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Vanishing at infinity on homogeneous spaces of reductive type

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Abstract

Let G be a real reductive group and $Z = G/H$ a unimodular homogeneous G space. The space Z is said to satisfy VAI (vanishing at infinity) if all smooth vectors in the Banach representations $L^p(Z)$ vanish at infinity, $1 \leqslant p < \infty$. For H connected we show that Z satisfies VAI if and only if it is of reductive type.

1. Introduction

In many applications of harmonic analysis of Lie groups it is important to study the decay of functions on the group. For example, for a simple Lie group G, the fundamental discovery of Howe and Moore [HM79, Theorem 5.1], that the matrix coefficients of non-trivial irreducible unitary representations vanish at infinity, is often seen to play an important role. In a more general context it is of interest to study matrix coefficients formed by a smooth vector and a distribution vector. If the distribution vector is fixed by some closed subgroup H of G, these generalized matrix coefficients will be smooth functions on the quotient manifold G/H. This leads to the question which is studied in the present paper, the decay of smooth functions on homogeneous spaces. More precisely, we are concerned with the decay of smooth L^p-functions on G/H.

Let G be a real Lie group and $H \subset G$ a closed subgroup. Consider the homogeneous space $Z = G/H$ and assume that it is unimodular, that is, it carries a G-invariant measure μ_Z. Note that such a measure is unique up to a scalar multiple.

For a Banach representation (π, E) of G, we denote by E^∞ the space of smooth vectors. In the special case of the left regular representation of G on $E = L^p(Z)$ with $1 \leqslant p < \infty$, it follows from the local Sobolev lemma that E^∞ is the space of smooth functions on Z, all of whose derivatives belong to $L^p(Z)$ (see [Pou72, Theorem 5.1]). Let $C_0^\infty(Z)$ be the space of smooth functions on Z that vanish at infinity. Motivated by the decay of eigenfunctions on symmetric spaces [RS91], the following definition was taken in [KS12].

Definition 1.1. We say that Z has the property VAI (vanishing at infinity) if for all $1 \leqslant p < \infty$ we have

$$L^p(Z)^\infty \subset C_0^\infty(Z).$$

By [Pou72, Lemma 5.1], $Z = G$ has the VAI property for G unimodular and $H = \{\mathbf{1}\}$. The main result of [KS12] establishes that all reductive symmetric spaces admit VAI. On the other

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hand, it is easy to find examples of homogeneous spaces without this property. For example, it is clear that a non-compact homogeneous space with finite volume cannot have VAI.

The main result of this article is as follows.

Theorem 1.2. *Let G be a connected real reductive group and $H \subset G$ a closed connected subgroup such that $Z = G/H$ is unimodular and of algebraic type. Then* VAI *holds for Z if and only if it is of reductive type.*

Here we recall the following definitions, in which G is a real reductive group (see [Wal88] for this notion), and for which we let Ad denote the adjoint representation of G on the Lie algebra $\mathfrak{g}$.

Definition 1.3. Let $H \subset G$ be a closed connected subgroup.

(1) We say that H is a *reductive subgroup* and that Z is of *reductive type* if H is real reductive and the representation Ad of H on $\mathfrak{g}$ is completely reducible.

(2) We say that H is an *algebraic subgroup* and that Z is of *algebraic type* if $\mathrm{Ad}(H)$ is the connected component of an algebraic subgroup of $\mathrm{Ad}(G)$.

In Theorem 1.2, the implication '*only if*' is valid without the assumption of algebraicity, and we do not know whether '*if*' is also valid without this assumption. Note that both (1) and (2) are fulfilled when H is semisimple. Note also that Z is unimodular when it is of reductive type.

If Z is of reductive and algebraic type and $B \subset G$ is a compact ball, then we show in § 5 (see also [LM00]) that

$$\inf_{z \in Z} \mathrm{vol}_Z(Bz) > 0.$$

In view of the invariant Sobolev lemma of Bernstein (see Lemma 3.2), this readily implies that Z has VAI.

The converse implication is established in Proposition 7.1. As a consequence of the proof, it is seen that in the non-reductive case the volume of the above-mentioned sets Bz can be made arbitrarily small by letting z tend to infinity in a suitable direction (see (7.6)).

2. Notation

Throughout, G is a connected real reductive group and $H \subset G$ is a closed connected subgroup such that $Z := G/H$ is unimodular. We write μ_Z for a fixed G-invariant measure and vol_Z for the corresponding volume function.

Let $\mathfrak{g}$ be the Lie algebra of G. We fix a Cartan involution θ of G. The derived involution $\mathfrak{g} \to \mathfrak{g}$ will also be called θ. The fixed point set of θ is a maximal compact subgroup K of G whose Lie algebra will be denoted $\mathfrak{k}$. Let $\mathfrak{p}$ denote the -1-eigenspace of θ on $\mathfrak{g}$; then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let κ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ such that

$$\kappa|_{\mathfrak{p}} > 0, \quad \kappa|_{\mathfrak{k}} < 0, \quad \mathfrak{k} \perp \mathfrak{p}.$$

Having chosen κ, we define an inner product on $\mathfrak{g}$ by

$$\langle X, Y \rangle = -\kappa(\theta(X), Y).$$

We denote by $\mathfrak{h}$ the Lie algebra of H and by $\mathfrak{q}$ its orthogonal complement in $\mathfrak{g}$.

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LEMMA 2.1. *The space Z is of reductive type if and only if there exists a Cartan involution θ of G which preserves H. With such a choice, we have $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$.*

Proof. See [Hel78, Exercise VI A8] or [Wol67, Theorem 12.1.4]. The last statement follows easily. □

Remark 2.2. Let Z be of reductive type and choose θ and κ as above. Then $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$ if and only if the pair $(\mathfrak{g}, \mathfrak{h})$ is symmetric, that is, if and only if

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \sigma(X) = X\}$$

for an involution σ of $\mathfrak{g}$. When $\mathfrak{g}$ is semisimple, it then follows that

$$\mathfrak{q} = \{X \in \mathfrak{g} \mid \sigma(X) = -X\}.$$

3. VAI versus volume growth

For a compact set $B \subset G$, we shall consider the volume function

$$F_B : G \to \mathbb{R}_{\geqslant 0}, \quad g \mapsto \mathrm{vol}_Z(Bg \cdot z_0).$$

For that, we recall some results from [Ber88]. By a *ball* in G, we will understand a compact symmetric neighborhood of $\mathbf{1}$. A continuous function $w : G \to \mathbb{R}_+$ is called a *weight* provided that for all balls $B \subset G$ there exists a constant $C_B > 0$ such that $w(xg) \leqslant C_B w(g)$ holds for all $x \in B$ and $g \in G$ (see [Ber88]). Two weights $G \to \mathbb{R}_+$ are called *comparable* if their mutual ratio is bounded from above and below by positive constants.

Let $Z(G)$ denote the center of G.

LEMMA 3.1. *Fix a ball $B \subset G$. Then:*

(1) *F_B is a weight;*
(2) *if $B' \subset G$ is another ball, then F_B is comparable to F'_B;*
(3) *F_B factors to a continuous function on $\mathrm{Ad}(G) \simeq G/Z(G)$.*

Proof. The last statement is easy. For the others, see [Ber88, p. 683, Lemma–Definition]. In the proof it is shown that $m_Z := F_B^{-1}\mu_Z$ is a so-called standard measure. □

Let $1 \leqslant p < \infty$. For every $k \in \mathbb{N}$, we let $\|\cdot\|_{p,k}$ be a kth Sobolev norm of $\|\cdot\|_p$, the L^p-norm on $L^p(Z)$ (see [BK14, § 2]). Note that the collection $\{\|\cdot\|_{p,k} : k \in \mathbb{N}\}$ determines the Fréchet topology on $L^p(Z)^\infty$.

For a subset $\Omega \subset Z$, we write $\|\cdot\|_{p,k,\Omega}$ for the seminorm on $L^p(Z)^\infty$, which is obtained by integrating the derivatives over Ω.

In this context we recall the invariant Sobolev lemma of Bernstein.

LEMMA 3.2. *Fix $k > (\dim G)/p$. Then for every ball B there is a constant $C_B > 0$ such that*

$$|f(z)| \leqslant C_B \,\mathrm{vol}_Z(Bz)^{-1/p} \|f\|_{p,k,Bz} \quad (z \in Z)$$

for all smooth functions f on Z.

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Proof. See [Ber88, 'Key lemma' on p. 686], and note that $m_Z := F_B^{-1}\mu_Z$ is a standard measure. The cited lemma has $p = 2$, but its proof is valid for $1 \leqslant p < \infty$ as well. □

For $v \in \mathcal{U}(\mathfrak{g})$ and $f \in L^p(Z)^\infty$, as $L_v f$ belongs to $L^p(Z)$, its norm over Bz will be arbitrarily small for z outside a sufficiently large compact set. Hence, for $f \in L^p(Z)^\infty$ with $1 \leqslant p < \infty$, we obtain that

$$\lim_{z\to\infty} \|f\|_{p,k,Bz} = 0.$$

Hence, we have shown the following result.

Proposition 3.3. *If* $\inf_{g\in G} F_B(g) > 0$ *for some ball* B*, then* VAI *holds.*

We shall establish this lower bound on F_B for spaces of reductive and algebraic type in the course of following two sections.

4. Algebraic lower bound of the volume function

In the following, we shall employ a complementary subspace to $\mathfrak{h}$,

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{h}.$$

Given such a subspace, we let $\pi_\mathfrak{v}$ denote the projection $\mathfrak{g} \to \mathfrak{v}$ along $\mathfrak{h}$, and accordingly identify $\mathfrak{v} \simeq \mathfrak{g}/\mathfrak{h}$ with the tangent space $T_{z_0}Z$ of Z at z_0. Given $g \in G$, we further note that the differential of the left multiplication $\tau_g : Z \to Z$ by g provides an isomorphism

$$d\tau_g : T_{z_0}Z = \mathfrak{v} \xrightarrow{\sim} T_{g\cdot z_0}Z. \tag{4.1}$$

We know from Lemma 3.1 that F_B factors through the adjoint representation $G \to \mathrm{Ad}(G)$. Let $F_{\mathrm{Ad}(B)}$ be the map corresponding to F_B for the space $\mathrm{Ad}(G)/\mathrm{Ad}(H)$. By replacing B with a ball which is the product of a ball in the semisimple part of G and a ball in the center $Z(G)$, we see from Lemma 3.1(2) that the factored map of F_B is comparable to $F_{\mathrm{Ad}(B)}$. In order to study F_B, we may hence assume that G is adjoint. In particular, there exists a semisimple linear complex algebraic group $G_\mathbb{C}$ with real points $G_\mathbb{R}$ such that $G = (G_\mathbb{R})_e$.

In addition, we assume in this section that Z is of algebraic type. Hence, there exists a connected complex algebraic subgroup $H_\mathbb{C} < G_\mathbb{C}$ such that $H = (H_\mathbb{C} \cap G)_e$. With $H_\mathbb{R} = G \cap H_\mathbb{C}$, we form $Z_\mathbb{R} = G/H_\mathbb{R}$ and observe that the volume functions of Z and $Z_\mathbb{R}$ are comparable. It is thus no loss of generality to assume in addition that $Z = Z_\mathbb{R}$ (by allowing H to have finitely many components). Note that then

$$Z = G/H \subset Z_\mathbb{C} := G_\mathbb{C}/H_\mathbb{C}.$$

Lemma 4.1. *Assume that* G/H *is of algebraic type and let* $B \subset G$ *be a ball. Then there exists a left* K*-invariant and right* H*-invariant algebraic function* F *on* G *such that* $F(\mathbf{1}) > 0$ *and*

$$0 \leqslant F(g) \leqslant F_B(g)^2 \tag{4.2}$$

for all $g \in G$.

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Proof. We need a few geometric preparations. As $\mathfrak{g}$ is reductive, the vector complement $\mathfrak{v}$ of $\mathfrak{h}$ can be chosen such that it has a basis consisting of ad-nilpotent elements $Y_1, \ldots, Y_n$. As explained above, we may assume that G is linear and semisimple. This implies in particular that $\exp(\mathbb{C}Y_j) < G_\mathbb{C}$ is a unipotent algebraic subgroup for each $1 \leqslant j \leqslant n$.

We define a map $\mathrm{Exp}\colon \mathfrak{v} \to G$ by

$$\mathrm{Exp}\left(\sum_{j=1}^n t_j Y_j\right) := \exp(t_1 Y_1) \cdot \ldots \cdot \exp(t_n Y_n)$$

and, for $g \in G$, we then consider the smooth map

$$\Phi_g : \mathfrak{v} \to Z, \quad Y \mapsto \mathrm{Exp}(Y) g \cdot z_0.$$

If for each $Y \in \mathfrak{v}$ we identify $T_{\Phi_g(Y)}Z$ with $\mathfrak{v}$ as in (4.1), we see that the differential of Φ_g at Y is given by

$$d\Phi_g(Y)(Y') = \pi_\mathfrak{v}\left(\mathrm{Ad}(g)^{-1} \sum_{j=1}^n t_j' \,\mathrm{Ad}(y_{j+1} \cdot \ldots \cdot y_n)^{-1} Y_j\right) \tag{4.3}$$

for $Y' = \sum_{j=1}^n t_j' Y_j$, $Y = \sum_{j=1}^n t_j Y_j$ and $y_i := \exp(t_i Y_i)$. In particular, $\Phi_\mathbf{1}$ defines a local diffeomorphism at $Y = 0$. We are concerned with the cardinality of the fibers $\Phi_g^{-1}(z) \subset \mathfrak{v}$ at generic elements $z \in Z$ and for generic $g \in G$.

Lemma 4.2. *There exists $N \in \mathbb{N}$ such that the generic fibers of Φ_g are bounded by N for generic elements $g \in G$.*

Proof. We recall the following result from algebraic geometry (see [Gro66, Proposition 15.5.1(i)]). Let Z_1, Z_2, Z_3 be complex irreducible algebraic varieties with $\dim Z_1 = \dim Z_3$ and further let

$$f : Z_1 \times Z_2 \to Z_3$$

be an algebraic map, such that for one $z_2' \in Z_2$ the map $f(\cdot, z_2')$ is dominant. Then there exists an $N \in \mathbb{N}$ such that the generic fibers of $f(\cdot, z_2)$ are bounded by N for all generic $z_2 \in Z_2$.

We apply this to $Z_1 = \exp(\mathbb{C}Y_1) \times \cdots \times \exp(\mathbb{C}Y_n)$, $Z_2 = G_\mathbb{C}$, $Z_3 = Z_\mathbb{C}$ and the map

$$f((z_1, \ldots, z_n), g) := z_1 \cdot \ldots \cdot z_n g \cdot z_0.$$

Observe that f is defined over $\mathbb{R}$. The assertion follows. □

We can now complete the proof of Lemma 4.1. Fix an open relatively compact neighborhood $V \subset \mathfrak{v}$ of zero with $\mathrm{Exp}(V) \subset B$ and for which $\Phi_\mathbf{1}$ restricts to a diffeomorphism onto its image. Set $\phi_g := \Phi_g|_V$. It follows from our formula (4.3) for the differential that the Jacobian

$$J_g(Y) := \det d\phi_g(Y) \quad (g \in G, Y \in V)$$

depends algebraically on g. Let ω_Z be a G-invariant differential form of Z and ω_g its pull-back to V under ϕ_g. Note that ω_g depends algebraically on g as an element of $\Omega^n(V)$, i.e. $\omega \in \mathbb{C}[G] \otimes \Omega^n(V)$. Define a function

$$f_V(g) := \int_V \omega_g \quad (g \in G).$$

Then it is clear that f_V is a polynomial function on G with $f_V(\mathbf{1}) > 0$.

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It follows from the uniform fiber bound that

$$|f_V(g)| \leqslant N \cdot F_B(g)$$

for $g \in G$ generic, and hence for all $g \in G$ by continuity. Hence, $F_V := f_V^2/N^2$ is a non-negative algebraic function which is dominated by F_B^2.

It follows from Lemma 3.1 that we can assume in addition that the ball B is right K-invariant, that is,

$$BK = B. \tag{4.4}$$

Then the volume function F_B is left K-invariant, and hence the average of F_V over K from the left is algebraic and satisfies (4.2). □

Corollary 4.3. *Let G/H be of algebraic type (see Definition 1.3(2)) and let $B \subset G$ be a ball. There is a finite-dimensional representation (π, W) of G with a cyclic K-fixed vector $v_K \in W$ and a cyclic H-fixed vector $v_H \in W$ such that $\langle v_H, v_K\rangle > 0$ and*

$$0 \leqslant \langle \pi(g)v_H, v_K\rangle \leqslant F_B(g)^2 \quad (g \in G). \tag{4.5}$$

Here $\langle\cdot\,,\cdot\rangle$ is an inner product on W which is θ-covariant: $\langle \pi(g)v, w\rangle = \langle v, \pi(\theta(g))^{-1}w\rangle$ for $g \in G$ and $v, w \in W$.

Proof. It follows from the remark at the beginning of this section that we may assume that G is linear semisimple algebraic. With the right action the algebraic function F of Lemma 4.1 generates a finite-dimensional representation W in which $v_H = F$ is H-fixed and cyclic. Moreover, evaluation at $\mathbf{1}$ is a K-fixed cyclic vector for the dual representation. Finally, the inner product $\langle\cdot\,,\cdot\rangle$ exists since θ is a Cartan involution, and with that we obtain v_K and $F(g) = \langle \pi(g)v_H, v_K\rangle$. □

5. Reductive spaces are VAI

For G and H both semisimple it was shown with analytic methods in [LM00] that $\inf_{g\in G} F_B(g) > 0$. In this section we give a geometric proof, which is valid more generally for spaces which are of both reductive and algebraic type. Combined with Proposition 3.3, this completes the proof of the implication 'if' of Theorem 1.2.

Lemma 5.1. *Let $Z = G/H$ be of reductive and algebraic type and let $B \subset G$ be a ball. Then there exists a constant $c > 0$ such that*

$$\mathrm{vol}_Z(Bz) \geqslant c \tag{5.1}$$

for all $z \in Z$.

Proof. By Lemma 3.1, it is no loss of generality to request in addition to (4.4) that B has the property

$$\theta(B) = B. \tag{5.2}$$

As Z is of reductive type, we can apply Lemma 2.1 and arrange that H is θ-stable. Then θ induces an automorphism on Z which is measure preserving. Hence, (5.2) implies that

$$F_B(g) = F_B(\theta(g)) \quad (g \in G). \tag{5.3}$$

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Let F be a matrix coefficient as in Corollary 4.3 such that

$$0 \leqslant F(g) \leqslant F_B(g)^2$$

for all $g \in G$. Because of (5.3), we also have

$$0 \leqslant F(\theta(g)) \leqslant F_B(g)^2$$

for all $g \in G$. Hence, it suffices to show that

$$\inf_{g \in G} [F(g) + F(\theta(g))] > 0.$$

We recall the following fact from convex geometry. Let $(W_{\mathbb{R}}, \langle \cdot\,, \cdot \rangle)$ be a Euclidean vector space and $C \subset W_{\mathbb{R}}$ a regular cone, i.e. C is convex, closed, contains no lines and has non-empty interior. Let $C^\star \subset W$ be the dual cone to C. Then $C^\star$ is regular as well. Fix an element $v^\star$ in the interior of $C^\star$. Then there exists a constant $c > 0$ such that

$$(\forall v \in C) \quad \langle v^\star, v \rangle \geqslant c\sqrt{\langle v, v \rangle}. \tag{5.4}$$

We wish to apply this fact to F and the representation W in Corollary 4.3. Note that W has a real structure $W_{\mathbb{R}}$ with $v_K, v_H \in W_{\mathbb{R}}$. As these vectors are cyclic, the closed convex cones C_H and C_K, generated by the G-orbit through the rays $\mathbb{R}^+ v_H$ and $\mathbb{R}^+ v_K$, respectively, both have non-empty interior. As F is non-negative, we clearly have $C_H \subset C_K^\star$ and $C_K \subset C_H^\star$. As C_K is regular, we conclude that C_H is regular as well. Further, v_K lies in the interior of C_K (see [HO97, Lemma 2.1.15]) and with (4.5) and (5.4) we obtain a constant $c > 0$ such that

$$F(g) \geqslant c\|\pi(g)v_H\| \tag{5.5}$$

for all $g \in G$.

For each $X \in \mathfrak{p}$, we let $v_H = v_H^+ + v_H^0 + v_H^-$ be the decomposition into sums of eigenvectors for X, with positive, fixed and negative eigenvalues, respectively. We obtain for $g = \exp X$ that

$$\|\pi(g)v_H\|^2 \geqslant \|v_H^0\|^2 + \|v_H^+\|^2$$

and

$$\|\pi(\theta(g))v_H\|^2 \geqslant \|v_H^0\|^2 + \|v_H^-\|^2.$$

Hence, by (5.5),

$$F(g) + F(\theta(g)) \geqslant c(\|v_H^0\|^2 + \|v_H^+\|^2 + \|v_H^-\|^2)^{1/2} = c\|v_H\|,$$

and the lemma is proved. □

Remark 5.2. If $Z = G/H$ is a reductive real spherical space (in particular, a reductive symmetric space), an upper volume bound of exponential type is also valid. See [KKSS14].

Remark 5.3. For a semisimple symmetric space the wave front lemma (see [EM93, Theorem 3.1]) shows that there exists an open neighborhood V of z_0, such that Bz contains a G-translate of V for all $z \in Z$. This implies (5.1) for this case.

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6. The differential of exp

Let $\mathfrak{v} \subset \mathfrak{g}$ be a complementary subspace to $\mathfrak{h}$, and consider the map

$$\Phi_g : \mathfrak{v} \to Z, \quad Y \mapsto \exp(Y)g \cdot z_0. \tag{6.1}$$

The following formula for its differential is well known.

Lemma 6.1. *The differential of Φ_g at $Y \in \mathfrak{v}$ is given by*

$$d\Phi_g(Y) = d\tau_{\exp(Y)g} \circ \pi_{\mathfrak{v}} \circ \mathrm{Ad}(g)^{-1} \circ \beta(Y) \circ \iota_{\mathfrak{v}}, \tag{6.2}$$

where

$$\beta(Y) = \frac{\mathbf{1} - e^{-\mathrm{ad}\,Y}}{\mathrm{ad}\,Y} \in \mathrm{End}(\mathfrak{g})$$

for $Y \in \mathfrak{v}$, and $\iota_{\mathfrak{v}} : \mathfrak{v} \to \mathfrak{g}$ is the inclusion map.

Remark 6.2. In fact, we shall apply the lemma in a more general situation where the complementary subspace $\mathfrak{v}$ splits in a direct sum of subspaces. For example, if $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$, we can replace (6.1) by

$$\Phi_g : \mathfrak{v}_1 \times \mathfrak{v}_2 \to Z, \quad (Y_1, Y_2) \mapsto \exp(Y_1)\exp(Y_2)g \cdot z_0.$$

Similar to (6.2), we find in this case for $W = (W_1, W_2) \in \mathfrak{v}$ that

$$d\Phi_g(Y)(W) = d\tau_{\exp(Y_1)\exp(Y_2)g}\pi_{\mathfrak{v}}\,\mathrm{Ad}(g)^{-1}(S_{Y,W}),$$

where

$$S_{Y,W} := \mathrm{Ad}(\exp(Y_2)^{-1})\beta(Y_1)(W_1) + \beta(Y_2)(W_2) \in \mathfrak{g}.$$

7. Non-reductive spaces are not VAI

In this section we prove that VAI does not hold on any homogeneous space $Z = G/H$ of G, which is not of reductive type. We maintain the assumptions in § 2 and establish the following result.

Proposition 7.1. *Assume that $Z = G/H$ is unimodular and not of reductive type. Then for all $1 \leqslant p < \infty$ there exists an unbounded function $f \in L^p(Z)^\infty$. In particular,* VAI *does not hold.*

Proof. As in Lemma 4.1, the key to the proof is the construction of a suitable vector complement $\mathfrak{v}$ to $\mathfrak{h}$ in $\mathfrak{g}$.

Let $\mathfrak{u}_H$ be the largest ideal of $\mathfrak{h}$ which acts by nilpotent morphisms on $\mathfrak{g}$. As H is not reductive in G, we have $\mathfrak{u}_H \neq \{0\}$. Let $L_H < H$ be a Levi complement to U_H. According to Borel and Tits (see [BT71] or [Hum75, § 30.3, Corollary A]), we find a parabolic subgroup Q of G with Levi decomposition $Q = LU$ such that $L_H \subset L$ and $U_H \subset U$. Let θ be a Cartan involution of G which fixes L and let $\overline{U} = \theta(U)$. We recall that according to the Bruhat decomposition,

$$\overline{U} \times L \times U \to G, \quad (\overline{u}, l, u) \mapsto \overline{u}lu \tag{7.1}$$

is a diffeomorphism onto its Zariski open image.

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Let $X \in \mathfrak{z}(\mathfrak{l})$ be an element in the center of $\mathfrak{l}$ such that $\operatorname{ad} X|_{\mathfrak{u}}$ has positive spectrum and set $a_t := \exp(tX)$ for $t \in \mathbb{R}$.

Notice that we cannot have $X \in \mathfrak{h}$, as in that case $\operatorname{ad} X$ would have a positive trace on $\mathfrak{h} = \mathfrak{l}_H + \mathfrak{u}_H$, contradicting that G/H is unimodular. It follows that $a_t \cdot z_0 \to \infty$ in $L/L \cap H$ and hence also in Z for $|t| \to \infty$.

We now construct a complementary subspace $\mathfrak{u}_X$ to $\mathfrak{u}_H$ as follows. If $\mathfrak{u}_H = \mathfrak{u}$, then $\mathfrak{u}_X = \{0\}$. Otherwise we choose an $\operatorname{ad} X$-eigenvector, say Y_1, in $\mathfrak{u} \backslash \mathfrak{u}_H$ with largest possible eigenvalue. If $\mathfrak{u}_H + \mathbb{R}Y_1 \subsetneq \mathfrak{u}$, we choose an eigenvector $Y_2 \in \mathfrak{u} \backslash (\mathfrak{u}_H + \mathbb{R}Y_1)$ with largest possible eigenvalue. We continue this procedure until $Y_1, Y_2, \ldots$ span a complementary subspace. This subspace we denote $\mathfrak{u}_X$.

Let $\mathfrak{l}_0 = \mathfrak{l}_H^{\perp_{\mathfrak{l}}}$ denote the orthocomplement of $\mathfrak{l}_H$ in $\mathfrak{l}$. Then

$$\mathfrak{v} = \overline{\mathfrak{u}} + \mathfrak{l}_0 + \mathfrak{u}_X$$

is an $\operatorname{ad} X$-stable complement to $\mathfrak{h}$ in $\mathfrak{g}$.

Before proceeding, we note some important consequences of this construction of $\mathfrak{v}$. Firstly, it follows that

$$\mathfrak{u}_X \to U/U_H, \quad Y \mapsto \exp(Y)U_H \tag{7.2}$$

is a diffeomorphism. This boils down to a general property of graded nilpotent Lie algebras that will be established in Lemma 7.5. Secondly, the following lemma holds.

Lemma 7.2. *With $\mathfrak{u}_X$ and $\mathfrak{v}$ defined as above, we have $\sup_{t<0}(M_t) < \infty$, where*

$$M_t := \sup_{W \in \mathfrak{g}, \|W\|=1} \|\operatorname{Ad}(a_t)\pi_{\mathfrak{v}} \operatorname{Ad}(a_t)^{-1} W\|.$$

Proof. For $W \in \mathfrak{v}$, we have

$$\operatorname{Ad}(a_t)\pi_{\mathfrak{v}} \operatorname{Ad}(a_t)^{-1} W = W$$

and, for $W \in \mathfrak{l}_H$, we have

$$\operatorname{Ad}(a_t)\pi_{\mathfrak{v}} \operatorname{Ad}(a_t)^{-1} W = 0.$$

Hence, we may assume that $W \in \mathfrak{u}_H$. We can write W as a combination of $\operatorname{ad} X$-eigenvectors $Y_\lambda \in \mathfrak{u}$ with eigenvalues λ. Then

$$\operatorname{Ad}(a_t)^{-1} W = \sum e^{-\lambda t} Y_\lambda.$$

If $Y_\lambda \in \mathfrak{u}_X$, then

$$\operatorname{Ad}(a_t)\pi_{\mathfrak{v}} e^{-\lambda t} Y_\lambda = Y_\lambda.$$

Finally, if Y_λ is not in $\mathfrak{u}_X$, then it is the sum of an element from $\mathfrak{u}_H$ and some eigenvectors $V_\mu \in \mathfrak{u}_X$. Moreover, all these V_μ must have eigenvalues $\mu \geqslant \lambda$, since otherwise Y_λ would have been preferred before such a V_μ in the construction of $\mathfrak{u}_X$. Thus,

$$\operatorname{Ad}(a_t)\pi_{\mathfrak{v}} e^{-\lambda t} Y_\lambda = \sum_{\mu \geqslant \lambda} e^{(\mu-\lambda)t} V_\mu,$$

which stays bounded for $t \to -\infty$. $\square$

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We now continue with the proof of Proposition 7.1. Let $V_0 \subset \mathfrak{l}_0$ be an open neighborhood of 0 such that $V_0 \to L/L_H$, $Y \mapsto \exp(Y)L_H$ is a diffeomorphism onto its image. It follows that the map

$$V_0 \times U/U_H \to Q/H, \quad (Y, uU_H) \mapsto \exp(Y)u \cdot z_0 \tag{7.3}$$

is a diffeomorphism onto its image.

Combining (7.3) and (7.2) with (7.1), we obtain a diffeomorphism

$$\begin{aligned} \Phi : \overline{\mathfrak{u}} \times V_0 \times \mathfrak{u}_X &\to G/H, \\ (Y^-, Y^0, Y^+) &\mapsto \exp(Y^-)\exp(Y^0)\exp(Y^+) \cdot z_0 \end{aligned}$$

onto its image.

Further, we let V^- and V^+ be open relatively compact convex neighborhoods of 0 in the vector spaces $\overline{\mathfrak{u}}$ and $\mathfrak{u}_X$. Set $V := V^- \times V^0 \times V^+$.

For $t \in \mathbb{R}$, we set $a_t := \exp(tX)$ and consider the map $\Phi_t : V \to G/H$,

$$\Phi_t(Y) := \exp(Y^-)\exp(Y^0)\exp(Y^+)a_t \cdot z_0,$$

where $Y = (Y^-, Y^0, Y^+) \in V$. It follows that Φ_t is a diffeomorphism onto its open image for all $t \in \mathbb{R}$. We need the following property for which we recall the identification (4.1) of the tangent spaces of Z with $\mathfrak{v}$.

Lemma 7.3. *There exists a linear map $L(Y) : \mathfrak{v} \to \mathfrak{g}$ such that*

$$d\Phi_t(Y) = \mathrm{Ad}(a_t)^{-1}(\mathbf{1}_{\mathfrak{v}} + \mathrm{Ad}(a_t)\pi_{\mathfrak{v}}\,\mathrm{Ad}(a_t)^{-1}L(Y)) \tag{7.4}$$

for all $t \leqslant 0$, and such that $\|L(Y)\| \to 0$ for $Y \to 0$.

Proof. Let $Y = (Y^-, Y^0, Y^+)$ and $X = (X^-, X^0, X^+)$ in $\mathfrak{v}$. It then follows from Remark 6.2 that

$$d\Phi(Y^-, Y^0, Y^+)(X^-, X^0, X^+) = d\tau_{y^- y^0 y^+ a_t}(z_0) \circ \mathrm{Ad}(a_t)^{-1}(S_{Y,X}),$$

where $y^- = \exp(Y^-)$ etc, and where $S_{Y,X} \in \mathfrak{g}$ is the element

$$\mathrm{Ad}(y_0 y^+)^{-1}\beta(Y^-)(X^-) + \mathrm{Ad}(y^+)^{-1}\beta(Y^0)(X^0) + \beta(Y^+)(X^+).$$

Defining $L(Y)$ by $L(Y)(X) = S_{Y,X} - X$ for $X \in \mathfrak{v}$, we obtain the expression in (7.4). It is easily seen that $\|L(Y)\| \to 0$ for $Y \to 0$. □

Let $J_t = |\det d\Phi_t|$. By Lemmas 7.3 and 7.2, there exists a constant $C > 0$ such that the following bound holds for V sufficiently small:

$$J_t(Y) \leqslant Ce^{t\lambda_X} \quad (t \leqslant 0, Y \in V) \tag{7.5}$$

with $\lambda_X = -\mathrm{trace}\,\mathrm{ad}_X|_{\overline{\mathfrak{u}}+\mathfrak{u}_X}$. Note that $\lambda_X > 0$ since $\mathfrak{u}_H$ is non-trivial.

Fix a function $\psi \in C_c^\infty(V)$ with $0 \leqslant \psi \leqslant 1$ and $\psi(0) = 1$. For all $t \in \mathbb{R}$, define $\chi_t \in C_c^\infty(Z)$ by $\chi_t(z) = \psi(\Phi_t^{-1}(z))$ and set

$$\chi := \sum_{n \in \mathbb{N}} n\chi_{-n}.$$

It is clear that $\chi \in C^\infty(Z)$ and that χ is unbounded. We claim that $\chi \in L^p(Z)^\infty$ for all $1 \leqslant p < \infty$.

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It follows from the estimate in (7.5) that for all $1 \leqslant p < \infty$ there exists $C > 0$ such that $\|\chi_t\|_p \leqslant Ce^{t\lambda_X/p}$ for all $t \leqslant 0$. Hence,

$$\chi = \sum_{n\in\mathbb{N}} n\chi_{-n} \in L^p(Z)$$

for all $1 \leqslant p < \infty$, and it only remains to be seen that also the derivatives of χ belong to $L^p(Z)$.

We first show this for first-order derivatives. Let $W \in \mathfrak{g}$ and consider the derivative $L(W)\chi_t$. At $z = \Phi_t(Y)$, this is given by

$$L(W)\chi_t(z) = d/ds|_{s=0}\chi_t(\exp(sW)ya_tz_0),$$

where $y = \exp(Y)$. For Y in a compact set, we can replace W by its conjugate by y without loss of generality, and thus we may as well consider the s-derivative of

$$\chi_t(y\exp(sW)a_tz_0).$$

We rewrite this as

$$\chi_t(ya_t\exp(s\operatorname{Ad}(a_t)^{-1}W)z_0)$$

and apply the projection along $\mathfrak{h}$. It follows that the derivative can be rewritten as

$$d/ds|_{s=0}\chi_t(ya_t\exp(s\pi_{\mathfrak{v}}\operatorname{Ad}(a_t)^{-1}W)z_0)$$

and then finally also as

$$d/ds|_{s=0}\chi_t(y\exp(s\operatorname{Ad}(a_t)\pi_{\mathfrak{v}}\operatorname{Ad}(a_t)^{-1}W)a_tz_0).$$

Note that $\operatorname{Ad}(a_t)\pi_{\mathfrak{v}}\operatorname{Ad}(a_t)^{-1}W \in \mathfrak{v}$. We conclude that the derivative is a linear combination of derivatives of ψ on V, with coefficients that are smooth functions on V. Furthermore, it follows from Lemma 7.2 that the coefficients are bounded for $t \leqslant 0$. As before, we conclude that $L(W)\chi_t \in L^p(Z)$ for all $t \leqslant 0$, with exponentially decaying p-norms. It follows that $L(W)\chi \in L^p(Z)$.

By repeating the argument for higher derivatives, we finally see that $\chi \in L^p(Z)^\infty$. This concludes the proof of Proposition 7.1. □

Remark 7.4. It follows from the proof of the proposition that

$$\lim_{t\to-\infty}\mathbf{v}_B(a_t\cdot z_0) = 0. \tag{7.6}$$

In fact, if we apply the invariant Sobolev lemma 3.2 to the function χ with $p = 1$, we get

$$n \leqslant \chi(a_{-n}\cdot z_0) \leqslant C_Bv_B(a_{-n}\cdot z_0)^{-1}\|\chi\|_{1,2\dim G} \quad (n\in\mathbb{N}).$$

Thus, for a constant $C > 0$,

$$\mathbf{v}_B(a_{-n}\cdot z_0) \leqslant \frac{C}{n} \quad (n\in\mathbb{N}).$$

The assertion (7.6) follows from the facts that the equivalence class of v_B is independent of the choice of the ball B and that $a_ta_{[t]}^{-1} \in B'$ for all $t \in \mathbb{R}$ and a certain ball B'.

The following general result was used in (7.2) above.

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Lemma 7.5. *Let $\mathfrak{u} = \bigoplus_{j>0} \mathfrak{u}^j$ be a positively graded nilpotent Lie algebra and $\mathfrak{h} < \mathfrak{u}$ a subalgebra. Let $\mathfrak{u}_0 \subset \mathfrak{u}$ be a graded vector complement to $\mathfrak{h}$, which is constructed as follows: if $\mathfrak{h} = \mathfrak{u}$, then $\mathfrak{u}_0 = \{0\}$. Otherwise we choose a vector, say Y_1, in $\mathfrak{u}^{j_1} \backslash \mathfrak{h}$, with j_1 as large as possible. If $\mathfrak{h} + \mathbb{R}Y_1 \subsetneq \mathfrak{u}$, we choose $Y_2 \in \mathfrak{u}^{j_2} \backslash (\mathfrak{h} + \mathbb{R}Y_1)$ with largest possible j_2. We continue this procedure until $Y_1, Y_2, \ldots$ span a complementary subspace. This subspace we denote $\mathfrak{u}_0$.*

Let U be a simply connected Lie group with Lie algebra $\mathfrak{u}$ and $H < U$ the connected subgroup associated to $\mathfrak{h}$. Then the map

$$\mathfrak{u}_0 \to U/H, \quad X \mapsto \exp(X)H$$

is a diffeomorphism.

Proof. This is by induction on $\dim \mathfrak{u}$. The one-dimensional case is trivial. Let $0 \neq Y$ be an element in $\mathfrak{u}$ of top degree, chosen as follows according to two cases. If $Y_1 \in \mathfrak{u}^{\mathrm{top}}$, we choose $Y = Y_1$. Otherwise $\mathfrak{u}^{\mathrm{top}} \subset \mathfrak{h}$, and we choose Y arbitrarily. Note that Y is central.

We consider the graded Lie algebra $\tilde{\mathfrak{u}} := \mathfrak{u}/\mathbb{R}Y$ and the subalgebra $\tilde{\mathfrak{h}} = (\mathfrak{h} + \mathbb{R}Y)/\mathbb{R}Y$. In both cases the assertion now follows easily by applying the induction hypothesis to this pair. Note that in the first case when $Y = Y_1$,

$$\exp(t_1Y_1 + \cdots + t_mY_m) = \exp(t_1Y_1)\exp(t_2Y_2 + \cdots + t_mY_m)$$

since Y is central. □

7.1 Final remarks

(1) We did not address here the case where G is not reductive. One might expect in general for G/H unimodular and algebraic that Z has VAI if and only if the nilradical of H is contained in the nilradical of G.

(2) The following may be an alternative approach to Theorem 1.2. To be more specific, assume $Z = G/H$ to be unimodular, algebraic and quasi-affine. Under these assumptions we expect that there are a rational G-module V and an embedding $Z \to V$ such that the invariant measure μ_Z, via pull-back, defines a tempered distribution on V. Note that if Z is of reductive type, then there exists a V such that the image of $Z \to V$ is closed, and hence μ_Z defines a tempered distribution on V. If Z is not of reductive type, then by Matsushima's criterion [BH62, Theorem 3.5] all images $Z \to V$ are non-closed and the expected embedding would imply that VAI does not hold. This is supported by a result in [Rao72], which asserts that for a reductive group G and $X \in \mathfrak{g} := \mathrm{Lie}(G)$ the invariant measure on the adjoint orbit $Z := \mathrm{Ad}(G)(X) \subset \mathfrak{g}$ defines a tempered distribution on $\mathfrak{g}$. Various particular results in the theory of prehomogeneous vector spaces provide additional support (see [BR05]).

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