ON \(q\)-GALOIS EXTENSIONS OF SIMPLE RINGS

HISAO TOMINAGA

To the memory of Professor Tadasi Nakayama

In 1952, the late Professor T. Nakayama succeeded in constructing the Galois theory for finite dimensional simple ring extensions \([7]\). And, we believe, the theory was essentially due to the following proposition: If a simple ring \(A\) is Galois and finite over a simple subring \(B\) then \(A\) is \(B'\)-completely reducible for any simple intermediate ring \(B'\) of \(A/B\) \([7,\) Lemmas 1.1 and 1.2\)]. Moreover, as was established in \([5]\), Nakayama's idea was still efficient in considering the infinite dimensional Galois theory of simple rings.

In this paper, we shall present first such a generalization of the proposition stated above that contains \([5,\) Lemma 2\] as well. And then, by the aid of this generalization, several facts obtained in \([6]\) and \([8]\) for division rings will be extended to simple rings. In fact, under the assumption that a simple ring extension in question is \(h\)-\(q\)-Galois and left locally finite, many important results previously obtained in \([2]\)-\([10]\) can be unified.

Throughout the present paper, \(A = \sum_{ij} D_{ij} E_{ij}\) will represent a simple ring where \(E = \{e_{ij}\}\) is a system of matrix units and \(D = V_A(E)\) a division ring, and \(B\) a simple subring of \(A\) containing the identity \(1\) of \(A\). And we use the following conventions: \(V\) and \(H\) mean \(V_A(B)\) and \(V_{B'}(B) = V_A(V_A(B))\), respectively. If \(H\) is a simple ring, we set \(H = \sum K d_{ik}\) where \(d = \{d_{ik}\}\) is a system of matrix units and \(K = V_H(H)\) a division ring. If \(T\) is a regular subring of \(A\) containing \(B\), \(\mathcal{G}(T, A/B)\) will mean the set of all the \(B\)-(ring) isomorphisms of \(T\) onto regular subrings of \(A\). And finally, \(A/B\) is said to be \(h\)-Galois\(^1\) if \(B\) is regular and \(\mathcal{G}A_r\) is dense in \(\text{Hom}_{\eta}(A, A)\), where \(\mathcal{G}\) is the

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Received April 26, 1965.

\(^1\) These lemmas were stated under the weaker assumption that \(A/B\) is (finite and) weakly normal.

\(^2\) In \([4]\), \(A/B\) was defined to be \(h\)-Galois if (i) \(B\) is regular, (ii) \(A\) is Galois over \(B'\) and \(V_{B'}(B')\) is simple for any regular subring \(B'\) of \(A\) left finite over \(B\), and (iii) \(A' = V_{B'}(A')\) and \(\{A' : H\} = [V : V_A(A')]\) for every regular subring \(A'\) of \(A\) left finite over \(H\), and it

(Continued on next page)
group of all the $B$-automorphisms of $A$. As to other notations and terminologies used here, we follow [4] and [5].

The following propositions previously known will play important roles in the present study.

**Proposition 1.** Let $B'$ be a subring of $A$ containing 1 of $A$, $V' = V_A(B')$ and $H' = V_A^2(B')$.

(a) If $A$ is $B' \cdot V'$-$A$-irreducible, then $A$ is homogeneously completely reducible as $B'$-$A$-module and as $V'$-$A$-module, both $V'$ and $H'$ are simple rings, $[A : B' \cdot A_r] = [V' : V']$ and $[A : V' \cdot A_r] = [H' : H']^B$.

(b) If $B'$ is an intermediate ring of $A/B$ left (resp. right) finite over $B$ and $A$ is $B' \cdot V'$-$A$-irreducible (resp. $A$-$B'$-$V'$-irreducible), then $[V : V'] \leq [B' : H^*]$; (resp. $[V : V'] \leq [B' : H^*]$) for any simple intermediate ring $H^*$ of $H \cap B'/B$.

(c) If $B'$ is an intermediate ring of $A/B[E]$ left (resp. right) finite over $B$ and $A$ is left (resp. right) locally finite over $B$, then $[V : V'] \leq [B' : B']$ (resp. $[V : V'] \leq [B' : B']$). ([2, Lemma 1 and Cor. 2].)

**Proposition 2.** Let $A$ be outer Galois and left locally finite over $B$, and $A'$ an intermediate ring of $A/B$.

(a) $A'$ is simple, $A/A'$ is (two-sided) locally finite, and each $B$-ring isomorphism of $A'$ into $A$ can be extended to an element of $G$.

(b) $A/B$ is $h$-Galois, and there exists a 1-1 dual correspondence between closed subgroups of $G$ and intermediate rings of $A/B$, in the usual sense of Galois theory.

(c) If $[A' : B] < \infty$ then $([A' : B], = [A' : B], = \#(G \cdot A')$ for any Galois group $G$ of $A/B$. ([3, Th. 1.1], [3, Cor. 1.4], [4, Lemma 1.8] and [9].)

**Proposition 3.** Let $A$ be Galois over $B$ with a regular Galois group $G$, and $H$ a simple ring left locally finite over $B$. And let $T$ be an intermediate ring of $A/B$ such that $[T : B] < \infty$ and $A$ is $T$-$A$-irreducible.

(a) If $[T : H \cap T] = [V : V_A(T)]$, then $\text{Hom}_B(T, A) = (G \cdot T)A$ and $G \cdot T = G \cdot T$.

was shown that if $A/B$ is $h$-Galois and left locally finite then $G \cdot A$ is dense in $\text{Hom}_B(A, A)$. And more recently, in [2], T. Nagahara has shown the converse implication. However, one will see later the converse implication to be true even under a somewhat weakened assumption. (Cf. Ths. 2 and 8.)

$\text{[a]}$ $[A : B' \cdot A_r]$ and $[V' : V']$ denote the length of the composition series of the $B'$-$A$-module $A$ and the length of the composition series of the $V'$-$module$ $V'$ (the capacity of the simple ring $V'$), respectively.
1. Preliminaries. The present section starts with the following brief lemma.

**Lemma 1.** Let $B'$ be a simple intermediate ring of $A/B$ with $[B'|B'] = n$ (= capacity of $A$). If $a$ is an arbitrary element of $A$ and $T$ an arbitrary simple intermediate ring of $A/B'$ then $[aB'|B'] \geq [aT|T]$. And, if $A/B$ is left locally finite and $[B':B] \leq \infty$ then there exists an intermediate ring $B''$ of $A/B'$ such that $[B'':B'' \leq \infty$ and $[aB'|B''] = [aA|A]$.

**Proof.** Without loss of generality, we may assume that $B'$ contains $E$ and $aB' = \sum a_\alpha B' = \bigoplus a_\alpha B'$ $(m = [aB'|B'])$. As each $e_iT = e_iB'T$ is a minimal right ideal of $T$, $aT = aB'T = \sum a_\alpha T$ implies then $[aT|T] \leq m$. Now, the rest of the proof will be obvious.

The proof of the next lemma proceeds in the usual way (cf. [4]), and may be omitted.

**Lemma 2.** Let $B'$ be a simple intermediate ring of $A/B$ with $[B'|B'] = n$, $\alpha$ and $\beta$ elements of $\mathfrak{S}(B', A/B)$, and $\mathfrak{S}$ a subset of $\mathfrak{S}(B', A/B)$.

(a) $\alpha A_r$ is $B'_1-A_r$-irreducible and $\alpha$ is linearly independent over $A_r$.

(b) Let $m$ be a $B'_1-A_r$-submodule of $\mathfrak{S}A_r$. $m$ is $B'_1-A_r$-irreducible if and only if $m = amA_r$ with some $a \in \mathfrak{S}$ and some non-zero $u \in V$.

(c) $\alpha A_r$ is $B'_1-A_r$-isomorphic to $\beta A_r$ if and only if $\alpha = \beta u$ with some $u \in V'$ (the multiplicative group of the regular elements of $V$), and so if $\alpha$ is contained in $\mathfrak{S}A_r$ then $\alpha = \sigma v$ with some $\sigma \in \mathfrak{S}$ and $v \in V'$.

We consider here the following conditions:

(1) $\text{Hom}_{B_r}(B', A) = \mathfrak{S}(B', A/B)A_r$ for any regular intermediate ring $B'$ of $A/B$ with $[B':B]_r < \infty$.

(1') $\text{Hom}_{B_r}(B', A) = \mathfrak{S}(B', A/B)A_l$ for any regular intermediate ring $B'$ of $A/B$ with $[B':B]_l < \infty$.

(2) $\mathfrak{S}(B_1, A/B)B_2 \subseteq \mathfrak{S}(B_2, A/B)$ for any regular subrings $B_1 \supseteq B_2$ of $A$ containing $B$ with $[B_1 : B]_l < \infty$.

(2') $\mathfrak{S}(B_1, A/B)B_2 \subseteq \mathfrak{S}(B_2, A/B)$ for any regular subrings $B_1 \supseteq B_2$ of $A$ containing $B$ with $[B_1 : B]_l < \infty$.

**Remark 1.** If the condition (1) is satisfied, then $J(\mathfrak{S}(B', A/B), B') = B$.
for any regular intermediate ring $B'$ of $A/B$ with $[B' : B]_{1} < \infty$. In fact, if $b'$ is an arbitrary element of $J(\mathfrak{S}(B', A/B), B')$ not contained in $B$ then $T = B[b']$ is a subring of $B'$ properly containing $B$. Since $\text{Hom}_{R_{1}}(B', A) = \mathfrak{S}(B', A/B)A_{r}$, we have $\text{Hom}_{R_{1}}(T, A) = \text{Hom}_{R_{1}}(B', A)|T = (\mathfrak{S}(B', A/B)|T)A_{r} = 1|T)A_{r}$, whence it follows a contradiction $[T : B]_{1} = 1$.

Now, we shall prove our first theorem which contains evidently the proposition cited at the opening as well as [5, Lemma 2].

**Theorem 1.** Let $A/B$ be left locally finite, and the condition (1) satisfied. If $T$ is a simple intermediate ring of $A/B$ with $[T : B]_{1} < \infty$ then $A$ is $T$-$A$-completely reducible. In particular, if $T$ is a regular subring of $A$ with $[T : B]_{1} < \infty$ then $A$ is homogeneously $T$-$A$-completely reducible with $[A|T_{1}A_{r}] = [V_{A}(T)|V_{A}(T_{1})]$ and $T$ is $f$-regular.

**Proof.** Let $M$ be an arbitrary minimal $T$-$A$-submodule of $A$ such that the composition series of $M$ as right $A$-module is of the shortest length among minimal $T$-$A$-submodules of $A$. Then, $M = eA$ with a non-zero idempotent $e$. In virtue of Lemma 1, we can find an intermediate ring $T^{*}$ of $A/T[E, e]$ with $[T^{*} : B]_{1} < \infty$ and $[eT^{*}]T^{*} = [eA|A]$. One may remark here that $TeT^{*} = eT^{*}$. In fact, for each $t \in T$ there exists some $a \in A$ with $ea = te \in T^{*}$, so that $te = e\cdot ea \in eT^{*}$. By Lemma 2 (a), $\text{Hom}_{R_{1}}(T^{*}, A) = \mathfrak{S}(T^{*}, A/B)A_{r}$ is $T^{*}$-$A_{r}$-completely reducible. Accordingly, the $T^{*}$-$A_{r}$-module $\text{Hom}_{R_{1}}(T^{*}, A) = \bigoplus_{j} M_{j}$ with $T^{*}$-$A_{r}$-irreducible $M_{j}$. By Lemma 2 (b), $M_{j} = \sigma_{j}u_{j}A_{r}$ with some $\sigma_{j} \in \mathfrak{S}(T^{*}, A/B)$ and non-zero $u_{j} \in V$. Since $M_{j} \subseteq \text{Hom}_{R_{1}}(T^{*}, A)$ and $TeT^{*} = eT^{*}$, each $M_{j} = (Te)M_{j}$ is a $T$-$A$-submodule of $A$. Further, there holds $M_{j} = u_{j} \cdot (Te)\sigma_{j} \cdot A = u_{j} \cdot (TeT^{*})\sigma_{j} \cdot A = u_{j} \cdot (eT^{*})\sigma_{j} \cdot A$, whence it follows $[M_{j}|A] = [u_{j} \cdot \sigma_{j} \cdot A|A] \leq [e\sigma_{j} \cdot A|A] \leq [e\sigma_{j} \cdot T^{*} \sigma_{j} | T^{*} \sigma_{j}] = [eT^{*} | T^{*}] = [M|A]$ by Lemma 1. Recalling here that $[M|A]$ is the least, we see that each $M_{j}$ is either 0 or $T$-$A$-irreducible. Finally, noting that $A$ is $T_{1}\cdot \text{Hom}_{R_{1}}(A, A)$-irreducible, there holds $A = e(T_{1}\cdot \text{Hom}_{R_{1}}(A, A)) = (Te)\sum M_{j} = \sum M_{j}$, which proves evidently the complete reducibility of $A$ as $T$-$A$-module. Now, the latter assertion will be evident by Prop. 1 (b).

The next has been proved in [2] and [5]. Nevertheless, according to the idea in [7], we shall present here another proof that needs only Lemma 2 and Th. 1.
COROLLARY 1. Let $A$ be left locally finite over a regular subring $B$, and $\mathfrak{H}A_r$ is dense in $\text{Hom}_{B_r}(A, A)$ for an automorphism group $\mathfrak{H}$ containing $\mathfrak{V}$. If $B'$ is a regular intermediate ring of $A/B$ with $[B' : B] < \infty$ then $\mathfrak{H}(B')A_r$ is dense in $\text{Hom}_{B_r}(A, A)$ and $J(\mathfrak{H}(B'), A) = B'$.

Proof. Let $T$ be an arbitrary intermediate ring of $A/B'[E]$ with $[T : B] < \infty$. Evidently, $\text{Hom}_{B_r}(T, A)$ is a $T_r - A_r$-submodule of $\text{Hom}_{B_r}(T, A) = (\mathfrak{H}T)A_r$. And then, by Lemma 2 (b), $\text{Hom}_{B_r}(T, A) = \oplus (\sigma_\mathfrak{H}u_{T}) T A_r$ with some $\sigma_\mathfrak{H} \in \mathfrak{H}$ and non-zero $u_{T} \in V$. In general, if $\tau w_{T} | T (\tau \in \mathfrak{H}, w \in V)$ is contained in $\text{Hom}_{B_r}(T, A)$, one will easily see that $\tau w_{T}$ is contained in $V_{\mathfrak{H}}(B') (\mathfrak{H} = \text{Hom}(A, A))$. Now, let $\sigma_\mathfrak{H} u_{T}$ be an arbitrary $\sigma_\mathfrak{H}u_{T}$. Since $A$ is homogeneously $B'-A$-completely reducible by Th. 1, a standard argument enables us to find such an invertible element $\nu \in V_{\mathfrak{H}}(B')$ that $\nu(a) = \nu(a)d_r$ for all $a \in A$. As $\nu^{-1} \sigma_\mathfrak{H} u_{T}$ is then contained in $V_{\mathfrak{H}}(B'_r A_r) = V_{\nu}(B'_r T A_r)$, $\sigma_\mathfrak{H} u_{T} = \nu v_{1} + \cdots + \nu v_{m}$ with some $v_{j} \in V_{\nu}(B')$. Noting that $T$ contains $E$, one will easily see that every $\nu v_{j} | T A_r$ is a $T_r - A_r$-irreducible submodule of $\text{Hom}_{B_r}(T, A)$, so that $(\nu v_{j} | T) A_r = (\nu v_{j} | T) A_r$ with some $\tau \in \mathfrak{H}$ and $v_{j} \in V$ (Lemma 2). We have then $A = v_{j} A = v_{j} \nu v_{j} | T A_r = v_{j} \nu v_{j} | T A_r = w_{j} A$, whence it follows $w_{j} \in V$. Hence, $\tau w_{j} = \tau w_{j} w_{j}^{-1}$ is contained in $V_{\mathfrak{H}}(B') \cap \mathfrak{H} = \mathfrak{H}(B')$. It follows therefore $\text{Hom}_{B_r}(T, A) = (\mathfrak{H}(B')) T A_r$, which forces $\mathfrak{H}(B') A_r$ to be dense in $\text{Hom}_{B_r}(A, A)$. Finally, to be easily verified, $B_r = V_{\mathfrak{H}}(B_r) = V_{\mathfrak{H}}(\mathfrak{A} A_r)$, which implies $J(\mathfrak{H}, A) = B$. And hence, by the fact proved above, $J(\mathfrak{H}(B'), A) = B'$. Patterning after the proof of [2, Lemma 2], we readily obtain the next:

LEMMA 3. Let $H$ be simple, and $T$ an intermediate ring of $A/B[A]$. If there exists an automorphism group $\mathfrak{H}$ of $H[T]$ with $J(\mathfrak{H}, H[T]) = T$ and $H\mathfrak{H} = H$, and if $H \cap T$ is simple, then $T$ is linearly disjoint from $H$.

The following proposition is a part of [2, Th. 1]. However, for the sake of completeness, we shall give here the proof.

PROPOSITION 4. If $B$ is a regular subring of $A$, the following conditions are equivalent to each other:

(A) $A$ is $h$-Galois and left locally finite over $B$.

(A') $\mathfrak{H}A_r$ is dense in $\text{Hom}_{B_r}(A, A)$ and $A/B$ is right locally finite.

(B) $A$ is Galois and left locally finite over $B$, and $B \cdot V - A$-irreducible.
(B') A is Galois and right locally finite over B, and $A \cdot B \cdot V$-irreducible.

(C) A is Galois and left locally finite over B, and $A \cdot B \cdot V$-irreducible.

(C') A is Galois and right locally finite over B, and $B \cdot V$ $A$-irreducible.

**Proof.** (A) $\Rightarrow$ (B) is obvious by Th. 1 and Cor. 1. Next, we shall prove (B) $\Rightarrow$ (C') $\Rightarrow$ (A'). As A is $B \cdot V \cdot A$-irreducible, H is simple by Prop. 1 (a). For an arbitrary intermediate ring $T$ of $A/B[E, \mathcal{A}]$ with $[T : B]_r < \infty$, we set $T' = J(\mathfrak{S}(T), A)$ and $H' = H \cap T'$. Then, $[V : V_A(T')]_r = [V : V_A(T)]_r = [T' : H]'_r \leq [V : V_A(T)]_r \leq [V : V_A(T')]_r < \infty$. On the other hand, noting that A is $A \cdot T'$-irreducible, Prop. 1 (b) yields also $[V : V_A(T')]_r \leq [T' : H]'_r < \infty$. Combining those above, we obtain $[T' : H]'_r = [V : V_A(T')]_r$. Since $[T' : B]_r = [T' : H]'_r \cdot [H' : B] < \infty$ by Prop. 3 (b), the proposition symmetric to Prop. 3 (a) yields $\text{Hom}_r(T', A) = (\mathfrak{S} | T')A_i$, which proves (C') $\Rightarrow$ (A').

In case the condition (B) is satisfied, for an arbitrary intermediate ring $T$ of $A/B[E, \mathcal{A}]$ with $[T : B]_r < \infty$ there holds $[V : V_A(T)]_r \leq [T : B]_r < \infty$ (Prop. 1 (c)). And so, repeating the above argument, we obtain $[T : B]_r \leq [T' : B]_r < \infty$, which means $A/B$ is right locally finite. We have proved thus (A) $\Rightarrow$ (B) $\Rightarrow$ (C') $\Rightarrow$ (A'), and symmetrically (A') $\Rightarrow$ (B') $\Rightarrow$ (C) $\Rightarrow$ (A).

**Corollary 2.** Let A be left locally finite over a regular subring B. If the condition (1) is satisfied, then (H is simple and) A is $h$-Galois and locally finite over H. And, if $A/B$ is Galois and the condition (1) is satisfied then $A/B$ is $h$-Galois, and conversely.

**Proof.** Let $B'$ be an arbitrary intermediate ring of $A/B[E, \mathcal{A}]$ with $[B' : B]_r < \infty$. Then, by Prop. 1 (c), we have $[V : V_A(B')]_r \leq [B' : B]_r < \infty$. Since A is $B \cdot V \cdot A$-irreducible (Th. 1), A is $V \cdot H \cdot A$-irreducible much more and H is simple by Prop. 1 (a). And then, by Prop. 1 (b), it follows $[V_A(B')]_r \leq [V : V_A(B')]_r < \infty$, which proves evidently the right local finiteness of $A/H$. Hence, Prop. 4 asserts that $A/H$ is locally finite and $h$-Galois. The latter assertion is a direct consequence of Th. 1 and Prop. 4.

The following theorem coincides essentially with [10, Th. 3].

**Theorem 2.** Let A be left locally finite over a regular subring B, and the condition (1) satisfied. If $A'$ is a simple intermediate ring of $A/H$ with $[A' : H]_r < \infty$, then $A'$ is $h$-regular and $V_A(A') = A'$. 

https://doi.org/10.1017/S0027763000026325 Published online by Cambridge University Press
Proof. By Cor. 2, $A/H$ is $h$-Galois and locally finite. If $A_0$ is an arbitrary intermediate ring of $A/A\{E\}$ with $[A_0 : H]_l < \infty$ then $A$ is $A_0$-$A$-irreducible and $A\cdot V \cdot H$-irreducible (Prop. 4). Hence, $[A_0 : H]_l \geq [V : V_\d(A_0)]_r \geq [V_\d(A_0) : H]_l$ by Prop. 1 (b), whence it follows $[A_0 : H]_l = [V : V_\d(A_0)]_r$. And then, Prop. 3 (a) asserts that $\text{Hom}_{B_1}(A_0, A) = (V | A_0)A_r$, which means that $V A_0$ is dense in $\text{Hom}_{B_1}(A, A)$. And then, the proof of [10, Th. 3] asserts that $A'$ is regular. Accordingly, $[V : V_\d(A')]_r \leq [A' : H]_l < \infty$ by Th. 1 and Prop. 1 (b), and $V_\d(A') = J(V (A'), A) = A'$ by Cor. 1.

**Lemma 4.** Let $A/B$ be left locally finite, and the condition (1) satisfied. If $\rho$ is a $B$-ring homomorphism of an intermediate ring $A_1$ of $A/B$ with $[A_1 : B]_l < \infty$ onto a simple intermediate ring $A_2$ of $A/B$ such that $V_\d(A_2)$ is a division ring, then $\rho$ is contained in $\otimes(A_0, A/B)|A_1$ for any regular intermediate ring $A_0$ of $A/A_1$ with $[A_0 : B]_l < \infty$.

**Proof.** Let $\mathfrak{H} = \otimes(A_0, A/B)$. Since $[A_2 : B]_l \leq [A_1 : B]_l < \infty$ and $V_\d(A_2)$ is a division ring, $A$ is $A_2$-$A$-irreducible (Th. 1). And, we have $\text{Hom}_{B_1}(A_1, A) = (\mathfrak{H} | A_1)A_r = \sum_i (a_i | A_1)A_r$ with some $a_i \in \mathfrak{H}$, for $[\text{Hom}_{B_1}(A_1, A) : A_r]_r = [A_1 : B]_l < \infty$. Now, the rest of the proof proceeds in the same way as in the proof of [4, Lemma 3.11].

**Theorem 3.** Let $A/B$ be left locally finite, and the conditions (1), (2) satisfied. If $B_1 \supseteq B_2$ are regular intermediate rings of $A/B$ with $[B_1 : B]_l < \infty$ then $\otimes(B_2, A/B)|B_2$.

**Proof.** Let $\sigma$ be an arbitrary element of $\otimes(B_2, A/B)$, and $B_3 = B\sigma$. We set $V_i = V_\d(B_i) = \sum_i m_i U_i g^{(i)}_{p_0}$ ($i = 2, 3$), where $\langle g^{(i)}_{p_0} \rangle$ is a system of matrix units and $U_i = V_{r_i}(|g^{(i)}_{p_0} \rangle)$ is a division ring. If $m_i \geq m_3$ then we can consider the subrings $A_3, A_3$ of $A$ defined as follows:

$$A_2 = \sum_i m_i B_i g^{(i)}_{p_0} + B_2 g, \quad \text{where} \quad g = \sum_i m_i g^{(i)}_{p_0}, \quad \text{and} \quad A_3 = \sum_i m_i B_3 g^{(i)}_{p_0}.$$

Evidently, $A_2$ is an intermediate ring of $A/B$ with $[A_2 : B]_l < \infty$, $A_3$ a simple intermediate ring of $A/B_3$, and $V_\d(A_3) = U_3$ a division ring. As $\langle g^{(i)}_{p_0} \rangle$ is linearly independent over $B_i$, we can define a $B$-linear map $\rho$ of $A_2$ onto $A_3$ by the following rule:

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Then, one will easily see that \( \rho \) is a ring homomorphism and \( \sigma = \rho | B_2 \). If \( A_0 \) is an arbitrary regular intermediate ring of \( A/A[J] \) with \( [A_0 : B] \) < \( \infty \) then \( \rho \) is contained in \( \mathcal{G}(A_0, A/B) | A_2 \) (Lemma 4), so that \( \sigma = \rho | B_2 \subseteq \mathcal{G}(A_0, A/B) | B_2 \) = \( (\mathcal{G}(A_0, A/B) | B_2) | B_2 \subseteq \mathcal{G}(B_1, A/B) | B_2 \) by (2). On the other hand, if \( m_2 < m_3 \) then the same argument applied to \( \sigma^{-1} \) (instead of \( \sigma \)) enables us to find a simple intermediate ring \( A_0 \) of \( A/B_2 \) with \( [A_0 : B] \) < \( \infty \) such that \( V_\alpha(A_0) \) is a division ring and \( \sigma^{-1} \) = \( \rho | B_3 \) for some \( \rho \in \mathcal{G}(A_0, A/B) \). Applying again the above argument to \( \sigma^{-1} \), we can find a simple intermediate ring \( A^* \) of \( A/(A_0 \rho) [B_1] \) with \( [A^* : B] \) < \( \infty \) such that \( V_\alpha(A^*) \) is a division ring and \( \sigma^{-1} = \tau | A_0 \rho \) for some \( \tau \in \mathcal{G}(A^*, A/B) \). Then, \( \sigma = \rho^{-1} \) = \( \tau | B_2 \) = \( \tau | B_3 \subseteq \mathcal{G}(A^*, A/B) | B_2 \subseteq \mathcal{G}(B_1, A/B) | B_2 \). Hence, in either cases, we have seen \( \mathcal{G}(B_1, A/B) \subseteq \mathcal{G}(B_1, A/B) | B_2 \), whence it follows eventually \( \mathcal{G}(B_1, A/B) = \mathcal{G}(B_1, A/B) | B_2 \).

**Corollary 3.** Let \( A \) be left locally finite over a regular subring \( B \), and \( \mathcal{G} \) an automorphism group of \( A \) containing \( \bar{\mathcal{V}} \). If \( \mathcal{G}|A_\alpha \) is dense in \( \text{Hom}_{\mathcal{G}}(A, A) \) then \( \mathcal{G}(B', A/B) = \mathcal{G}|B' \) for each regular intermediate ring \( B' \) of \( A/B \) with \( [B' : B] \) < \( \infty \). In particular, if \( A/B \) is \( h \)-Galois and left locally finite, then the condition (2) is fulfilled. (Cf. [4, Cor. 3.7].)

**Proof.** If \( B_2 = B'|E] \), then \( \mathcal{G}(B_2, A/B) \subseteq \text{Hom}_{\mathcal{G}}(B_2, A) = (\mathcal{G} | B_2) A_\alpha \), whence it follows \( \mathcal{G}(B_2, A/B) = \mathcal{G} | B_2 \) (Lemma 2 (c)). Now, the same argument as in the proof of Th. 3 enables us to see that \( \mathcal{G}(B', A/B) \subseteq \mathcal{G}(B_2, A/B) | B' = \mathcal{G} | B' \), whence it follows \( \mathcal{G}(B', A/B) = \mathcal{G} | B' \).

2. \( q \)-Galois Extensions. \( A/B \) is said to be \( q \)-Galois (resp. right \( q \)-Galois) if \( B \) is regular and the conditions (1), (2) (resp. (1'), (2')) are satisfied. To be easily verified, if \( A \) is a division ring, the notion of \( q \)-Galois coincides with that of quasi-Galois defined in [8] provided \( A/B \) is left locally finite (cf. [8] and Remark 2). And, \( A/B \) is said to be locally \( h \)-Galois if for each finite subset \( F \) of \( A \) there exists such an intermediate ring \( A' \) of \( A/B[F] \) that \( A'/B \) is \( h \)-Galois. Needless to say, if \( A/B \) is \( h \)-Galois or locally Galois then it is locally \( h \)-Galois.

**Proposition 5.** If \( A/B \) is locally \( h \)-Galois and left locally finite then it is \( q \)-Galois.
Proof. Let $B_1 \supseteq B_2$ be regular intermediate rings of $A/B$ with $[B_i : B]_1 < \infty$, and let $\sigma$ be an arbitrary element of $\mathfrak{S}(B_i, A/B)$. Then, the simple rings $V_\alpha(B_i)$, $V_\delta(B_i)$ and $V_{\alpha'}(B_\sigma)$ are represented as the complete matrix rings over division rings with the systems of matrix units $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, respectively. Now, for an arbitrary finite subset $F$ of $A$, choose an intermediate ring $A^*$ of $A/B[\mathcal{B}_\sigma, F, E, \Gamma_1, \Gamma_2, \Gamma_3]$ such that $A^*/B$ is $\mathfrak{h}$-Galois. Then, by Cor. 3, $\sigma$ can be extended to an automorphism $\sigma^*$ of $A^*$. Since $V_{\alpha'}(B)$ and $V_{\alpha'}(B_\sigma) = V_{\alpha'}(B_\sigma)\sigma^*$ are simple rings, they are the complete matrix rings over division rings with the systems of matrix units $\Gamma^*$ and $\Gamma^*_1$, respectively. If we set $B^* = B_2[\mathcal{B}_\sigma, F, E, \Gamma^*, \Gamma^*_2]$, $B^*$ is a regular subring of $A$ left finite over $B$ such that $V_{\alpha'}(B)$ and $V_{\alpha'}(B_\sigma)$ are simple. Hence, we have seen that there exists a directed set $\{B^*_i\}$ of regular intermediate rings $B^*_i$ of $A/B[\mathcal{B}_\sigma, F, E, \Gamma^*, \Gamma^*_2]$ such that $[B^*_i : B]_1 < \infty$, $A = \cup B^*_i$ and that $V_{\alpha'}(B)$ and $V_{\alpha'}(B_\sigma)$ are simple. It follows therefore $V = \cup V_{\alpha'}(B)$ and $V_{\alpha'}(B_\sigma) = \cup V_{\alpha'}(B_\sigma)$ are simple by [4, Lemma 1.1], which proves (2). Moreover, noting that $B^*$ contains $E$, we see that $\text{Hom}_{\alpha'}(B^*_i, A) = \text{Hom}_{\alpha'}(B^*, A^*)_{A^*} = ((\mathfrak{S}(A^*/B)\mid B^*)_{A^*}A_r \subseteq \mathfrak{S}(B^*, A/B)A_r$.

We insert here [4, Th. 2.3] as an easy consequence ofCors. 2 and 3.

Proposition 6. If $A/B$ is Galois and locally Galois then $A/B$ is $\mathfrak{h}$-locally Galois, and conversely.

Proof. $A/B$ is $\mathfrak{h}$-Galois by Cor. 2, so that for each shade $B'$ we have $\mathfrak{S}(B'/B) \subseteq \mathfrak{S}(B', A/B) = \mathfrak{S}|B'(\text{Cor. 3}).$ And the converse part is obvious.

By the validity of Th. 1, the proof of the next lemma proceeds just like that of [5, Lemma 8] did.

Lemma 5. Let $A/B$ be left locally finite, the condition (1) satisfied, and $A^*$ a regular subring of $A$ containing $B$. If $F$ is an arbitrary finite subset of $A^*$, then $A^*$ contains a regular subring $B'$ of $A$ such that $B' \supseteq B[F]$ and $[B' : B]_1 < \infty$.

Lemma 6. Let $A/B$ be $q$-Galois and left locally finite. If $A'$ is an $f$-regular intermediate ring of $A/B$ then $(H \cap A')\mathfrak{S}(A', A/B) \subseteq H$.

Proof. Let $\sigma$ be an arbitrary element of $\mathfrak{S}(A', A/B)$, and $h$ an arbitrary one of $H \cap A'$. And, choose a simple intermediate ring $B'$ of $A'/B[h]$ such...
that \( V_a(B') = V_a(A') \) and \([B' : B] < \infty\). Then, by Lemma 5 the regular subring \( A' \sigma \) contains a simple subring \( B^* \) containing \( B' \sigma \) such that \( V_a(B^*) \) is simple and \([B^* : B] < \infty\). Here, needless to say, \( B'' = B^* \sigma^{-1} \) is a regular subring of \( A \) as an intermediate ring of \( A'/B' \). And so, \( I_{B''} = \sigma^{-1}|B^* \) is contained in \( \mathfrak{S}(B^*, A/B) \). If \( v \) is an arbitrary element of \( V \), \( I_v = \tau|B^* \) with some \( \tau \in \mathfrak{S}(B^*[E, v], A/B) \) (Th. 3). As \( \nu \tau \) is contained in \( V \), we have \( h \cdot \nu \tau = \nu \tau \cdot h \), whence it follows \( h \sigma \cdot v = v \cdot h \sigma \). We see therefore \( h \sigma \in H \).

Now, we can prove the following theorem that corresponds to [8, Cor. 1].

**Theorem 4.** If \( A \) is \( q \)-Galois and left locally finite over \( B \), then \( H/B \) is outer Galois and \( \mathfrak{S}(H, A/B) = \mathfrak{S}(H/B) \).

**Proof.** Let \( B' \) be an arbitrary intermediate ring of \( H/B[A] \) with \([B' : B] < \infty \) (Cor. 2). Since \( B' \mathfrak{S}(B', A/B) \subseteq H \) (Lemma 6), Lemma 2 (a) yields \([\mathfrak{S}(B', A/B) H_r : H_r] \leq [B' : B] < \infty \). Hence, \( \mathfrak{S}(B', A/B) H_r = \oplus \sigma_i H_r \) with some \( \sigma_i \in \mathfrak{S}(B', A/B) \) and so \( \mathfrak{S}(B', A/B) = \mathfrak{S}(B', H/B) = \{\sigma_1, \ldots, \sigma_l\} \) by Lemma 2 (c). Now, we set \( H = \bigcup B_a \), where \( B_a \) ranges over all the intermediate rings of \( H/B[A] \) with \([B_a : B] < \infty \). We can consider then the inverse limit \( \mathfrak{S} = \lim \mathfrak{S}(B_a, A/B) \), that may be regarded as a set of \( B \)-isomorphisms of \( H \) into \( H \). Since every \( \mathfrak{S}(B_a, A/B) \) is finite and \( \mathfrak{S}(B_a, A/B)|B_a = \mathfrak{S}(B_a, A/B) \) for each \( B_a \supset B_0 \) (Th. 3), we obtain \( \mathfrak{S}|B_a = \mathfrak{S}(B_a, A/B) \) ([1, Cor. 3.9]). If \( T \) is an arbitrary subring of \( H \) properly containing \( B \) with \([T : B] < \infty \) then there exists some \( B_a \) containing \( T \) and then \( J(\mathfrak{S}(B_a, A/B), B_a) = B \) by Remark 1. Combining this with \( \mathfrak{S}|B_a = \mathfrak{S}(B_a, A/B) \), we readily see that \( J(\mathfrak{S}, H) = B \). Further, if \( \sigma \) is in \( \mathfrak{S} \) then for each \( B_a \) we can find a positive integer \( n_a \) such that \( \sigma^{n_a}|B_a = 1 \), which proves \( H\sigma = H \), that is, \( \sigma \) is an automorphism of \( H \). Finally, if \( \tau \) is an arbitrary element of \( \mathfrak{S}(H, A/B) \) then \( H\tau \subseteq H \) (Lemma 6), and so we obtain \( \mathfrak{S}(H, A/B) = \mathfrak{S}(H/B) \) by Prop. 2 (a).

**Corollary 4.** Let \( A/B \) be \( q \)-Galois and left locally finite. If \( A' \) is a simple intermediate ring of \( A/H \) with \([A' : H] < \infty \) then \( A' \) is \( f \)-regular and \( \mathfrak{S}(A', A/B)|H = \mathfrak{S}(H/B) \).

**Proof.** The first assertion is contained in Th. 2, and then \( H\mathfrak{S}(A', A/B) \subseteq H \) (Lemma 6). Recalling now that \( H/B \) is outer Galois (Th. 4), the latter is obvious by Prop. 2 (a).
3. $h$-$q$-Galois Extensions. $A/B$ is said to be $h$-$q$-Galois (resp. right $h$-$q$-Galois) if $B$ is regular and $A/B'$ is $q$-Galois (resp. right $q$-Galois) for each regular intermediate ring $B'$ of $A/B$ with $[B' : B]_q < \infty$ (resp. $[B' : B]_q < \infty$). If $A/B$ is left locally finite and locally $h$-Galois then it is $h$-$q$-Galois by Prop. 5 and Cor. 1. Moreover, in case $A$ is a division ring, the notion of $q$-Galois coincides with that of $h$-$q$-Galois (Lemma 2).

Now, assume that $A/B$ is $h$-$q$-Galois and left locally finite. If $B'$ is a regular intermediate ring of $A/B$ with $[B' : B]_q < \infty$, then $A/B'$ is $q$-Galois and $V^*_A(B')/B'$ is outer Galois (Th. 4), and so $H[B']$ is a simple ring (Prop. 2). Recalling that $A/H$ is locally finite (Cor. 2), Th. 2 yields $H[B'] = V^*_A(B')$. (This fact will be used often without mention in the sequel.) Since $\mathfrak{S}(V^*_A(B')/B') \subseteq \mathfrak{S}(H/B)$ (Cor. 4), $\sigma \rightarrow \sigma|H$ is a continuous monomorphism of compact $\mathfrak{S}(V^*_A(B')/B')$ into $\mathfrak{S}(H/H \cap B')$ and its image is a Galois group of $H/H \cap B'$.

Hence, we see that $\sigma \rightarrow \sigma|H$ is an isomorphism onto $\mathfrak{S}(H/H \cap B')$. (Cf. [4] or [9]). By the aid of this fact, the same argument as in the proof of [5, Lemma 9] enables us to see that if $A$ is $h$-$q$-Galois and left locally finite over $B$ and $A'$ is a regular intermediate ring of $A/B$ with $[H[A'] : H]_q < \infty$ then $H[A']$ is outer Galois and locally finite over $A'$ and $\mathfrak{S}(H[A']/A') \approx \mathfrak{S}(H/H \cap A')$ by contraction. Accordingly, by the validity of Lemma 5, we can apply the same argument as in the proof of [5, Th. 6] to obtain the next theorem that is stated without proof.

**Theorem 5.** Let $A$ be $h$-$q$-Galois and left locally finite over $B$. If $A'$ is a regular intermediate ring of $A/B$, and $H'$ an intermediate ring of $H/B$ that is Galois over $B$, then $H'[A']$ is outer Galois and locally finite over $A'$ and $\mathfrak{S}(H'[A']/A') \approx \mathfrak{S}(H'/H \cap A')$ (algebraically and topologically) by contraction.

As the first corollary to Th. 5, we shall remark that if $A/B$ is $h$-$q$-Galois and left locally finite then the condition (2) can be sharpened as follows:

$$(2^*) \mathfrak{S}(A_1, A/B) \cap A \subseteq \mathfrak{S}(A_2, A/B)$$

for each $f$-regular intermediate rings $A_1 \supseteq A_2$ of $A/B$.

To prove $(2^*)$, let $\sigma$ be an arbitrary element of $\mathfrak{S}(A_1, A/B)$, and $B_1$ a simple intermediate ring of $A_1/B$ with $[B_1 : B]_q < \infty$ and $V^*_A(B_1) = V^*_A(A_1)$. If $B_2$ is an arbitrary regular subring of $A$ between $A_1$ and $B$ with $[B_2 : B]_q < \infty$, then we can find a regular subring $B^*$ of $A$ between $A \sigma$ and $(B_1[B_2]) \sigma$ with $[B^* : B]_q < \infty$ (Lemma 5). Evidently $B' = B^* \sigma^{-1}$ is regular as an intermediate
ring of $A_2/B_1$. Hence, $\sigma' = \sigma|B'$ is in $\mathcal{G}(B', A/B)$, and so $B_2\sigma = B_2\sigma'$ is regular by the condition (2). Now, let $B_2$ be specialized as a simple intermediate ring of $A_2/B$ with $[B_2 : B_1] < \infty$ and $V_\delta(B_2) = V_\delta(A_2)$. Since $A_2 = (H \cap A_2)[B_2]$ by Th. 5 and Prop. 2, Lemma 6 yields $V_\delta(B_2\sigma) = V_\delta((H \cap A_2)\sigma)[B_2\sigma] = V_\delta(A_2\sigma)$. Hence, $V_\delta(B_2\sigma)$ being simple by the above remark, it follows that $\sigma|A_2$ is contained in $\mathcal{G}(A_2, A/B)$.

**Corollary 5.** Let $A/B$ be $h$-$q$-Galois and left locally finite. If $B'$ is a regular intermediate ring of $A/B$ with $[B' : B_1] < \infty$ then $\mathcal{G}(B', A/B) = \mathcal{G}(V_\delta(B'), A_2/B)|B'$.

**Proof.** By Th. 5, $H^* = V_\delta(B') = H[B']$ is outer Galois over $B'$. We set here $H^* = \cup B'_\sigma$, where $B'_\sigma$ ranges over all the $\mathcal{G}(H^*/B')$-invariant shades. Now, let $\rho$ be an arbitrary element of $\mathcal{G}(B', A/B)$. Then, the set $\mathcal{G}_\sigma = \{\rho' \in \mathcal{G}(B'_\sigma, A/B) : \rho'|B' = \rho\}$ is non-empty (Th. 3). If $\rho'$ and $\rho''$ are in $\mathcal{G}_\sigma$ then $\rho'' = \rho' \cdot$ with some $B'_\rho$-(ring) isomorphism $\cdot$ between regular subrings $B'_\rho$ and $B'_\sigma\rho'$. As $B'_\sigma = (H \cap B'_\sigma)[B']$ (Th. 5 and Prop. 2), $B'_\rho \subseteq H[B'_\rho] = V_\delta(B'_\rho)$ by Lemma 6. And so, recalling that $A$ is $q$-Galois and left locally finite over $B'_\rho$ and $B'_\rho/B'_\rho$ is Galois, by [4, Cor. 3.9], Lemma 6 and Prop. 2 (a), we see that $\mathcal{G}(B'_\rho/B'_\rho) = \mathcal{G}(V_\delta(B'_\rho)/B'_\rho)|B'_\rho = \mathcal{G}(B'_\rho, A/B')$. Consequently, $\mathcal{G}(B'_\rho, A/B') = \mathcal{G}(B'_\rho/B'_\rho) \simeq \mathcal{G}(B'_\rho/B')$ is finite, and so $\mathcal{G}_\sigma$ is finite, too. Thus, by [1, Th. 3.6], the inverse limit $\mathcal{C} = \lim \mathcal{G}_\sigma$ is non-empty, which means that $\rho \in \mathcal{G}(B', A/B)$ can be extended to an isomorphism $\rho^*$ of $H^*$ into $A$. Since $(H \cap B'_\rho)\rho \subseteq H$ for each $\rho \in \mathcal{G}_\sigma$ (Lemma 6), $H^*\rho^* = (\cup (H \cap B'_\sigma)[B'])\rho^*$ is to be regular. Hence, we have seen $\mathcal{G}(B', A/B) \subseteq \mathcal{G}(H^*, A/B)|B'$. The converse inclusion is secured by (2*).

**Corollary 6.** Let $A/B$ be $h$-$q$-Galois and left locally finite. If $B'$ is a regular intermediate ring of $A/B[\Delta]$ with $[B' : B] < \infty$ then $H^*[B'] = H^*B$ and $[H^*[B'] : H^*] = [A^* : H \cap A^*]|B'[H \cap B']|B'$ for each intermediate ring $H^*$ of $H/H \cap B'$ and each intermediate ring $A^*$ of $H[B']/B'$.

**Proof.** We set $H' = H \cap B'$ and $\mathcal{G}' = \mathcal{G}(H[B']/B')$. Then, $H'$ is simple by Th. 4 and Prop. 2. If $M$ is an arbitrary $\mathcal{G}(H/H')$-invariant shade then $\mathcal{G}(M[B']/B') = \mathcal{G}'[M[B']] \simeq \mathcal{G}'[M = \mathcal{G}(M/H')]$ (Th. 5), which implies $[M[B'] : B'] = [M : H']$. Accordingly, we obtain $[M[B'] : M]|B'[H \cap B']$. On the other hand, by the validity of Th. 5, Lemma 3 applies to obtain $[M \cdot B' : M]|B'$. 

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It follows therefore \$M[B'] = M \cdot B'\$. Now, it will be easy to see that \$H[B'] = H \cdot B' = \bigoplus_i H b'_i\$, where \{(b'_i)'s\} is an arbitrary linearly independent left \$H\$-basis of \$B'\$. And so, we have \$H^a[B'] = f(\bigoplus_i (H b'_i))\$ \(= \bigoplus_i H^a b'_i\) (Prop. 2 (b)), whence \$H^a[B'] = H^a \cdot B'\$. And, at the same time, the latter assertion is also obvious by Th. 5 and Prop. 2 (b).

If \$A/B\$ is \$k\$-\$q\$-Galois and left locally finite, we can prove the following sharpening of Th. 3, which is at the same time an extension of [6, Th. 5] to simple rings.

**Theorem 6.** Let \$A/B\$ be \$k\$-\$q\$-Galois and left locally finite. If \$A \supsetneq A_2\$ are \$f\$-regular intermediate rings of \$A/B\$ then \$\mathfrak{S}(A_2, A/B) = \mathfrak{S}(A_1, A/B) | A_2\$.

**Proof.** (I) We shall prove first our theorem for regular intermediate rings \$A_1 \supsetneq A_2\$ of \$A/H\$ with \$[A_1 : H]_l < \infty\$. By the validity of (2*), it suffices to prove that \$\mathfrak{S}(A_i, A/B) \subseteq \mathfrak{S}(A_i, A/B) | A_2\$. Choose a simple intermediate ring \$B'_i\$ of \$A_2/B\$ with \$[B'_i : B_1] < \infty\$ and \$V_2(B'_i) = V_2(A_2)\$ (Th. 2). And then, between \$A_1\$ and \$B'_i\$ there exists a regular subring \$B_i\$ of \$A\$ with \$[B_i : B]_l < \infty\$ and \$A_1 = V_2(A_i) = H[B_i]\$. If \$B_2 \supseteq B_1\$ then \$B_2 \supseteq B_2 \supseteq A_2 = V_2(A'_2)\$, and hence \$B_1\$ is a regular subring of \$A\$ left finite over \$B\$ (Th. 4 and Prop. 2 (a)) and \$A_2 = V_2(A_i) = H[B_i]\$. Since \$\mathfrak{S}(A_2, A/B) | B_2 = \mathfrak{S}(B_2, A/B) = \mathfrak{S}(A_1, A/B) | B_2 = \mathfrak{S}(A_1, A/B) | B_1\$ (Cor. 5 and Th. 3), for each \$\sigma \in \mathfrak{S}(A_i, A/B)\$ with \$\sigma | B_2 = \sigma | B_2\$ we can find some \$\rho \in \mathfrak{S}(A_1, A/B)\$ with \$\rho | B_2 = \sigma | B_2\$ as \$A_2\$ is finite over \$H[B_2] = H[B_3]\$ (Cor. 4), \$\rho^{-1}\$ is contained in \$\mathfrak{S}(A_2/B_2) = \mathfrak{S}(A_2/A_2 \cap B_1) = \mathfrak{S}(A_1/B_1) | A_2\$ (Th. 5). Hence, \$\sigma\$ is in \$\mathfrak{S}(A_1, A/B) | A_2\$.

(II) Now, assume that \$A_i\$ be \$f\$-regular, and take simple intermediate rings \$B_i\$ of \$A_i/B\$ with \$[B_i : B_i]_l < \infty\$ and \$V_2(B_i) = V_2(A_i)\$ \((i = 1, 2)\$. Then, \$A'_1 = V_2(A_i) = H[B_i]\$ are finite over \$H\$ (Cor. 2), \$A'_1 \supsetneq A'_2 \supset H$ and \$A'_2 \supsetneq A_2 \supset B_1\$. Now, let \$\sigma_i\$ be arbitrary elements of \$\mathfrak{S}(A_i, A/B)\$. Then, by Cor. 5 and (2*), \$\sigma_i | B_1 = \tau_i | B_i\$ for some \$\tau_i \in \mathfrak{S}(A'_1, A/B)\$. Recalling that \$A_i = (H \cap A_i)[B_i]\$ (Th. 5 and Prop. 2), we see that \$A_i | a_i = ((H \cap A_i) a_i) | B_i = H[B_i \tau_i] = A'_1 \tau_i\$ (Lemma 6). And so, \$\sigma_i \tau_i^{-1}\$ is contained in \$\mathfrak{S}(A'_i/B_i) | A_i$ (Th. 4 and Prop. 2 (a)), whence it follows \$\sigma_i \in \mathfrak{S}(A'_i, A/B) | A_i$. Combining this with (2*), we obtain \$\mathfrak{S}(A_i, A/B) = \mathfrak{S}(A'_i, A/B) | A_i$. On the other hand, there holds \$\mathfrak{S}(A_i, A/B) = \mathfrak{S}(A'_i, A/B) | A_i$ by (I). Hence, it follows \$\mathfrak{S}(A_2, A/B) = \mathfrak{S}(A'_1, A/B) | A_2 = \mathfrak{S}(A'_1, A/B) | A_1 | A_2 = \mathfrak{S}(A_1, A/B) | A_2\$, completing the proof.

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Remark 2. Let $A$ be a division ring, and left locally finite over $B$. Then, $\mathfrak{G}(B', A/B)$ is nothing but the set of all $B$-ring isomorphisms of $B'$ into $A$, and the condition (2) is superfluous. Following [6] and [8], we consider the following conditions:

(1°) $\mathfrak{G}(B', A/B) \cong 1$ for each subring $B'$ of $A$ properly containing $B$ with $[B': B] < \infty$, and $\mathfrak{G}(B_1, A/B)|B_2 = \mathfrak{G}(B_2, A/B)$ for each intermediate rings $B_1 \supseteq B_2$ of $A/B$ with $[B_1 : B_2] < \infty$.

(2°) $H/B$ is Galois, and $\mathfrak{G}(B_1, A/B)|B_2 = \mathfrak{G}(B_2, A/B)$ for each intermediate rings $B_1 \supseteq B_2$ of $A/B$ with $[B_1 : B_2] < \infty$.

(3°) $H/B$ is Galois, and $\mathfrak{G}(A_1, A/B)|A_2 = \mathfrak{G}(A_2, A/B)$ for each intermediate rings $A_1 \supseteq A_2$ of $A/H$ with $[A_1 : H] < \infty$.

(4°) $J(\mathfrak{G}(B', A/B), B') = B$ for each intermediate ring $B'$ of $A/B$ with $[B' : B] < \infty$.

If $A/B$ is $q$-Galois (and necessarily $h$-$q$-Galois by Lemma 2), then all the conditions (1°)-(4°) are fulfilled by Remark 1 and Ths. 4, 6. Conversely, if (4°) is satisfied then $A/B$ is $q$-Galois. To see this, it will suffice to prove that if $\{x_1, \ldots, x_n\}$ is a subset of $B'$ that is linearly left independent over $B$ then there exists an element $\xi \in \mathfrak{G}(B', A/B)A_r$ such that $x_i\xi = 0$ for all $i \neq n$ and $x_n\xi \neq 0$, where $B'$ is an arbitrary intermediate ring of $A/B$ with $[B' : B] < \infty$. If $n = 2$, by (4°) there exists some $\rho \in \mathfrak{G}(B', A/B)$ with $(x_1x_1^{-1})\rho \neq x_1x_1^{-1}$, and then one will easily see that $\xi = \rho - 1(x_1x_1^{-1})\rho$ is an element requested. Now, assume that we can find $\xi_1, \ldots, \xi_{n-1} \in \mathfrak{G}(B', A/B)A_r$ such that $x_i\xi_j = \delta_{ij}x_i$ ($i, j = 1, \ldots, n - 1$). There holds then $x_i(\sum \xi_j - 1) = 0$ for $i = 1, \ldots, n - 1$. If $x_n(\sum \xi_j - 1) \neq 0$, our assertion is true for $\xi = \sum \xi_j - 1$. If otherwise $x_n = \sum_i^{-1}x_n\xi_i$ then, say, $\{x_1, x_n\}$ is linearly left independent over $B$. We set here $\xi_1 = \sum_i \rho_p\alpha_p$ with $\rho_p \in \mathfrak{G}(B', A/B)$ and $\alpha_p \in A$. If $B'' = B'[\cup B'|p_r, \cup a_p\xi_i|]$, then by the case $n = 2$ there exists an element $\xi' \in \mathfrak{G}(B'', A/B)A_r$ such that $x_i\xi' = 0$ and $x_n\xi' = 0$. Now, it will be easy to see that $x_i\xi_i' = 0$ for $i = 1, \ldots, n - 1$, so that $\xi = \xi_i\xi_i'$ contained in $\mathfrak{G}(B', A/B)A_r$ is an element requested.

Next, we shall prove the implications $(2°) \Rightarrow (4°)$ and $(3°) \Rightarrow (4°)$. In any rate, we have $J(\mathfrak{G}(B', A/B), B') \subseteq J(\mathfrak{G}(H/B'), B') = H \cap B'$. If $(2°)$ is satisfied then $\mathfrak{G}(H/B)|H \cap B' \subseteq \mathfrak{G}(B', A/B)|H \cap B'$, whence it follows $J(\mathfrak{G}(B', A/B), B') = B$. On the other hand, if $(3°)$ is satisfied then $\mathfrak{G}(H/B) \subseteq \mathfrak{G}(H[B'], A/B)|H$.
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Since the implication $(1°) \implies (4°)$ is obvious, we have proved that $A$ is $q$-
Galois if and only if any of the equivalent conditions $(1°)-(4°)$ is satisfied (cf.
[6, Th. 1] and [8, Th. 3]).

In case $A/B$ is an algebraic field extension, it is well-known that $A/B$ is
Galois (in our sense) if and only if it is normal and separable. The next
theorem may be regarded as an extension of this fact to simple rings, and
contains [6, Cor. 3] as well as [4, Th. 3.5].

**Theorem 7.** If $A$ is $h$-$q$-Galois and left locally finite over $B$ and $[A : H]_l < \infty$, then $A/B$ is $h$-Galois and $\mathfrak{G}(A', A/B) = \mathfrak{G}|A'$ for each $f$-regular inter-
mediate ring $A'$ of $A/B$. In particular, if $A$ is locally Galois over $B$ and $[A : H]_l < \infty$, then $A/B$ is $\mathfrak{G}$-locally Galois.

**Proof.** Since $A'$ is $f$-regular, we can find an intermediate ring $A''$ of
$A/H[E, A']$ with $[A' : H]_l < \infty$ (Cor. 2). Now, by the validity of Cors. 2, 4
and Th. 6, we can apply the same argument as in the proof of [4, Lemma 3.9]
to see that $\mathfrak{G}(A'', A/B) = \mathfrak{G}|A''$. Then, we obtain $\mathfrak{G}|A' = \mathfrak{G}(A'', A/B)|A' =
\mathfrak{G}(A', A/B)$ (Th. 6), and in particular $\mathfrak{G}|H = \mathfrak{G}(H, A/B) = \mathfrak{G}(H/B)$ (Th. 4).
Hence, there holds $J(\mathfrak{G}|A') = J(\mathfrak{G}|H, H) = B$. And so, $A$ being $B$-$V$-$A$-
irreducible (Th. 1), $A/B$ is $h$-Galois by Prop. 4. The latter assertion is [4,
Th. 4.4] itself, and is clear by the former and Prop. 6.

Next, we shall prove an extension of the latter half of [2, Th. 1], that
contains completely [6, Cor. 2].

**Theorem 8.** Let $A/B$ be $h$-$q$-Galois and left locally finite. If $B'$ is a regular
intermediate ring of $A/B$ with $[B' : B]_l < \infty$ then $\infty > [B' : B]^t \geq [V : V_A(B')] = [V_A(B') : H] = [B' : H \cap B']$, and in particular $A/B$ is (two-sided) locally
finite.

**Proof.** We set $V_A(B') = \sum K'd'_{k,k'}$, where $d' = \{d'_{k,k'}\}$ is a system of
matrix units and $K' = V_{r_{A}(B')} (d')$ is a division ring (Cor. 2), and consider $T
= B'[E, d', d']$ and $H' = H \cap T$ (simple by Th. 4 and Prop. 2). Since $H \mathfrak{G}(V_A(B')
(T)/T) = H$ (Cor. 4) and $A$ is $B$-$V$-$A$-irreducible (Th. 1), Prop. 1 and Lemma

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*In case $[B : B]_l$ coincides with $[B' : B]_l$, the equal dimensions will be denoted as $[B : B]_l$. 

https://doi.org/10.1017/S00277630000026325 Published online by Cambridge University Press
And then, $A$ being $A\cdot V \cdot H$-irreducible by Cor. 2 and Prop. 4, we obtain $[T : H'] \geq [V : V_{a}(T)] \geq [V'_{a}(T) : H'] \geq [T : H']$, again by Prop. 1 and Lemma 3. Hence, it follows $[T : H'] = [V : V_{a}(T)] = [V'_{a}(T) : H]$ and $[T : B] = [T : H'] \cdot [H' : B] = [T : H'] \cdot [H' : B] = [T : B]$, by Prop. 2 (c).

Since $A/B'$ is $h$-$q$-Galois, by the same reason, we have $[V_{a}(B') : V_{a}(T)] = [V'_{a}(T) : V'_{a}(B')]$ and $[T : B] = [T : B']$. Combining those above with the fact that $A$ is $B' \cdot V' \cdot A$-irreducible (Th. 1), it follows at once $[T : B] = [B' : B] = [V : V_{a}(B')] = [V'_{a}(T) : H]$ by Prop. 1 (b). Now, we shall prove $[B' : H \cap B'] = [V'_{a}(B') : H]$. If $H^* = (H \cap B')[A]$ and $B^* = H^*[B']$ then $B^*$ is regular as an intermediate ring of $V'_{a}(B')/B'$ (Th. 4 and Prop. 2 (a)). Hence, Cor. 6 yields $[B^* : H \cap B^*] = [V'_{a}(B^*) : H'] = [V'_{a}(B') : H']$.

Recalling here that $\mathcal{O} = \mathcal{O}(V'_{a}(B')/B') = \mathcal{O}(H[B']/B') \approx \mathcal{O}(H/H \cap B')$ by contraction (Th. 5), Prop. 2 (c) yields $[B^*[B'] = \#(\mathcal{O} \mid B^*) = \#(\mathcal{O} \mid H^*) = \#(\mathcal{O} \mid H \cap B^*) = [H \cap B^* : H \cap B']$, whence it follows $[B' : H \cap B'] = [B^* : H \cap B^*]$. We have proved therefore $[B' : H \cap B'] = [V'_{a}(B') : H]$.

**Lemma 7.** Let $A$ be $h$-$q$-Galois and left locally finite over $B$. If $A'$ is an $f$-regular intermediate ring of $A/B$ then $A/A'$ is left locally finite and $[A^* : H \cap A'] = [V : V_{a}(A')]$.

**Proof.** Let $N$ be an arbitrary $\mathcal{O}(H/B)$-invariant shade of $A$. Then, by Th. 5 and Prop. 2 (b), we have $[N[A'] : A'] = [N : N \cap A'] < \infty$ and $H \cap N[A'] = H \cap (N[H \cap A'])[A'] = N[H \cap A']$. Since $H \cap A'$ is also a regular intermediate ring of $A/B$ (Prop. 2 (a)), we obtain $[H \cap N[A'] : H \cap A'] = [N[H \cap A'] : H \cap A'] = [N : N \cap A'] = [N[A'] : A'] < \infty$ again by Th. 5 and Prop. 2 (b). We choose here a simple intermediate ring $B'$ of $A'/B$ with $[B' : B] < \infty$ and $V_{a}(B') = V_{a}(A')$, and set $B^* = N[B']$. Then, $B^*$ is a regular subring of $A$ with $[B^* : B] < \infty$ as an intermediate ring of $V'_{a}(B')/B'$ (Th. 4 and Prop. 2). Recalling that $H[B^*] = V'_{a}(B^*) \supseteq N[A'] \supseteq B^* \supseteq A$, Cor. 6 and Th. 8 imply $[N[A'] : H \cap N[A']] = [B^* : H \cap B^*] = [V : V_{a}(B^*)] = [V : V_{a}(B')] < \infty$. Combining this with $[H \cap N[A'] : H \cap A'] = [N[A'] : A'] < \infty$, it follows at once $[A' : H \cap A'] = [N[A'] : H \cap N[A']] = [V : V_{a}(B')] = [V : V_{a}(A')]$, which is the latter assertion. Next, we shall prove the first half. Here, without loss of generality, we may assume that $A' \subseteq H$. For an arbitrary finite subset $F$ of $A$, we set $B_{F} = B[E, A, F]$. Then, $[A'[H \cap B_{F}] : A'] < \infty$ by Prop. 2 and
Theorem 9. Let $A$ be $h$-$q$-Galois and left locally finite over $B$. If $A'$ is an $f$-regular intermediate ring of $A/B$ then $A$ is $h$-$q$-Galois, right $h$-$q$-Galois and locally finite over $A'$ and $[A': H \cap A'] = [V: V_d(A')] = [V_d^2(A') : H]$.

Proof. To prove the first assertion, we may restrict our attention to the case that $A' \subseteq H$. If $A''$ is a regular intermediate ring of $A/A'$ with $[A'': A']$ < $\infty$ then, to be easily verified, $A''$ is $f$-regular. Since $A_0 = A''[E, J]$ is left finite over $A'$ (Lemma 7), $\mathfrak{S}(A'', A/A') A_r = (\mathfrak{S}(A_0, A/A')| A'') A_r$ (Th. 6). And so, we see that it suffices to prove that Hom$_{A_r}(A'', A) = \mathfrak{S}(A'', A/A') A_r$ for each intermediate ring $A''$ of $A/A'[E, J]$ with $[A'': A']$ < $\infty$. By Th. 4 and Prop. 2 (a), $H'' = A'' \cap H$ is a simple subring of $H$. As $\mathfrak{S}(H/A')| H'' = \mathfrak{S}(H, A/B)| H'' = \mathfrak{S}(V_d(A''), A/B)| H''$ (Ths. 4 and 6), it follows $\mathfrak{S}(H/A')| H'' = \mathfrak{S}(V_d^2(A''), A/A')| H''$ (Prop. 2 (b)). Recalling that $\mathfrak{S}(H/A') H_r$ is dense in Hom$_{A_r}(H, H)$ (Prop. 2) and that $[H'': A']$ < $\infty$ (Prop. 2 (c) or Th. 8), we have then Hom$_{A_r}(H'', H) = (\mathfrak{S}(V_d^2(A''), A/A')| H'') H_r = \sum (a_i| H'') H_r$ with some $a_i \in \mathfrak{S}(V_d^2(A''), A/A')$ (Lemma 2). Since $a_i| H'' \neq a_j| H''$ if $i \neq j$, irreducible $(a_i| A''') A_r$ is not $A''$-$A_r$-isomorphic to $(a_j| A'') A_r$ (Lemma 2), which implies $\sum (a_i| A'') A_r = \sum (a_i| H'') H_r$. By [4, Lemma 1.5] and Th. 8, there holds $[\sum (a_i| A'') A_r : A_r] = [V : V_d(A'')] = [V_d^2(A') : H'']$. On the other hand, the same reason together with Ths. 4 and 6 implies $\infty > [\sum (a_i| A'') A_r : A_r] = [\sum (a_i| A'') A_r : A_r] = [V : V_d(A'')] = [V_d^2(A') : H'']$. It follows therefore $[\sum (a_i| A'') A_r : A_r] = [V : V_d(A'')] = [\sum (a_i| A'') A_r : A_r]$, whence we obtain $[\sum (a_i| A'') A_r : A_r] = s \cdot [V : V_d(A'')] = [\sum (a_i| H'') H_r, H] : H_r$, $[V : V_d(A'')] = [H'' : A'] \cdot [A'' : H'']$, $[A'' : A']$ by Lemma 7. We have proved therefore Hom$_{A_r}(A'', A) = \sum (a_i| V'') A_r = \mathfrak{S}(A'', A/A') A_r$ by (2*), and $A/A'$ is locally finite by Lemma 7 and Th. 8. The final equalities are now direct consequences of Lemma 7 and Th. 8, for $A' \cap H$ is $f$-regular. In particular, noting that $[A' : H \cap A'] = [V : V_d(A')]$, we can repeat a symmetric argument to see that $A/A'$ is right $h$-$q$-Galois.

Corollary 7. The following conditions are equivalent to each other:

(Q) $A/B$ is $h$-$q$-Galois and left locally finite.
(Q') \( A/B \) is right \( h \)-\( q \)-Galois and right locally finite.

Combining Th. 9 with Th. 7, we readily obtain the following:

**Corollary 8.** Let \( A \) be \( h \)-\( q \)-Galois and left locally finite over \( B \) and \([A : H] \leq 8\). If \( A' \) is an \( f \)-regular intermediate ring of \( A/B \) then \( A/A' \) is \( h \)-Galois and locally finite.

Now, we shall add to Prop. 4 other equivalent conditions to complete \([2, \text{Th. } 1]\).

**Proposition 7.** Let \( B \) be a regular subring of \( A \). \( A/B \) is \( h \)-Galois and left locally finite over \( B \) if and only if any of the following conditions is satisfied:

(D) \( A \) is Galois and left locally finite over \( B \), \( H \) is simple, and \([V : V_d(B')] : H] = [V : V_d(B')]_r \) for every regular intermediate ring \( B' \) of \( A/B \) with \([B' : B] < \infty\).

(D') \( A \) is Galois and right locally finite over \( B \), \( H \) is simple, and \([V : V_d(B')] : H] = [V : V_d(B')]_r \) for every regular intermediate ring \( B' \) of \( A/B \) with \([B' : B] < \infty\).

(E) \( A \) is left locally finite over \( B \) and Galois over every regular subring left finite over \( B \), \( H \) is simple, and \([A' : H] = [V : V_d(A')]_r \) for every regular intermediate ring \( A' \) of \( A/H \) with \([A' : H] < \infty\).

(E') \( A \) is right locally finite over \( B \) and Galois over every regular subring right finite over \( B \), \( H \) is simple, and \([A' : H] = [V : V_d(A')]_r \) for every regular intermediate ring \( A' \) of \( A/H \) with \([A' : H] < \infty\).

**Proof.** Since \((A) \Rightarrow (D)\) and \((E)\) is evident by Cor. 1 and Th. 9, it is left to prove the converse. Now, let \( T \) be an arbitrary intermediate ring of \( A/B \) \([E, A] \) with \([T : B] < \infty\), and set \( T' = f(\mathfrak{G}(T), A) \) and \( H' = H \cap T' \). Then, \([H' : B] < \infty \) by Prop. 3 (b). Noting that \( A \) is \( H'[T] \)-\( A \)-irreducible, Prop. 1 (b) yields \( \infty > [H'[T] : H'] = [V_d(H') : V_d(H'[T])]_r = [V : V_d(T')]_r \), whence it follows \([T' : H'] = [V : V_d(T')]_r \). In case (D), Lemma 3 yields then \([T' : H'] = [H' : T'] = [V : V_d(T')]_r \). Hence, we have \([T' : H'] = [V : V_d(T')]_r < \infty \), so that it follows \( \text{Hom}_{B'}(T', A) = (\mathfrak{G} | T')_A \) by Prop. 3 (a), which proves \((D) \Rightarrow (A)\). Now, we shall prove \((E) \Rightarrow (A)\). If \( N \) is an arbitrary \( \mathfrak{G}(H/B) \)-invariant shade of \( H' \), then \( \mathfrak{G}(T) | N[T] \) and \( \mathfrak{G}(T)|N \) are (outer) Galois groups of \( N[T]/T \) and \( N/H' \), respectively. There holds then \([N : H'] = \# (\mathfrak{G}(T)|N) = \# (\mathfrak{G}(T)|N[T]) \)
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\( [N[T] : T] \) (Prop. 2 (c)), and so Lemma 3 yields \( [N \cdot T : H'] = [N \cdot T : N] \), \( [N : H'] = [T : H'] \cdot [N[T] : T] = [N[T] : H'] \), whence we obtain \( N \cdot T = N[T] \). We readily see then \( H \cdot T \) is a regular intermediate ring of \( A/H \) with \( [H \cdot T : H] = [T : H'] < \infty \). It follows therefore \( [T : H'] = [H \cdot T : H] = [V : V_n(T)] \), and we have \( \text{Hom}_B(T, A) = (\mathcal{G} \cdot T)A \), again by Prop. 3 (a).

We shall present here a notably short proof to [4, Lemma 2.2]\(^1\).

**Proposition 8.** If \( A \) is Galois and left locally finite over \( B \) and \( [V : C] < \infty \), then \( A/B \) is \( \mathcal{G} \)-locally Galois.

**Proof.** By the validity of Prop. 6, it suffices to prove that \( A/B \) is locally Galois. To be easily seen, \( (H \text{ is simple and} \, [V \cdot (B') : H'] = [V : V_n(B')] ) \), for each regular intermediate ring \( B' \) of \( A/B \) with \( [B' : B] < \infty \). \( A/B \) is therefore \( h \)-Galois by Prop. 7. We set here \( V = \sum U g_{pq} \), where \( I' = \langle g_{pq} \rangle \) is a system of matrix units and \( U = V_{i'}(I') \) a division ring. Now, let \( B' \) be an arbitrary intermediate ring of \( A/B[E, I'] \) with \( [B' : B] < \infty \). Since \( J(\mathcal{G}|B', B') = B \), there exists a finite subset \( \bar{\mathcal{G}} \) of \( \mathcal{G} \) with \( J(\mathcal{G} \cap B', B') = B \). If \( N \) is an arbitrary \( \mathcal{G}(H/B) \)-invariant shade of \( B' \cap B' \cap H \) then \( B'[\bigcup_{\bar{\mathcal{G}}} B' \cap B' \cap H] \) is contained in the simple ring \( M = N[B'] \) (Th. 5 and Prop. 2 (b)). And so, \( \mathcal{G} = \mathcal{G}(B'[\bar{\mathcal{G}}]) \) induces an automorphism group of \( M \). Since \( J(\mathcal{G}|M, M) = B \) and \( V_{n}(B) \) is evidently simple, \( M/B \) is Galois, which implies that \( A/B \) is locally Galois.

We shall conclude this section with the following theorem, whose first assertion is [4, Lemma 4.2].

**Theorem 10.** (a) If \( A/B \) is locally Galois then \( H \) is simple and for each finite subset \( F \) of \( A \) there exists a simple intermediate ring \( A' \) of \( A/H[F] \) such that \( [A' : H] < \infty \) and \( A'/B \) is Galois, and conversely provided \( A/B \) is left locally finite.

(b) If \( A/B \) is locally Galois then so is \( A/A' \) for every \( f \)-regular intermediate ring \( A' \) of \( A/B \).

\(^1\) The proof of Prop. 8 given in [4] enabled us moreover to see that there exists a Galois group \( \mathcal{G} \) of \( A/B \) with the property that \( (\mathcal{G}, A/B) \) is l.f.d. for each finite subset \( \bar{\mathcal{G}} \) of \( \mathcal{G} \), which was needed only to prove the following: If \( A \) is Galois and left locally finite over \( B \) and \( [V : C] < \infty \), then every \( (+) \)-regular subgroup of \( \mathcal{G} \) is regular. However, in [2] and [10], we have proved directly an extension of the last proposition (cf. also Th. 11 (a)).
Proof. (a) Let $V = \sum U g_p$, where $\Gamma = \{ g_p \}$ is a system of matrix units and $U = V_r(\Gamma)$ a division ring. If $B'$ is an arbitrary shade of $B[E, \Gamma]$, then $A' = V_{\eta}(B') = H[B'] = \bigcup N\eta[B']$, where $N\eta$ ranges over all the $\Theta(H/B)$-invariant shades. Now, let $B''$ be a shade of $N\eta[B']$, and $\Theta' = \{ \sigma \in \Theta(B''/B) : B'\sigma = B' \}$. Then, noting that $\Theta(B'/B) \subseteq \Theta' \mid B'$, Th. 5 together with Lemma 6 and Prop. 2 proves that $N\eta[B']/B$ is Galois. Hence, $A'/B$ is locally Galois, and so it is Galois by Th. 7, for $[V_\eta(B) : V_\eta(A')] = [V_\eta(H) : V_\eta(A')] \leq [A' : \Theta] < \infty$ (Prop. 1). And, by the fact used just above, the converse part will be an easy consequence of Prop. 8.

(b) If $B'$ is an intermediate simple ring of $A'/B$ with $[B' : B]_i < \infty$ and $V_\eta(B') = V_\eta(A')$, then $A/B'$ is locally Galois. And so, by (a), for each finite subset $F$ of $A$ there exists a simple intermediate ring $A''$ of $A/V_\eta(B'[F])$ such that $A''/B'$ is Galois and $[V_\eta(B') : V_\eta(A'')] \leq [A'' : V_\eta(B')] < \infty$. Prop. 8 implies then that $A''/B'$ is $\Theta(A''/B')$-locally Galois. Since $A''/A'$ is $h$-Galois and locally finite by Cor. 8, $A''/A'$ is locally Galois again by Prop. 8. We have proved therefore $A/A'$ is locally Galois.

4. ($*/$)-Regular Subgroups. By the validity of Ths. 4, 9 and Cor. 2 (and Lemma 3 if necessary), the proofs of Lemmas 2, 3 of [10] are applicable without any change to those of the following lemmas.

Lemma 8. Let $A$ be $h$-$q$-Galois and left locally finite over $B$, and $\Theta'$ a ($*_{/}$)-regular subgroup of $\Theta$. If $A' = J(\Theta', A)$ then $[A' : H \cap A']_i < \infty$.

Lemma 9. Let $A$ be $h$-$q$-Galois and left locally finite over $B$, and $V'$ a simple subring of $V$ with $[V : V']_r < \infty$. If $V_\eta(V_\eta(V'[F]) \subseteq V'$ for some finite subset $F$ of $A$ then $V_\eta(V')$ is a simple ring.

The first assertion of the following theorem contains [10, Th. 2].

Theorem 11. Let $A$ be $h$-$q$-Galois and left locally finite over $B$, and $\Theta'$ a ($*_{/}$)-regular subgroup of $\Theta$ with $A' = J(\Theta', A)$.

(a) $\Theta'$ is $f$-regular (i.e. $A'$ is simple) and dense in $\Theta(A')$.

(b) $\overline{V \cdot \Theta'} = \Theta(H \cap A')$.

(c) If $\overline{\Theta}$ is an open subgroup of $\Theta$ then $(\text{Cl} \Theta') : (\Omega \cap \text{Cl} \Theta') \Theta < \infty$.

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6) $\text{Cl} \Theta'$ is the topological closure of $\Theta'$ in $\Theta$. https://doi.org/10.1017/S0027763000026325 Published online by Cambridge University Press
Proof. One may remark here that $H' = H \cap A'$ is $f$-regular (Th. 4 and Prop. 2). As $[V : V_{\mathfrak{G}}]_r < \infty$ and $V_{\mathfrak{G}} = V_{\mathfrak{G}}(V_{\mathfrak{G}})$, $V_{\mathfrak{G}}(A') = V_{\mathfrak{G}}(V_{\mathfrak{G}})$ is simple by Lemma 9. Further, by Lemma 8, there holds $[A' : H']_r < \infty$. Since $A/H'$ is locally finite (Th. 9), $V_{r, A'}(A')$ coincides with the center of $V_{r, A'}(A')$ and $J(\mathfrak{G}' \mid V_{r, A'}(A')) = A'$, [10, Lemma 1] proves that $A'$ is simple. And so, $A/A'$ is $h$-q-Galois and locally finite (Th. 9). If $T$ is an arbitrary intermediate ring of $A/A'[E]$ with $[T : A'] < \infty$, then $A$ is $T$-A-irreducible and $[T : V_{r, A'}(A') \cap T] = [V_{r, A'}(A') : V_{r, A'}(T)]$ (Th. 8). Hence, $A/A'$ is $h$-Galois and $\mathfrak{G}'$ is dense in $\mathfrak{G}(A')$ by Prop. 3 (a), which completes the proof of (a). Recalling here that $[T : H']_r = [T : A']_r [A' : H']_r < \infty$ (Lemma 8), for each $\sigma \in \Cl(V \cdot \Cl \mathfrak{G}')$ we can find such an element $\tau \in V \cdot \Cl \mathfrak{G}'$ that $\tau \mid T = \rho \mid T$. And then $\sigma^{-1}$ is contained in $\mathfrak{G}(T) \subseteq \mathfrak{G}(A') = \Cl \mathfrak{G}'$ by (a). Hence, $\sigma$ is contained in $\mathfrak{V} \cdot \Cl \mathfrak{G}'$, which means that $\mathfrak{V} \cdot \Cl \mathfrak{G}'$ is a closed ($*_f$)-regular subgroup of $\mathfrak{G}$ with $J(\mathfrak{V} \cdot \Cl \mathfrak{G}', A) = H'$. Accordingly, (b) is a consequence of (a). Finally, we shall prove (c). Since $J(\mathfrak{V} \cdot \mathfrak{G}', A) = A'$ and $V_{r, \mathfrak{G}'} = V_{\mathfrak{G}}$, it suffices to prove our assertion for closed $\mathfrak{G}' = \mathfrak{G}(A')$. Moreover, without loss of generality, we may assume that $\mathfrak{G} = \mathfrak{G}(A')$ for some intermediate ring $B'$ of $A/B[E]$ with $[B' : B]_r < \infty$. If $T = A[B']$ (finite over $A'$) then $\mathfrak{G}(T)$ is a closed ($*_f$)-regular subgroup of $\mathfrak{G}'$ with $J(\mathfrak{G}(T), A) = T$ by Cor. 1 or [5, Theorem 1]. And so, by (b), it follows $(\mathfrak{G} \cap \mathfrak{G}')(\mathfrak{G} \cap \mathfrak{G}')(\mathfrak{G} \cap \mathfrak{G}) = \mathfrak{G}((V_{r, A'}(A') \cap T))$. Hence, by Th. 4 and Prop. 2 (c), we obtain $(\mathfrak{G}' : (\mathfrak{G} \cap \mathfrak{G}')(\mathfrak{G} \cap \mathfrak{G})) = (\mathfrak{G}' : (\mathfrak{G}((V_{r, A'}(A') \cap T)))) = \# (\mathfrak{G}' \mid V_{r, A'}(A') \cap T] = [V_{r, A'}(A') \cap T : A'] < \infty$.

As a direct consequence of Th. 11 (a) andCors. 1, 8, we readily obtain the following theorem.

Theorem 12. If $A$ is $h$-q-Galois and left locally finite over $B$ and $[A : H]_r \leq 8$ then there exists a 1-1 dual correspondence between closed ($*_f$)-regular sub-

A/B, in the usual sense of Galois theory.

Remark 3. Evidently, Th. 12 is nothing but [2, Th. 5], and the assumption cited in Th. 12 is the best one obtained by now to allow the existence of Galois correspondence.

Let $A/B$ be $h$-q-Galois and left locally finite. If $T$ is an intermediate ring of $A/B$ left finite over $B$ such that $A$ is $T$-A-irreducible and $J(\mathfrak{G}(T), A) = T$, then $T$ is a simple ring by Th. 11 (a). In particular, if $A/B$ is $h$-Galois then
the assumption $I(\mathcal{G}(T), A) = T$ is automatically enjoyed by [5, Th. 1] (cf. [2, Cor. 6]). The next will be an easy consequence of the above remark, Th. 1 and [4, Lemma 1.1].

**Proposition 9.** Let $A/B$ be locally $h$-Galois and left locally finite. If $V$ is a division ring then every intermediate ring of $A/B$ is simple.

**Remark 4.** Let $A$ be left algebraic over $B$ (that is, $[B[a] : B] < \infty$ for every $a \in A$). If every intermediate ring of $A/B$ left finite over $B$ is a simple ring then $V$ is a division ring. In fact, for an arbitrary non-zero element $v \in V$, $B[v]$ is a simple ring, and so the center of $B[v]$ is a field. Hence, $v$ belonging to the center of $B[v]$ is regular and $v^{-1}$ is contained in $V$.

We shall conclude our study with the following (cf. [2, Th. 2]).

**Theorem 13.** Let $A$ be $h$-q-Galois and left locally finite over $B$, and $\mathcal{G}'$ an $N$-regular subgroup of $\mathcal{G}$. Then, $\mathcal{G}'$ is $(*_f)$-regular if and only if $[V : I(\mathcal{G}')] < \infty$, $V^*_A(I(\mathcal{G}')) = I(\mathcal{G}') = I(Cl(\mathcal{G}'))$ and $(Cl(\mathcal{G}')) : (\mathcal{G} \cap Cl(\mathcal{G}')) I(\mathcal{G}')) < \infty$ for every open subgroup $\mathcal{G}$ of $\mathcal{G}$.

**Proof.** If $\mathcal{G}'$ is $(*_f)$-regular then $I(\mathcal{G}')$ coincides with $V_{\mathcal{G}'}$, so that the only if part is obvious by Th. 11. To prove the if part, we may restrict our proof to the case that $\mathcal{G}'$ is closed. By Th. 11 (a), $V_\mathcal{G}(I(\mathcal{G}'))$ is simple and there exists a finite subset $F$ of $V_\mathcal{G}(I(\mathcal{G}'))$ with $V_\mathcal{G}(B[F]) = I(\mathcal{G}')$. If we set $\mathcal{G} = \mathcal{G}(B[F]), \mathcal{G}^* = \mathcal{G} \cap \mathcal{G}'$ is a subgroup of $\mathcal{G}$ containing $I(\mathcal{G}')$. And so, there holds $B[F] \subseteq J(\mathcal{G}^*), A \subseteq V_\mathcal{G}(I(\mathcal{G}'))$, which implies $I(\mathcal{G}') = V_\mathcal{G}(B[F]) \supseteq V_{\mathcal{G}'} \supseteq V^*_A(I(\mathcal{G}')) = I(\mathcal{G}')$. We see therefore $\mathcal{G}^*$ is a closed $(*_f)$-regular subgroup of $\mathcal{G}$ with $V_{\mathcal{G}'} = I(\mathcal{G}')$. By assumption, $(\mathcal{G}' : \mathcal{G}^*) < \infty : \mathcal{G}' = \cup_i \mathcal{G}^* a_i$. Now, we set $A^* = J(\mathcal{G}^*, A)$ and $A' = J(\mathcal{G}', A)$. Then $\mathcal{G}^* = \mathcal{G}(A^*)$ and $A$ is $h$-Galois and locally finite over $A^*$ (Th. 11 (a) and its proof). And hence, by Th. 4 and Prop. 2, $A^{**} = A^* \cup A^* a_i$ is a $\mathcal{G}'$-invariant simple ring as an intermediate ring between $V^*_A(A^*) = V_\mathcal{G}(V_{\mathcal{G}'} = V_\mathcal{G}(I(\mathcal{G}'))$ and $A^*$. If an element $\sigma \in \mathcal{G}'$ induces an inner automorphism in $A^{**} : \sigma|A^{**} = \tilde{\sigma}|V_{A^{**}}(v \in V_{A^{**}}(A'))$ then $\sigma|\mathcal{G} \cap A^{**} = 1$, and so $\sigma$ is contained in $\mathcal{G}(H \cap A^*) = \mathcal{G}^* \mathcal{P}$ (Th. 11 (b)): $\sigma = \tau_u (\tau \in \mathcal{G}^*, \bar{u} \in \mathcal{P})$. But then, $\tau^{-1}\sigma = \bar{u} \in \mathcal{G}' \cap \mathcal{P} = I(\mathcal{G}')$ implies $\sigma \in \mathcal{V}(I(\mathcal{G}')) \subseteq \mathcal{G}^*$, so that $v$ is contained in $V_{A^{**}}(A^*) = V_{A^{**}}(A^{**})$. Hence, $\sigma|A^{**} = \bar{v}|A^{**} = 1$, which means $\mathcal{G}'|A^{**}$ is an outer group of finite order. Accordingly, as is well-known, $A^{**}$ is outer Galois and finite over the simple ring $A'$. Moreover, noting that $\mathcal{G}^* = \mathcal{G}^*(\mathcal{P})$.

https://doi.org/10.1017/S0027763000026325 Published online by Cambridge University Press
$\cap (\mathfrak{S}') = \mathfrak{S}'(H \cap A^*) \cap \mathfrak{S}' = \mathfrak{S}'(H \cap A^*)$, we obtain $[A^* : A'] = \# (\mathfrak{S}'|A^*)$

$= (\mathfrak{S}' : \mathfrak{S}^*) = \# (\mathfrak{S}'|A'[H \cap A^*]) = [A'[H \cap A^*] : A']$ by Prop. 2 (c), whence there holds $A^* = A'[H \cap A^*]$. We see therefore our assertion $I(\mathfrak{S}') = V_{\mathfrak{S}'} = V_{\mathfrak{S}}$.

**References**


Department of Mathematics,
Hokkaido University