G$ for some group G.*

Proof See [3, Lemma 4.4]. ∎

Evidently the group G in Lemma 6.3 is isomorphic to G_α for any $\alpha \in Y$. We can now make the desired conclusion.

Proposition 6.4 *A semigroup S is a direct product of a semilattice monoid and a group if and only if S is a Clifford monoid in which all structure homomorphisms are bijective.*

A comparison of Propositions 1.1 and 6.4 clearly exhibits the difference between TG and $T \times G$ for a semilattice monoid T and a group G.

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21420 Bol, Brač, Croatia