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LOOPS AS INVARIANT SECTIONS IN GROUPS, AND THEIR GEOMETRY

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ABSTRACT. We investigate left conjugacy closed loops which can be given by invariant sections in the group generated by their left translations. These loops are generalizations of the conjugacy closed loops introduced in [13] just as Bol loops generalize Moufang loops. The relations of these loops to common classes of loops are studied. For instance on a connected manifold we construct proper topological left conjugacy closed loops satisfying the left Bol condition but show that any differentiable such loop must be a group. We show that the configurational condition in the 3-net corresponding to an isotopy class of left conjugacy closed loops has the same importance in the geometry of 3-nets as the Reidemeister or the Bol condition.

0. **Introduction.** The development of the theory of loops has shown that there are particularly fruitful connections to the foundation of geometry and to the theory of groups; both theories have delivered methods for studying loops. From a group theoretical point of view the groups G generated by left or right translations of a loop have played an important role since the time of Albert's work (*cf. e.g.* [1]). A loop can be seen as a certain section in the group G (*cf. e.g.* [14], p. 216) and the investigation of loops can be transported to the study of these sections. Hence, for the same reason as loops with weaker associativity conditions have been considered one can study loops such that the corresponding sections have nice properties in the group G. From this point of view significant classes are loops with the left inverse property, left Bol loops, A-loops (*cf.* [6] and [14], pp. 222–223) and conjugacy closed loops (*cf.* [13]).

The class of conjugacy closed loops is given by two conditions: the set of left translations is invariant in the group generated by left translations, and the same holds for the set of right translations in the group generated by the right translations. Another situation in which a class of loops is defined by two dual properties occurs for instance in the case of Moufang loops which are given by two dual Bol conditions. This motivated us to introduce the class of left conjugacy closed loops which contains the class of conjugacy closed loops properly but lies in the class of left A-loops. In contrast to the class of conjugacy closed loops (which are G-loops) the loops isotopic to a left conjugacy closed loop are not necessarily left conjugacy closed. Hence, from a geometric point of view, we are justified in introducing the subclass of universal conjugacy closed loops

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which is closed with respect to isotopisms. This class is really wide in contrast to Theorem 1 in [24].

Any class of loops which is defined by algebraic identities and which is closed with respect to isotopisms can be characterized by configurational conditions in its associated 3-net. It turns out that the configurational conditions determining 3-nets associated with left or right conjugacy closed loops, respectively, play an outstanding role in the theory of configurations. Together with the Reidemeister and Bol conditions, these are the only configurations which can be characterized by certain sets Σ of projectivities of length ≤ 6 of a line onto itself such that every projectivity of Σ having a fixed point is already the identity. In contrast to the Bol conditions our configurational conditions with respect to two different families of parallel lines do not imply the condition with respect to the third family.

From a group theoretical point of view, we have found very often that for a left conjugacy closed loop L the group G of left translations contains one or more (abelian or non-abelian) normal subgroups operating sharply transitively on L. If L is a differentiable left conjugacy closed loop then G is a Lie transformation group on L which contains a normal subgroup N operating also sharply transitively on L such that N and the submanifold of left translations have the same tangent space at 1.

An interesting subclass of left conjugacy closed loops are the loops which also satisfy the left Bol identity; we call them Burn loops. We want to stress that every connected differentiable Burn loop is a group but we have examples of proper differentiable Burn loops living on a manifold with two connected components.

In the first section we clarify the relations between the class of left conjugacy closed loops and other classes of loops: Bol and Moufang loops, commutative loops, loops with inverse property, conjugacy closed loops, left A-loops and G-loops. We show that the class of universal left conjugacy closed loops is smaller than the class of left conjugacy closed loops.

The second section is devoted to the study of differentiable left A-loops, left conjugacy closed loops, Burn loops and to the Lie groups generated by the left translations of these loops.

In the third section we study the class of Burn loops L such that every loop isotopic to L is a Burn loop. We characterize such loops by configurational conditions and by properties of the left and middle nucleus.

In Section 4 we give a complete classification of configurational conditions which are related to sets of projectivities of small length of a line onto itself. In addition we show that there exist conjugacy closed loops having neither the left nor the right inverse property, but for which corresponding configurational conditions hold in the associated 3-net for all three families of lines.

NOTATION. For elements x, y of a quasigroup one defines $y \setminus x := \lambda_y^{-1}x$ and $x/y := \rho_y^{-1}x$, where $\lambda_y: x \longmapsto yx$ and $\rho_y: x \longmapsto xy$ denote the left and the right translation by y respectively. An autotopism of a quasigroup is an isotopism onto itself. Often we call lines in a 3-net parallel to indicate that they belong to the same family.

1. Left conjugacy closed loops, Burn loops and extra loops.

1.1. *Loops as invariant sections.*

THEOREM 1.1.1. *Let L be a loop and let G be the group generated by the left translations $\lambda_g\colon L \longrightarrow L$, $g \in L$. For the loop L the following conditions are equivalent:*

(i) *the set $\{\lambda_g, g \in L\}$ is invariant under the inner automorphisms of G;*

(ii) $x \backslash (yz) = x \backslash (yx) \cdot x \backslash z$ *for all* $x, y, z \in L$;

(iii) $x \cdot yz = (xy)/x \cdot xz$ *for all* $x, y, z \in L$;

(iv) *the triples* $(\lambda_x^{-1}\varrho_x, \lambda_x^{-1}, \lambda_x^{-1})$ *are autotopisms of L for all* $x \in L$;

(v) *the triples* $(\varrho_x^{-1}\lambda_x, \lambda_x, \lambda_x)$ *are autotopisms of L for all* $x \in L$.

PROOF. If we put $(xy)/x = u$ and $xz = v$ we obtain the equivalence of the conditions (ii) and (iii). By the definition of an autotopism of L the conditions (ii) and (iv) respectively (iii) and (v) are equivalent.

If we have the condition (i) then for any $x, y \in L$ there exists an element $z \in L$ such that $\lambda_x\lambda_y\lambda_x^{-1} = \lambda_z$. It follows $\lambda_x\lambda_y\lambda_x^{-1}(x) = xy = zx$ and $z = (xy)/x$ or $\lambda_x\lambda_y\lambda_x^{-1}(u) = \lambda_{xy/x}(u)$. With $z = \lambda_x^{-1}u$ the last relation implies $x \cdot yz = (xy)/x \cdot xz$. From the identity (iii) we get immediately (i). ∎

The condition (i) motivates the following:

DEFINITION 1.1.2. A loop L is called *left conjugacy closed* if L satisfies one of the conditions of Theorem 1.1.1.

The class of left conjugacy closed loops generalizes in a natural way the class of conjugacy closed loops in which both of the sets of left and right translations are invariant under the groups generated by the left and the right translations, respectively. These loops are characterized for example by the identity (iv) and the dual identity $(zy)/x = z/x \cdot (xy)/x$ for all x, y, z.

Conjugacy closed loops were introduced by E. G. Goodaire and D. A. Robinson in [13] as examples of loops all of whose isotopes are isomorphic; such loops are called *G-loops*. In contrast to conjugacy closed loops there exist left conjugacy closed loops which are isotopic but not isomorphic. In the class of finite left Bol loops there are many left conjugacy closed loops having non-isomorphic isotopes. Already the six non-isomorphic proper Bol loops of the smallest order 8 are left conjugacy closed, since the sets of left translations consist of unions of conjugacy classes in the left translation group (*cf.* the opposite right Bol loops given by the right translation in [7], p. 382, and the proof of the Corollary on p. 384 or [9], p. 71). These 6 non-isomorphic left conjugacy closed Bol loops form two isotopy classes ([7], p. 385); representatives of the isotopy classes of the corresponding opposite loops are given by the following sets of permutations:

(a) $\rho_1 = \mathrm{id}$, $\rho_2 = (1234)(5678)$, $\rho_3 = \rho_2^2$, $\rho_4 = \rho_2^3$, $\rho_5 = (1537)(2648)$, $\rho_6 = (1638)(2547)$, $\rho_7 = \rho_5^3$, $\rho_8 = \rho_6^3$;

(b) $\rho_1 = \mathrm{id}$, $\rho_2 = (1234)(5678)$, $\rho_3 = \rho_2^2$, $\rho_4 = \rho_2^3$, $\rho_5 = (1537)(2648)$, $\rho_6 = (16)(25)(38)(47)$, $\rho_7 = \rho_5^3$, $\rho_8 = (18)(27)(36)(45)$.

It is clear that the group G of the right translations is generated in the first case by ρ_2, ρ_5, ρ_6, and in the second case by ρ_2, ρ_5, ρ_6 and ρ_8.

Among these generators we obtain in the first case the following relations:

$$\rho_6^{-1}\rho_2\rho_6 = \rho_2^{-1}, \quad \rho_5\rho_2 = \rho_2\rho_5, \quad \rho_6\rho_5 = \rho_5\rho_6, \quad \rho_2^2 = \rho_5^2 = \rho_6^2.$$

Hence the non-abelian group G contains the abelian normal subgroup $\Lambda = \langle\rho_5\rangle \times \langle\rho_5\rho_2\rangle$ since for the involution $\rho_5\rho_2$ one has $\rho_6^{-1}(\rho_5\rho_2)\rho_6 = \rho_5\rho_2^{-1}$. The group Λ is the direct product of a cyclic group of order 4 with a group of order 2. Since ρ_6 commutes with ρ_5 the group G is the direct product of the dihedral group D of order 8 and the cyclic group $\langle\rho_5\rangle$ with identified subgroups $\langle\rho_2^2\rangle$ and $\langle\rho_5^2\rangle$ (*cf.* [17], p. 349). Since any subgroup of the group G containing the commutator subgroup $G' = \langle\rho_2^2\rangle$ is a normal subgroup of G, the group G has the following normal subgroups, which operate sharply transitively on the loop L: the abelian group Λ and the group $\langle\rho_2, \rho_6\rangle$.

In the second case we have the following relations:

$$\rho_5\rho_2 = \rho_2\rho_5, \quad \rho_2^2 = \rho_5^2, \quad \rho_6^{-1}\rho_2\rho_6 = \rho_2^{-1} = \rho_8^{-1}\rho_2\rho_8,$$
$$\rho_6\rho_5\rho_6 = \rho_5^{-1} = \rho_8\rho_5\rho_8, \quad \rho_6\rho_8 = \rho_8\rho_6.$$

Hence the non-abelian group G contains the normal subgroup $\Lambda = \langle\rho_2\rangle \times \langle\rho_6\rho_8\rangle$. From this it follows that G is the direct product $\langle\rho_2, \rho_6\rangle \times \langle\rho_6\rho_8\rangle$ (*cf.* [17], p. 349), since G is not metacyclic. As $G' = \langle\rho_2^2\rangle$, the group G has the following normal subgroups which operate sharply transitively on the loop L: the abelian group Λ and the dihedral group $\langle\rho_2, \rho_6\rangle$.

Since principal isotopic loops have the same left translation groups G (*cf.* [21], p. 65) for each Bol loop of order 8 the group G contains an abelian and a non-abelian sharply transitive normal subgroup.

For any $n \in \mathbb{N}$, $n \geq 2$, R. P. Burn (*cf.* [8], pp. 446–447) found in the group

$$G_{8n} = \langle\alpha, \beta, \gamma\rangle \text{ with } \alpha^{2n} = \beta^2 = \gamma^2 = (\alpha\beta)^2 = 1, \quad \alpha\gamma = \gamma\alpha, \quad \beta\gamma = \gamma\beta,$$

suitable unions of conjugacy classes which are sets of right translations of right-isotopic right Bol loops of cardinality $2n$ but do not satisfy the Moufang identity. If n is even then these loops have exactly three non-isomorphic principal isotopes; if n is odd then they have exactly two non-isomorphic principal isotopes. In all these cases the group G_{8n} is the group generated by the right translations. From the construction of these loops, it follows immediately that the abelian subgroup $\langle\alpha, \gamma\rangle$ and the dihedral subgroup $\langle\alpha, \beta\rangle$ are sharply transitive normal subgroups of G_{8n}.

The loops considered by R. P. Burn are right conjugacy closed but not conjugacy closed. Hence the opposite loops $(L, *)$ with $x * y = yx$ of these loops are left conjugacy closed but not conjugacy closed.

It is well known (*cf.* [21], p. 65) that every loop isotopic to a loop L is isomorphic to a loop L^* which is a principal isotope of L and in which the multiplication $(x, y) \longmapsto x * y$ is given by

$$x * y = x/a \cdot b \backslash y,$$

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where a, b are fixed elements in L.

The loops with multiplication $x * y = x/a \cdot y$ or $x * y = x \cdot b \setminus y$ are said to be *left* or *right isotopic* to L, respectively. Clearly every principal isotopism is a composition of a left and a right isotopism.

REMARK 1.1.3. The class of left conjugacy closed loops is closed with respect to left isotopisms. Moreover, the left translations are isomorphisms between left isotopic left conjugacy closed loops.

PROOF. Let $\tilde{L}$ be a loop which is left isotopic to the loop L. Then the left translations $\tilde{\lambda}_x$ of $\tilde{L}$ are given by $\tilde{\lambda}_x = \lambda_{x/a}$ for some fixed $a \in L$. Hence the set $\{\tilde{\lambda}_x, x \in \tilde{L}\}$ of left translations of $\tilde{L}$ coincides with the set $\{\lambda_x, x \in L\}$ of left translations of L.

The multiplication $y \circ z = y/x \cdot z$ which defines a loop left isotopic to L is related to the multiplication of L by the isomorphism $y \longmapsto xy$ because of the identity in Theorem 1.1.1 (iii). ∎

M. Kikkawa (*cf.* [18]) introduced the notions of left A-loops and homogenous loops as follows:

DEFINITION 1.1.4. A loop L is called a *left A-loop* if every map $h_{x,y} = \lambda_{xy}^{-1}\lambda_x\lambda_y : L \longrightarrow L$ $(x, y \in L)$ is an automorphism of L. If a left A-loop has the left inverse property then it is called a *homogenous loop*.

The class of left A-loops is characterized by the property that the set $\{\lambda_x, x \in L\}$ of left translations of L is invariant under the conjugations with the elements $h_{x,y}$ $(x, y \in L)$ (*cf.* [14], p. 223). Hence every left conjugacy closed loop is a left A-loop. The existence of conjugacy closed loops which are not groups contradicts Theorem 1 in [24], since they are G-loops.

DEFINITION 1.1.5. A loop L is called an *extra loop* if it satisfies the identity

$$x \cdot (yx)z = xy \cdot xz.$$

A loop L is an extra loop if and only if it is a Moufang loop in which for every $x \in L$ the element x^2 lies in the nucleus of L ([11], Theorem 1).

THEOREM 1.1.6. *A loop L with the inverse property is left conjugacy closed if and only if it is an extra loop. In particular, such a loop is conjugacy closed.*

PROOF. The identity $x \setminus (yz) = x \setminus (yx) \cdot x \setminus z$ for a loop with the inverse property is equivalent to $x^{-1} \cdot yz = (x^{-1} \cdot yx)(x^{-1}z)$. We take $yx = v, x^{-1} = u$ and obtain the equivalent identity $u \cdot (vu)z = uv \cdot uz$. In an extra loop we have

$$z^{-1}(u^{-1}v^{-1}) \cdot u^{-1} = z^{-1}u^{-1} \cdot v^{-1}u^{-1}.$$

Putting $w^{-1} = u^{-1}v^{-1}$ we obtain $z^{-1}w^{-1} \cdot w^{-1} = z^{-1}u^{-1} \cdot (uw^{-1}u^{-1})$. From [13], Theorem 2.1, it follows that L is conjugacy closed. ∎

REMARK 1.1.7. A finite left conjugacy closed loop of odd order has the inverse property if and only if it is a group.

PROOF. This follows from Theorem 1.1.6 since the map $x \longmapsto x^2$ is a permutation of the loop. ∎

THEOREM 1.1.8. *A commutative loop L is left conjugacy closed if and only if L is an abelian group.*

PROOF. For commutative loops the identity $x \backslash (yz) = x \backslash (yx) \cdot x \backslash z$ is equivalent to $x \backslash (zy) = x \backslash z \cdot y$ which gives the associative identity $x \backslash (xu \cdot y) = uy$ putting $x \backslash z = u$. ∎

1.2. *Examples.* From the classification of loops of order ≤ 5, it follows by checking the sets of right translations in [9], p. 70, (*cf.* also [21], p. 61) that the smallest proper left conjugacy closed loop has order at least 6.

Indeed, we show that there exists already a loop L of order 6 which is even conjugacy closed, but not a group. This loop does not satisfy any Bol identity since the smallest order for a proper Bol loop is 8 ([7], [8]).

To be precise, the loop L has elements e, f, f^2, g, gf, gf^2 whose corresponding right translations are the permutations

$$\rho_1 = \text{identity}, \quad \rho_2 = (123)(456), \quad \rho_3 = (132)(465),$$
$$\rho_4 = (142635), \quad \rho_5 = (152436), \quad \rho_6 = (162534)$$

(*cf.* [1], pp. 416–419). The elements e, f, f^2 form a normal subgroup N of index 2 in L and one has $(gf^i)f^j = gf^{i+j}, f^i(gf^j) = gf^{2i+j}$ and $(gf^i)(gf^j) = f^{1+2i+j}$ for all $i, j, k \in \mathbb{Z}/3\mathbb{Z}$.

By an easy computation we obtain that N is the nucleus of L. Theorem 3.1 in [13] implies that L is a conjugacy closed loop such that any loop isotopic to L is even isomorphic to L; in particular L is an A-loop ([6], Theorem 4.7) and hence a counterexample to Theorem 2 in [24]. Let G be the group generated by the right translations of L. Then the nucleus N is a normal subgroup of G (*cf.* [1], p. 416). Since we have $\rho_4^2 = \rho_5^2 = \rho_6^2 = (123)(465)$ and $\rho_4\rho_5 = \rho_5\rho_6 = \rho_6\rho_4 = (132)$ and $\rho_5\rho_4 = \rho_6\rho_5 = \rho_4\rho_6 = (456)$ the commutators $\rho_4^{-1}\rho_5^{-1}\rho_4\rho_5$, $\rho_5^{-1}\rho_6^{-1}\rho_5\rho_6$ and $\rho_6^{-1}\rho_4^{-1}\rho_6\rho_4$ are elements of the normal subgroup N. Hence the factor group G/N is an abelian group.

We want to show that the group G/N is a cyclic group of order 6 generated by the element $N\rho_4$. First, we notice that $\rho_3\rho_4^3 = \rho_4^2\rho_5$ and hence $N\rho_4^2\rho_5 = N\rho_4^3$. Since $\rho_6 = \rho_4\rho_5\rho_4^{-1}$ and the element $N\rho_5$ is contained in the group generated by $N\rho_4$ and $N\rho_4^2\rho_5 = N\rho_4^3$ the group G/N is generated by the element $N\rho_4$ of order 6. The group G has order 18 and contains precisely one normal subgroup Θ of order 6, namely the semidirect product of N with the involution ρ_4^3. This normal subgroup Θ is isomorphic to the symmetric group S_3 and hence operates on L sharply transitively. Consequently, for the opposite loop L^* with the multiplication $x * y = yx$ the group generated by the left translations has a sharply transitive normal subgroup.

This loop L just considered of order 6 is a special case of conjugacy closed loops constructed by V. D. Belousov [3] (*cf.* also [13], p. 669). If we take $F = \mathrm{GF}(3)$ to be the field with 3 elements, denote by F^* the multiplicative group of F and define on $F \times F^*$ the multiplication by

$$(x, \xi)(y, \eta) = \big(xy, (x-1)(y-1) + \xi y + \eta\big)$$

then we obtain L, with $e = (1,0), f = (1,1)$ and $g = (-1,0)$.

Any conjugacy closed loop L is power-associative and power-commutative, that is $(x^m x^n)x^p = x^m(x^n x^p)$ and $x^m x^n = x^n x^m$ for all $x \in L$ and $m,n,p \in \mathbb{Z}$, since it is true already for A-loops (*cf.* Theorem 2.4 in [6]). Now, we present a wide class of left conjugacy closed loops which contains non-power-associative examples.

THE LOOP $L(f)$. Let V and W be abelian groups and $f\colon V \times V \longrightarrow W$ be a mapping with $f(x,0) = f(0,x) = 0$ for any $x \in V$. Then the multiplication $\circ$ on $V \times W$ given by

$$(x,\xi) \circ (y,\eta) = \big(x+y, \xi+\eta+f(x,y)\big)$$

defines a loop with the identity $(0,0)$. Indeed, if $(x,\xi) \circ (y,\eta) = (z,\zeta)$ then we have

$$\begin{aligned}(x,\xi) &= (z,\zeta)/(y,\eta) = \big(z-y, \zeta-\eta-f(z-y,y)\big),\\ (y,\eta) &= (x,\xi)\setminus(z,\zeta) = \big(z-x, \zeta-\xi-f(x,z-x)\big).\end{aligned}$$

Such a loop which we will denote by $L(f)$ is a left conjugacy closed loop if and only if (*cf.* Theorem 1.1.1 (iii))

$$(x,\xi) \circ [(y,\eta) \circ (z,\zeta)] = \big\{[(x,\xi) \circ (y,\eta)]/(x,\xi)\big\} \circ [(x,\xi) \circ (z,\zeta)],$$

for all $x,y,z \in V$ and $\xi,\eta,\zeta \in W$. This identity is equivalent to the relation

$$f(y,z) + f(x,y+z) = f(x,y) - f(y,x) + f(x,z) + f(y,x+z) \tag{0}$$

for all $x,y,z \in V$. In particular if $f(x,.)\colon V \longrightarrow W$ is a homomorphism of abelian groups for any fixed $x \in V$ then $L(f)$ is a left conjugacy closed loop. If V and W are abelian Lie groups and f is an analytical (differentiable) function satisfying (0) $L(f)$ is an analytical (differentiable) left conjugacy closed loop.

THE LOOP $L(a,p,r)$. Let K be a commutative field and let V and W be vector spaces over K. Let $a\colon V \times V \longrightarrow W$ and $p,r\colon V \times V \times V \longrightarrow W$ be mappings with the following properties:

a is bilinear and skew-symmetric $\big(a(x,x) = 0\big)$,
p is trilinear and symmetric in the first two variables,
r is trilinear and symmetric in all three variables if the characteristic of K is $\neq 2$; otherwise symmetric in the last two variables.

On $V \oplus W$ we define a loop $L(a,p,r)$ by

$$(x,\xi) \circ (y,\eta) := \big(x+y, \xi+\eta+a(x,y)+p(x,x,y)+r(x,y,y)\big).$$

Defining $f(x,y) = a(x,y)+p(x,x,y)+r(x,y,y)$, we note that the relation (0) is satisfied. Hence $L(f) = L(a,p,r)$ is a left conjugacy closed loop.

In the loop $L(a,p,r)$ one has the identity $(x,\xi)^2 \circ (x,\xi) = (x,\xi) \circ (x,\xi)^2$ if and only if $2p(x,x,x) = 2r(x,x,x)$ for all $x \in V$. If for some $x \in V$ this condition is not fulfilled then the loop $L(a,p,r)$ is not power-associative. Consequently such examples are not A-loops and therefore not conjugacy closed (*cf.* [6], Theorem 2.4).

If $p(x,x,x) = 0$ and $r(x,x,x) = 0$ for all $x \in V$ then the loop $L(a,p,r)$ is clearly power-associative. Such a loop $L(a,p,r)$ has the left inverse property

$$(-x,-\xi) \circ \big((x,\xi) \circ (y,\eta)\big) = (y,\eta)$$

if and only if $2p(x,x,y) = 2r(x,x,y)$ for all $x, y \in V$.

If K is a field of characteristic different from 2 then the map $(x,\xi) \longmapsto (x,\xi)^2 = (2x, 2\xi)$ is surjective and if in the loop $L(a,p,r)$ every square is contained in the left nucleus then L coincides with its left nucleus, that is L is a group. From Theorem 1.4.4 it will follow that in case of $\operatorname{Char} K \neq 2$ a loop $L(a,p,r)$ is a left Bol loop if and only if it is a group.

If we choose for K the field of real or complex numbers then any loop $L(a,p,r)$ is analytical.

The left translation $\lambda_{(x,\xi)}$ in $L(a,p,r)$ is the mapping

$$(y,\eta) \longmapsto \big(x+y, \xi+\eta+a(x,y)+p(x,x,y)+r(x,y,y)\big).$$

The mapping $y \longmapsto a(x,y)+p(x,x,y)$ is a linear map $\Lambda_x \colon V \longrightarrow W$. The mapping $y \longmapsto r(x,y,y)$ is a quadratic map and can be represented by a symmetric tensor M_x such that $y \longmapsto r(x,y,y) = M_x(y,y)$. Hence the group G generated by the left translations of the loop $L(a,p,r)$ is a subgroup of the transformation group Γ consisting of the mappings

$$(y,\eta) \longmapsto \big(y+t, \eta+\tau+\Lambda(y)+M_t(y,y)\big)$$

where $t \in V, \tau \in W$, Λ is a linear map $V \longrightarrow W$ and M_t is a symmetric tensor on V such that the correspondence $t \longmapsto M_t$ is a homomorphism from V into the space of symmetric 2-tensors on V with values in W. The mappings

$$\gamma_{(t,\tau)} \colon (y,\eta) \longmapsto \big(t+y, \tau+\eta+a(t,y)+r(t,t+y,y)\big)$$

form a sharply transitive normal subgroup N of G. Namely, a straightforward calculation shows that

$$\gamma_{(s,\sigma)}\gamma_{(t,\tau)} = \gamma_{\gamma_{(s,\sigma)}(t,\tau)},$$

and this means that the multiplication $(t,\tau) * (y,\eta) := \gamma_{(t,\tau)}(y,\eta)$ is associative. The normality of N in $\Gamma \supseteq G$ we can see if we put

$$\varphi(y,\eta) = \big(y+b, \eta+\beta+\Lambda(y)+r(b,y,y)\big)$$

and verify

$$\varphi\gamma_{(t,\tau)}\varphi^{-1}(y,\eta) = \big(y+t, \eta+\tau+\kappa+a(t,y)+r(t,t+y,y)\big)$$

with a suitable vector $\kappa \in W$.

1.3. *Universal properties.*

THEOREM 1.3.1. *All loops isotopic to a loop L are left conjugacy closed if and only if for all $r, x, y, z \in L$ one of the following identities is satisfied:*

$$[rx] \cdot [r \backslash (yz)] = [(rx \cdot r \backslash y)/x] \cdot [r \backslash (rx \cdot z)], \tag{1}$$

$$r(x \backslash yz) = \{r[x \backslash (y \cdot r \backslash x)]\} \cdot (x \backslash rz), \tag{2}$$

$$r \cdot x \backslash (y \cdot r \backslash (xz)] = [r \cdot x \backslash (y \cdot r \backslash x)]z. \tag{3}$$

PROOF. First we observe that a loop L satisfies the identity (1) with $r = 1$ if and only if L is left conjugacy closed (*cf.* Theorem 1.1.1 (iii)). Since the class of left conjugacy closed loops is closed with respect to left isotopisms, it is sufficient to investigate loops L^* which are right isotopic to L. In any such loop L^* the multiplication $(x, y) \longmapsto x * y$ is given by $x * y = x \cdot r \backslash y$ for a fixed $r \in L$. The loop L^* is left conjugacy closed if and only if $u * (y * v) = [(u * y)/^* u] * (u * v)$ for all $u, y, v \in L^*$ (*cf.* Theorem 1.1.1 (iii)). This is equivalent to

$$u[r \backslash (y \cdot r \backslash v)] = [(u \cdot r \backslash y)/(r \backslash u)] \cdot [r \backslash (u \cdot r \backslash v)].$$

Putting $r \backslash u = x$ and $r \backslash v = z$ we obtain the equivalent identity (1).

From the identity $x \backslash^* (y * z) = x \backslash^* (y * x) * (x \backslash^* z)$ analogously we obtain

$$r[x \backslash (y \cdot r \backslash z)] = \{r[x \backslash (y \cdot r \backslash x)]\} \cdot x \backslash z.$$

Replacing the variable z by rz we have the identity (2).

Similarly, replacing the variable z by $r \backslash (xz)$ in the identity (2) we obtain (3). ■

DEFINITION 1.3.2. A loop L is called *universal left conjugacy closed* if every loop isotopic to L is left conjugacy closed.

A loop L is universal left conjugacy closed if and only if L satisfies one of the identities of Theorem 1.3.1.

Any Bol loop of order 8 and the Bol loops of order $2n$ with the left translation groups G_{8n} ($n \geq 2$), the opposite loops of those mentioned following Theorem 1.1.1, are universal left conjugacy closed loops since these loops are left conjugacy closed and any isotopic loop belongs to the same class.

Now we look for the shape of the identity (1) in the class of loops $L(f)$. A straightforward computation shows that the identity

$$\begin{aligned}&[(r,\rho)(x,\xi)]\{(r,\rho) \backslash [(y,\eta)(z,\zeta)]\}\\ &\quad = \Big(\{[(r,\rho)(x,\xi)] \cdot [(r,\rho) \backslash (y,\eta)]\}/(x,\xi)\Big) \cdot \Big((r,\rho) \backslash \{[(r,\rho)(x,\xi)] \cdot (z,\zeta)\}\Big)\end{aligned}$$

is equivalent to the identity

$$\begin{aligned}&f(y,z) - f(r, y+z-r) + f(r+x, y+z-r)\\ &\quad = f(r,x) - f(r, y-r) + f(r+x, y-r) - f(y,x) + f(r+x, z) - f(r, x+z) + f(y, x+z)\end{aligned}$$

for all $r, x, y, z \in V$. For $r = 0$ this identity is reduced to the identity (0). From this identity, it follows that the function $w(x,y,z) = f(x,y+z) - f(x,y) - f(x,z)$ is symmetric with respect to the variables x, y and z. Using this fact we can reduce the previous identity

$$\begin{aligned}
&[f(y,z)+f(y,x)-f(y,x+z)] + [f(r+x,y-r+z)-f(r+x,y-r)-f(r+x,z)] \\
&\qquad - [f(r,y-r+z)-f(r,y-r)-f(r,z)] + [f(r,x+z)-f(r,x)-f(r,z)] \\
&\quad = -w(y,x,z) + w(r+x,y-r,z) - w(r,y-r,z) + w(r,x,z) = 0
\end{aligned}$$

to the trivial identity

$$\begin{aligned}
0 &= -w(z,y,x) + w(z,r+x,y-r) - w(z,r,y-r) + w(z,r,x) \\
&= -[f(z,y+x)-f(z,y)-f(z,x)] + [f(z,x+y)-f(z,r+x)-f(z,y-r)] \\
&\qquad - [f(z,y)-f(z,r)-f(z,y-r)] + [f(z,r+x)-f(z,r)-f(z,x)] = 0.
\end{aligned}$$

Hence we obtain:

PROPOSITION 1.3.3. *If the loop $L(f)$ is left conjugacy closed, then $L(f)$ is universal left conjugacy closed.*

Now we show that a slight modification of a loop constructed by V. D. Belousov (*cf.* [3], p. 184) and investigated by E. G. Goodaire and D. A. Robinson [13] as a conjugacy closed loop gives a new class of universal left conjugacy closed loops.

THE LOOP $L(F,\sigma,\psi)$. Let F be a commutative field and F^* its multiplicative group. We define on $F^* \times F$ a loop $L = L(F,\sigma,\psi)$ by

$$(x,\xi)\circ(y,\eta) = \big(xy,\, y^\sigma\xi + \psi(x)\eta + (\psi(x)-1)(y^\sigma-1)\big),$$

where $\sigma\colon F^* \longrightarrow F^*$ is an endomorphism of F^* and ψ is any function $F^* \longrightarrow F^*$ satisfying $\psi(1) = 1$. Clearly $(1,0)$ is the identity of L and one has

$$(x,\xi)\setminus(z,\zeta) = \Big(x^{-1}z,\, \psi(x)^{-1}\big[\zeta - x^{-\sigma}z^\sigma\xi - \big(\psi(x)-1\big)\big(x^{-\sigma}z^\sigma - 1\big)\big]\Big),$$

and

$$(z,\zeta)/(y,\eta) = \Big(zy^{-1},\, y^{-\sigma}\big[\zeta - \psi(zy^{-1})\eta - \big(\psi(zy^{-1})-1\big)\big(y^\sigma-1\big)\big]\Big).$$

We verify the identity (3) in Theorem 1.3.1. Indeed, we have on the left side

$$\begin{aligned}
&(r,\rho)\cdot(x,\xi)\setminus\big[(y,\eta)\cdot(r,\rho)\setminus\big((x,\xi)(z,\zeta)\big)\big] \\
&\quad = \Big(yz,\, \big(y^\sigma r^{-\sigma}z^\sigma - \psi(x)^{-1}\psi(y)r^{-\sigma}x^\sigma z^\sigma\big)\rho + \psi(r)\psi(x)^{-1}r^{-\sigma}x^\sigma z^\sigma\eta \\
&\qquad + \big(\psi(x)^{-1}\psi(y)z^\sigma - \psi(r)\psi(x)^{-1}y^\sigma r^{-\sigma}z^\sigma\big)\xi + \psi(y)\zeta + \psi(y)z^\sigma - \psi(y) \\
&\qquad - \psi(y)\psi(x)^{-1}z^\sigma + \psi(x)^{-1}\psi(y)r^{-\sigma}x^\sigma z^\sigma - \psi(r)\psi(x)^{-1}r^{-\sigma}x^\sigma z^\sigma \\
&\qquad + \psi(r)\psi(x)^{-1}y^\sigma r^{-\sigma}z^\sigma - y^\sigma r^{-\sigma}z^\sigma + 1\Big).
\end{aligned}$$

Putting $(z,\zeta) = (1,0)$ we obtain

$$\begin{aligned}(r,\rho)\cdot(x,\xi)\setminus[(y,\eta)\cdot(r,\rho)\setminus(x,\xi)]\\ = \Big(y, \big(y^\sigma r^{-\sigma} - \psi(x)^{-1}\psi(y)r^{-\sigma}x^\sigma\big)\rho + \psi(r)\psi(x)^{-1}r^{-\sigma}x^\sigma z^\sigma\eta\\ + \big(\psi(x)^{-1}\psi(y) - \psi(r)\psi(x)^{-1}y^\sigma r^{-\sigma}\big)\xi - \psi(x)^{-1}\psi(y) + \psi(x)^{-1}\psi(y)r^{-\sigma}x^\sigma\\ - \psi(r)\psi(x)^{-1}r^{-\sigma}x^\sigma + \psi(r)\psi(x)^{-1}y^\sigma r^{-\sigma} - y^\sigma r^{-\sigma}z^\sigma + 1\Big).\end{aligned}$$

Now, we multiply this expression by (z,ζ) from the right and obtain the identity (3).

A direct computation shows that in contrast to the example of Belousov, the loop $L(F,\sigma,\psi)$ is conjugacy closed if and only if $\psi\colon F^* \longrightarrow F^*$ is an endomorphism, which is satisfied if and only if the loop $L(F,\sigma,\psi)$ is a group.

It is easy to see that the group generated by the left translations of the loop $L(F,\sigma,\psi)$ is a subgroup of the group consisting of the mappings

$$(x,\xi) \longmapsto (ax, \alpha x^\sigma + \beta\xi + \gamma)$$

where $a,\beta \in F^*$ and $\alpha,\gamma \in F$.

Now we consider a construction given in [23]. Let G, H be groups and f a mapping from G into the automorphism group $\operatorname{Aut} H$ of H with $f(1) = \mathrm{id}$. On $G \times H$ one can define a multiplication $\big((x,\xi),(y,\eta)\big) \longmapsto (x,\xi)\circ(y,\eta) = \big(xy, \xi\cdot f(x)\eta\big)$. With this multiplication $G \times H$ is a loop $L = L(G,H,f)$ with identity $(1,1)$ such that

$$(x,\xi)\setminus(z,\zeta) = \big(x^{-1}z, f(x)^{-1}(\xi^{-1}\zeta)\big)$$

and

$$(z,\zeta)/(y,\eta) = \big(zy^{-1}, \zeta\cdot f(zy^{-1})(\eta^{-1})\big).$$

This loop L is left universal conjugacy closed if and only if the identity (3) of Theorem 1.3.1 is satisfied:

$$(r,\rho)\circ(x,\xi)\setminus\{(y,\eta)\circ(r,\rho)\setminus[(x,\xi)\circ(z,\zeta)]\} = \{(r,\rho)\circ(x,\xi)\setminus[(y,\eta)\circ(r,\rho)\setminus(x,\xi)]\}\circ(z,\zeta).$$

This is equivalent to the relation

$$\begin{aligned}&\Big(rx^{-1}yr^{-1}xz, \rho\cdot f(r)f(x)^{-1}\Big(\xi^{-1}\cdot\eta\cdot f(y)f(r)^{-1}\big(\rho^{-1}\cdot\xi\cdot f(x)(\zeta)\big)\Big)\Big)\\ &= \Big(rx^{-1}yr^{-1}xz, \rho\cdot f(r)f(x)^{-1}\big(\xi^{-1}\cdot\eta\cdot f(y)f(r)^{-1}(\rho^{-1}\cdot\xi)\big)\cdot f(rx^{-1}yr^{-1}x)\zeta\Big)\end{aligned}$$

for all $r,x,y,z \in G$, $\rho,\xi,\eta,\zeta \in H$.

Hence L is universal left conjugacy closed if and only if the mapping $f\colon G \longrightarrow \operatorname{Aut} H$ satisfies the identity

$$f(r)f(x)^{-1}f(y)f(r)^{-1}f(x) = f(rx^{-1}yr^{-1}x). \tag{4}$$

In particular L is left conjugacy closed if and only if this identity is satisfied for $r = 1$; this means

(5) $$f(x)^{-1}f(y)f(x) = f(x^{-1}yx).$$

Consequently if the mapping $f\colon G \longrightarrow \mathrm{Aut}\, H$ satisfies the identity (5) but does not satisfy the identity (4) then the loop L is left conjugacy closed but not universal left conjugacy closed.

THEOREM 1.3.4. *For a loop $L = L(G, H, f)$ the following holds true:*

(i) L is left conjugacy closed if and only if in L the identity (5) is satisfied;

(ii) L is universal left conjugacy closed if and only if in L the identity (4) is satisfied;

(iii) L is right conjugacy closed if and only if L is a group which is equivalent to the fact that $f\colon G \longrightarrow \mathrm{Aut}\, H$ is a homomorphism.

PROOF. After the previous discussion we only have to prove the assertion (iii). The loop L is right conjugacy closed if and only if the identity

$$[(z,\zeta)(y,\eta)]/(x,\xi) = (z,\zeta)/(x,\xi) \cdot [(x,\xi)(y,\eta)]/(x,\xi)$$

is satisfied. This leads to the equivalent identity

$$f(z)\eta \cdot f(zyx^{-1})\xi^{-1} = f(zx^{-1})f(x)\eta \cdot f(zx^{-1})f(xyx^{-1})\xi^{-1}.$$

Since ξ^{-1} and η are arbitrary elements in H, putting $\xi = 1$ we obtain $f(z) = f(zx^{-1})f(x)$. Substituting $u = zx^{-1}$ we obtain the homomorphic property $f(ux) = f(u)f(x)$. But if f is a homomorphism then a direct computation shows that the associativity law in L holds. ∎

THEOREM 1.3.5. *(a) There exist left conjugacy closed loops which are not universal left conjugacy closed.*

(b) There exist left conjugacy closed loops which are not conjugacy closed but which are G-loops.

PROOF. (a) Let G be a non-commutative group and let α, β be two commuting automorphisms of H both different from $id \in \mathrm{Aut}\, H$. Now we define the mapping $f\colon G \longrightarrow \mathrm{Aut}\, H$ in the following way: $f(1) = \mathrm{id}$; $f(g) = \alpha$ for $g \in G' \setminus \{1\}$, where G' is the commutator group of G and $f(g) = \beta$ for $g \in G \setminus G'$. Then we have for all $x, y \in G$: $f(y^{-1}xy) = f(x) = f(y)^{-1}f(x)f(y)$, since $y^{-1}xy$ is in G' if and only if $x \in G'$. Let x, y be two elements of G such that $x^{-1}y^{-1}xy \neq 1$. Then $\alpha = f(x^{-1}y^{-1}xy)$ but $f(x)^{-1}f(y)^{-1}f(x)f(y) = \mathrm{id}$ since α and β commute. Hence the loop multiplication on $G \times H$ determined by the mapping f satisfies the identity (5) but it does not satisfy the identity (4) for $y = 1$.

(b) Let G be the multiplicative group of the Galois field GF(5) and let H be its additive group. We denote by k a generator of the cyclic group G and by $\lambda_x\colon H \longrightarrow H$ the automorphism $\lambda_x h = xh$ for $x \in G$. Let $\psi\colon G \longrightarrow G$ be the following function:

$$\psi(1) = \psi(k) = 1, \quad \psi(k^2) = \psi(k^3) = k^2.$$

Now we consider the loop $L = L(G, H, f)$ with $f(x) = \lambda_{\psi(x)}$ for $x \in G$. Since the groups G and H are commutative the function f satisfies the identity (5) and the loop L is left conjugacy closed. Clearly L is not associative.

Now we have to show that L is a G-loop. We know from Remark 1.1.3 that the loops which are left isotopic to L are isomorphic to L. Multiplication in a loop right isotopic to L can be given by

$$(x, \xi) \circ (y, \eta) = (x, \xi) \cdot [(r, \rho) \setminus (y, \eta)], \quad (r, \rho) \in G \times H.$$

The loop defined by this multiplication is isomorphic to $L = L(G, H, f)$ with respect to the mapping $\Phi: (x, \xi) \longmapsto (rx^{\tau_r}, \xi + \rho)$, where $r \longmapsto \tau_r: G \longrightarrow \operatorname{Aut} H$ is a homomorphism with $\tau_1 = \tau_{k^2} = \lambda_1$ and $\tau_k = \tau_{k^3} = \lambda_{-1}$. Indeed

$$\Phi\big((x, \xi)(y, \eta)\big) = \big(rx^{\tau_r}y^{\tau_r}, \xi + f(x)\eta + \rho\big)$$

and

$$\Phi(x, \xi) \circ \Phi(y, \eta) = (rx^{\tau_r}, \xi + \rho)[(r, \rho) \setminus (ry^{\tau_r}, y + \rho)] = \big(rx^{\tau_r}y^{\tau_r}, \xi + \rho + f(rx^{\tau_r})f(r)^{-1}\eta\big).$$

Hence Φ is an isomorphism if and only if

$$f(rx^{\tau_r}) = f(r)f(x) \quad \text{for all } x \text{ (and any } r).$$

Since $f(x) = \lambda_{\psi(x)}$ we have to show

$$\psi(rx^{\tau_r}) = \psi(r) \cdot \psi(x).$$

This is clearly satisfied for $r = 1$. If $r = k^2$ then one has $\psi(k^2x) = \psi(k^2)\psi(x) = k^2\psi(x)$ because of the definition of ψ. In the same manner we can check this relation for the cases $r = k, k^3$. ■

1.4. *Burn loops.* Since not any left conjugacy closed loop is a Bol loop we are justified in introducing the following class of loops:

DEFINITION 1.4.1. A loop L is called a *Burn loop* if it is a left Bol loop and left conjugacy closed. A Burn loop which is universal conjugacy closed we call a *universal Burn loop*.

Now in the class of universal conjugacy closed loops $L(F, \sigma, \psi)$ we want to construct examples of universal Burn loops which are not Moufang loops.

PROPOSITION 1.4.2. *A loop $L(F, \sigma, \psi)$ is a Burn loop if and only if the function $\psi: F^* \longrightarrow F^*$ satisfies the identity $\psi(x^2y) = \psi(x)^2\psi(y)$ for all $x, y \in F^*$.*

PROOF. It is a straightforward calculation that

$$(x,\xi)\{(y,\eta)[(x,\xi)(z,\zeta)]\} = \Big(xyxz, \big(y^\sigma x^\sigma z^\sigma + \psi(x)\psi(y)z^\sigma\big)\zeta + \psi(x)x^\sigma z^\sigma \eta + \psi(x)^2\psi(y)\zeta$$
$$+ \psi(x)^2\psi(y)z^\sigma - \psi(x)^2\psi(y) - \psi(x)\psi(y)z^\sigma$$
$$+ \psi(x)\psi(y)x^\sigma z^\sigma - \psi(x)x^\sigma z^\sigma + \psi(x)x^\sigma y^\sigma z^\sigma - x^\sigma y^\sigma z^\sigma + 1\Big).$$

Putting in the previous expression $(z,\zeta) = (1,0)$ and multiplying the result by (z,ζ) from the right we obtain

$$\Big(xyxz, z^\sigma\Big[\big(x^\sigma y^\sigma + \psi(x)\psi(y)\big)\xi + \psi(x)x^\sigma\eta - \psi(x)\psi(y) + \psi(x)\psi(y)x^\sigma$$
$$-\psi(x)x^\sigma + \psi(x)y^\sigma x^\sigma - y^\sigma x^\sigma + 1\Big] + \psi(xyx)\zeta + \psi(xyx)z^\sigma - \psi(xyx) - z^\sigma + 1\Big).$$

Comparing the coefficients in the both expressions we obtain the identity $\psi(x^2y) = \psi(x)^2\psi(y)$. ■

Let Q be the subgroup of squares in the multiplicative group F^* of F. If $Q \neq F^*$ and $|F^*| \geq 3$ then the map ψ with $\psi(q) = 1$ for any $q \in Q$ and $\psi(t) = c \in F^* \setminus \{1,-1\}$ for any $t \in F^* \setminus Q$ is not an endomorphism of F^*. For any such map $\psi\colon F^* \longrightarrow F^*$ we get a universal Burn loop which is not a group. None of these Burn loops is conjugacy closed, hence they are not Moufang loops (*cf.* Theorem 1.1.6).

Certainly in this way we have constructed proper finite universal Burn loops of order $s = p^n(p^n - 1)$ for any prime $p \neq 2$ and natural number n if $s > 6$.

If F is the field $\mathbb{Q}$ of rational numbers and if $x = \varepsilon\prod_{i=1}^n p_i^{k_i}$ ($\varepsilon = \pm 1$, $p_i > 0$, $i = 1,\ldots,m$ are prime numbers) is the prime power decomposition of x then we can put $\psi(x) = \varepsilon\prod_{i=1}^n p_i^{f(k_i)}$, where $f(k) = \begin{cases} k & \text{if } k \equiv 0 \bmod 2 \\ 2k & \text{if } k \equiv 1 \bmod 2 \end{cases}$.

Clearly we have $\psi(x^2y) = \psi(x^2)\psi(y)$ but $\psi(p^2) = p^2 \neq \psi(p)\psi(p) = p^4$ for any prime $p \geq 2$, and the loop $L(\mathbb{Q},\sigma,\psi)$ is a universal Burn loop.

Let Q be the subgroup of squares in the multiplicative group F^* of a field F with $|F^*| \geq 3$ and $Q \neq F^*$. Then the map ψ with $\psi(q) = q$ for any $q \in Q$ and $\psi(t) = ct$ with $c \in F^* \setminus \{1,-1\}$ for any $t \in F^* \setminus Q$ is not an endomorphism of F^*, but satisfies $\psi(x^2y) = \psi(x)^2\psi(y)$. In particular if F is the field $\mathbb{R}$ of real numbers then it follows from this construction:

THEOREM 1.4.3. *There exist topological (even differentiable) universal Burn loops which are homeomorphic to the manifold* $\mathbb{R} \times \mathbb{R}^*$.

This information is interesting also from analytical point of view since there are no proper differentiable Burn loops on connected manifolds (*cf.* Theorem 2.4).

THEOREM 1.4.4. *A left Bol loop L is a Burn loop if and only if for every $x \in L$ the element x^2 is contained in the left nucleus of L.*

PROOF. The left Bol identity can be written as $\lambda_x\lambda_y\lambda_x = \lambda_{x\cdot yx}$. If L is left conjugacy closed then from the identity (ii) in Theorem 1.1.1 one has $\lambda_x^{-1}\lambda_y\lambda_x = \lambda_{x\backslash yx}$. Multiplying this identity from left by λ_x^2 we obtain $\lambda_{x\cdot yx} = \lambda_x\lambda_y\lambda_x = \lambda_x^2\lambda_{x\backslash yx}$. Taking $x \setminus yx = u$ we have $\lambda_{x\cdot xu} = \lambda_x^2\lambda_u$. In a left Bol loop one has $x \cdot xu = x^2 \cdot u$ and hence $\lambda_{x^2\cdot u} = \lambda_x^2\lambda_u$ which means $(x^2 \cdot u)z = x^2 \cdot uz$.

If we have $(x^2 \cdot u)z = x^2 \cdot uz$ for all $x, u, z \in L$ then we obtain that L is a Burn loop by reversing the steps of the first part of the proof. ∎

THEOREM 1.4.5. *A left conjugacy closed loop L is a Burn loop if and only if L satisfies the identity*

$$x(x \cdot ts) = (x \cdot xt)s \quad \text{for all } x, t, s \in L.$$

PROOF. The identity $x(x \cdot ts) = (x \cdot xt)s$ is equivalent to $\lambda_x^2\lambda_t = \lambda_{x\cdot xt}$. Putting $t = x \setminus yx$ we obtain $\lambda_x^2\lambda_{x\backslash yx} = \lambda_{x\cdot yx}$. Since $\lambda_{x\backslash yx} = \lambda_x^{-1}\lambda_y\lambda_x$ we obtain from the previous identity the Bol identity $\lambda_x^2(\lambda_x^{-1}\lambda_y\lambda_x) = \lambda_{x\cdot yx}$. ∎

The paper [24] suggests a study of loops L such that every loop isotopic to L is a left A-loop, or homogenous loop, respectively.

Bruck and Paige give in [6] an example of a commutative A-loop L which is an extension of a cyclic group N of order 2 by the elementary abelian group of order 4; L is not diassociative but every element has order 2. In this loop N is the nucleus and hence every loop isotopic to L is an A-loop, too (*cf.* Theorem 4.7 in [6]). From Theorem 1.1.8, it follows that this loop is not left conjugacy closed.

Let L be a commutative Moufang loop of nilpotency class 2 (*cf.* [21], p. 111); for example every commutative Moufang loop generated by 3 elements has this property. Bruck proved in [4], p. 298, that any commutative Moufang loop is an A-loop. Since L is nilpotent of class 2 any loop isotopic to L is an A-loop by [6], Theorem 4.7. But as a commutative loop L cannot be left conjugacy closed (*cf.* Theorem 1.1.8).

Any conjugacy closed loop is a left and right A-loop. This is proved in [13], Theorem 2.2, where one finds many examples of such loops.

2. **Differentiable left conjugacy closed loops.** Let L be a loop defined on a C^∞-differentiable manifold. L is called a *differentiable loop* if all of its operations $(x, y) \longmapsto xy$, $(x, y) \longmapsto x \setminus y$ and $(x, y) \longmapsto x/y$ are C^∞-differentiable mappings.

THEOREM 2.1. *The group G generated by the left translations of a differentiable left A-loop is a Lie transformation group on L.*

PROOF. We introduce on L the ternary operation $(x, y, z) \longmapsto m(x, y)z := \lambda_y\lambda_{y\backslash x}\lambda_y^{-1}z$. We prove that for any $u \in L$ the left translation λ_u is an automorphism of this operation. Namely, we have

$$\lambda_u m(x, y)z = \lambda_u\lambda_y\lambda_{y\backslash x}\lambda_y^{-1}z = \lambda_{uy}(\lambda_{uy}^{-1}\lambda_u\lambda_y)(y \setminus x \cdot y \setminus z).$$

Since L is a left A-loop we have

$$\begin{aligned} \lambda_u m(x, y)z &= \lambda_{uy}(\lambda_{uy}^{-1}\lambda_u x \cdot \lambda_{uy}^{-1}\lambda_u z) = \lambda_{uy}(uy \setminus ux \cdot uy \setminus uz) \\ &= \lambda_{uy}\lambda_{uy\backslash ux}\lambda_{uy}^{-1}\lambda_u z = m(\lambda_u x, \lambda_u y)\lambda_u z. \end{aligned}$$

Since the transitive family of diffeomorphisms $\{m(x,y), x, y \in L\}$ satisfies the conditions of Definition 2.1 in [15] one can associate to this family $\{m(x,y), x, y \in L\}$ a linear connection ∇ given by

$$\nabla_X Y = \frac{d}{dt}\Big|_{t=0}\Big[\big(T_x m(\gamma(t), x)\big)^{-1} Y_{\gamma(t)}\Big]$$

with a smooth curve γ which locally uniquely solves the initial value problem $\gamma'(t) = X_{\gamma(t)}, \gamma(0) = x$.

We prove that for any $u \in L$ the map λ_u is an affine transformation relative to ∇:

$$\begin{aligned}(T\lambda_u)(\nabla_X Y) &= \frac{d}{dt}\Big|_{t=0}(T\lambda_u)\big[T_x m\big(\gamma(t), x\big)\big]^{-1} Y_{\gamma(t)}\\ &= \frac{d}{dt}\Big|_{t=0} T_x\Big(\lambda_u\big[m\big(\gamma(t), x\big)\big]^{-1}\Big) Y_{\gamma(t)}\\ &= \frac{d}{dt}\Big|_{t=0} T_x\Big(m\big(\lambda_u\gamma(t), \lambda_u x\big)^{-1}\Big) T\lambda_u Y_{\gamma(t)},\end{aligned}$$

where $\lambda_u\gamma(t)$ is the solution of $[\lambda_u\gamma(t)]' = (T\lambda_u)X_{\lambda_u\gamma(t)}$.

It follows that G is a subgroup of the affine Lie transformation group A of $\{L, \nabla\}$ (*cf.* [19], Chapter VI. §1. Theorem 1.5). Since in A limits of automorphisms of the differentiable ternary operation m are again automorphisms the group G is a closed subgroup of A. Hence G is a Lie transformation group on L. ∎

It is well known that for locally compact connected loops the mapping $\sigma: x \longmapsto \lambda_x: L \longrightarrow G$ which associates to any element $x \in L$ its left translation $\lambda_x \in G$ is a topological embedding (*cf.* [9], p. 218). If L is a differentiable loop then the smooth version of this fact holds.

PROPOSITION 2.2. *Let L be a connected differentiable loop such that the group G generated by its left translations is a Lie group. The the mapping $\sigma: x \longmapsto \lambda_x: L \longrightarrow G$ is a differentiable embedding, i.e. the section $\sigma(L) = \{\lambda_x, x \in L\}$ is an embedded differentiable submanifold in G.*

PROOF. Let x_0 be a point of L. We have to prove that the mapping σ is differentiable at x_0.

First we show that for λ_{x_0} there exists a neighbourhood $U_{\lambda_{x_0}}$ in G and a finite set of elements $z_1, \ldots, z_r$ in L such that for the pointwise stabilizer Θ of $z_1, \ldots, z_r$ in G one has $(U_{\lambda_{x_0}}^{-1} U_{\lambda_{x_0}}) \cap \Theta = \{1\}$.

Let z_1 be an arbitrary element of L. If Θ_i is the pointwise stabilizer of the points $z_1, \ldots, z_i$ and $\dim(\Theta_i) > 0$ then we choose as z_{i+1} a point which is not fixed under the connected component of Θ_i. Clearly $\dim(\Theta_{i+1}) < \dim(\Theta_i)$. Since G is finite dimensional there exist points $z_1, \ldots, z_r$ such that $\Theta = \Theta_r$ is a discrete Lie subgroup of G. Hence there exists a neighbourhood W of $1 \in G$ such that $W \cap \Theta = \{1\}$, and we choose $U_{\lambda_{x_0}}$ satisfying $U_{\lambda_{x_0}}^{-1} U_{\lambda_{x_0}} \subset W$.

If for $u, u' \in U_{\lambda_{x_0}}$ one has $uz_i = u'z_i$ for $i = 1, \ldots, r$ then $u'^{-1}u(z_i) = z_i$ for $i = 1, \ldots, r$ and hence $u = u'$. Now, in the neighbourhood $U_{\lambda_{x_0}}$ the function $\sigma(x) = \lambda_x$ is the solution of the system of equations $\lambda_x z_i = xz_i$ $(i = 1, \ldots, r)$. Since the mapping $g \longmapsto (gz_1, \ldots, gz_r)\colon U_{\lambda_{x_0}} \longrightarrow L^r$ is a differentiable injection the solution λ_x is differentiable by the implicite function theorem. ∎

In Sections 1.1 and 1.2 we have seen many examples of left conjugacy closed loops L whose groups G generated by the left translations contain normal subgroups operating sharply transitively on L. In the next theorem we prove that for differentiable left conjugacy closed loops this is always the case.

THEOREM 2.3. *Let L be a differentiable connected left conjugacy closed loop. Then the group G generated by the left translations is a Lie group; in G there exists a normal subgroup N which operates sharply transitively on L and the tangent space of N at* 1 *coincides with the tangent space of the section $\sigma(L) = \{\lambda_x, x \in L\}$.*

PROOF. By Theorem 2.1 the group G is a Lie group. We denote by g its Lie algebra. By Proposition 2.2 the set $\sigma(L) = \{\lambda_x, x \in L\}$ is a differentiable submanifold in G. Since $\sigma(L)$ is invariant with respect to inner automorphisms of G the vector subspace $\mathcal{M} = T_{\sigma(e)}\sigma(L)$ is invariant in g under the adjoint representation $\mathrm{Ad}(G)$ of G. Hence $\mathcal{M}$ is an ideal in the Lie algebra g and it is a complement of the Lie algebra h of the stabilizer group G_e. Hence g is a semidirect product of $\mathcal{M}$ with h. If M is the normal subgroup of G corresponding to the ideal $\mathcal{M} \subset g$ then G is the product $M \cdot G_e$ such that $M \cap G_e$ is a central normal subgroup of G (*cf.* [25], Satz 2, p. 158]). Since G operates effectively on L the group G_e cannot contain a nontrivial normal subgroup of G. Hence $M \cap G_e = \{1\}$.
Since $G = MG_e$ operates transitively on L, the group M is sharply transitive on L. ∎

In Theorem 1.4.3 we have seen the existence of proper topological universal Burn loops defined on a manifold. In contrast to this we have:

THEOREM 2.4. *Every connected differentiable Burn loop is a group.*

PROOF. Every Burn loop L is a Bol loop and hence power-associative (*cf.* [21], IV.6.6. Corollary). Using a result of Holmes [16] (*cf.* also [20], Theorem 5.5.(v)), it follows that in any differentiable Burn loop there exists a suitable neighbourhood of the identity e which is simply covered by 1-dimensional subgroups. Hence the mapping $x \longmapsto x^2$ is locally surjective. Since in a Burn loop every square is contained in the left nucleus (*cf.* Theorem 1.4.4) there exists a neighbourhood of e which is a local Lie group U. The subgroup $N \leq L$ generated by U has the same dimension as L. Since N is an open and closed subset of L, it follows $N = L$. ∎

COROLLARY 2.5. *Every connected differentiable left conjugacy closed loop with the inverse property is a group.*

The assertion follows immediately from Theorem 1.1.6 and 2.4.

For any analytical loop $L(a,p,r)$ defined in Section 1.2 (where the field of scalars is $\mathbb{R}$ or $\mathbb{C}$) the left translations $\lambda_{(x,\xi)}$ generate a closed Lie subgroup of the Lie transformation group

$$\Gamma = \Big\{(y,\eta) \longmapsto \big(y+t, \eta+\tau+\Lambda(y)+M_t(y,y)\big)\Big\}$$

where $t \in V$, $\tau \in W$, $\Lambda \in \mathrm{Hom}(V,W)$ and M_t is a symmetric tensor on W such that $t \longrightarrow M_t$ is a homomorphism from V into the space of symmetric 2-tensors on V with values in W. The sharply transitive normal subgroup described in Theorem 2.3 consists in the case of an analytical loop $L(a,p,r)$ of the mappings

$$(y,\eta) \longmapsto \big(t+y, \tau+\eta+a(t,y)+r(t,t+y,y)\big)$$

with $t \in V$ and $\tau \in W$.

In contrast to the opinion expressed in [24], the examples of loops $L(a,p,r)$ show moreover that to force an A-loop or a left A-loop to be a group we have to work within classes of loops satisfying relatively strong associativity conditions.

THEOREM 2.6. *Let L be a connected differentiable Moufang loop which is an A-loop. Then L is a Lie group.*

PROOF. From Theorem 4B in [4], p. 298, it follows that all elements $uvu^{-1}v^{-1}$, $v^{-1}u^{-1}vu$ and u^3 for $u,v \in L$ are contained in the nucleus N of L. Since N is a normal subloop of L (*cf.* [4], p. 301) the factor loop L/N is a commutative differentiable Moufang loop in which every element different from 1 has order 3. Since any commutative connected differentiable Moufang loop is a Lie group one has $L = N$. ∎

THEOREM 2.7. *Any locally compact connected topological Moufang loop L such that every loop isotopic to L is an A-loop must be a group.*

PROOF. From Theorem 4.7 in [6], it follows that the derived subloop L' is contained in the nucleus N of L. Since L/L' is an abelian group L/N is also a connected abelian group. It follows from Theorem 7A in [4], p. 302, that L/N has exponent 3. Hence L must be equal to N and L is a group. ∎

3. Geometry of universal Burn loops.

3.1. *The Bol condition.* Let L be a Bol loop and let N be the 3-net associated with L (*cf.* [22], [3], [21]). If we choose a vertical line G then we may define the following mapping: $\sigma_G : N \longrightarrow N : \sigma_G(x) = x$ for all $x \in G$; if $x \notin G$ let $\sigma_G(x)$ be the unique point of N such that the transversal and horizontal lines through x and $\sigma_G(x)$ intersect each other on G.

PROPOSITION 3.1.1. *Let L be a Bol loop. The mappings σ_G are involutory collineations of N preserving the family of vertical lines and interchanging the families of horizontal and transversal lines. If we choose a unit point e on L then the corresponding loop multiplication on G and the isostrophic loop multiplication arising by interchanging the families of horizontal and vertical lines are the same.*

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Conversely, if in an arbitrary 3*-net* M *all loop multiplications on vertical lines are the same independent of the choice of the other coordinate axes as the horizontal or transversal lines, then* M *satisfies the Bol condition.*

PROOF. The first assertion is a direct consequence of the Bol condition. It follows that the construction of the product $x \cdot y$ for points $x, y \in G$ is mapped by σ_G onto the construction of the product in the isostrophic loop arising by interchanging the families of horizontal and transversal lines.

Now let M be a 3-net. Constructing the products $x_1 \cdot_h x_2$ and $x_1 \cdot_t x_2$ on a vertical line L with unit point x_0 we have to use the following points and lines:

Let H_i and T_i respectively be the horizontal or transversal line through the point x_i for $i = 0, 1, 2$. Let $y_i = T_i \cap H_0$ for $i = 1, 2$ and let $y_j' = H_j \cap T_0$ for $j = 1, 2$. The points x_i, y_i, y_i', e, the lines L, H_i, T_i and the vertical lines through x_i, y_i form a non-closed Bol configuration which closes if and only if $x_1 \cdot_h x_2 = x_1 \cdot_t x_2$. ∎

3.2. *The Burn configuration.* Let L_0, L_1, L_2, L_3 be vertical lines and let x_0, y_0, x_0', y_0', $x_1, y_1, x_1', y_1', x_2, x_2', x_3, x_3'$ be points in N such that the following conditions are satisfied:

(i) $x_0, x_0', y_0, y_0' \in L_0, x_1, x_1', y_1, y_1' \in L_1$;

(ii) $x_2, x_2' \in L_2, x_3, x_3' \in L_3$;

(iii) each of the pairs $\{x_0, x_1\}$, $\{y_0, x_1\}$, $\{y_0, y_1\}$, $\{y_1, x_2\}$, $\{x_2, x_3\}$, $\{x_0, x_3\}$, $\{x_0', x_1'\}$, $\{y_0', x_1'\}$, $\{y_0', y_1'\}$, $\{y_1', x_3'\}$, $\{x_2', x_3'\}$, $\{x_0', x_3'\}$ determines a line if the pair consists of different points;

(iv) the lines $x_0x_1, y_0y_1, x_0'x_1', y_0'y_1', x_2x_3, x_2'x_3'$ belong to the same family, the same holds for the lines x_0x_3, $x_0'x_3'$, y_1x_2, $y_1'x_2'$, y_0x_1, $y_0'x_1'$, but these two families are different.

This configuration we call the *Burn configuration.*

If we replace in the defining properties of a Burn configuration the condition (ii) by the weaker condition

(ii)' $x_2, x_2' \in L_2$ and $x_3 \in L_3$;

then we call this configuration a *non-closed Burn configuration*. We say that N satisfies the Burn condition if in any non-closed Bol configuration the incidence $x_3' \in L_3$ is satisfied, too. In this situation we say that every non-closed Burn configuration can be completed within N to a Burn configuration.

PROPOSITION 3.2.1. *If a* 3*-net* N *satisfies the Burn condition then it is a Bol net.*

PROOF. If we consider the non-closed Burn configurations with $x_3 \in L_0 = L_3$ then we obtain non-closed Bol configurations. These can be completed by the Burn condition to (closed) Bol configurations. ∎

PROPOSITION 3.2.2. *If a* 3*-net* N *satisfies the Burn condition then in any loop associated with* N *every element* x^2 $(x \in L)$ *is in the intersection of the left and the middle nucleus.*

PROOF. Let C be a Burn configuration such that the lines x_0x_1, y_0y_1, $x'_0x'_1$, $y'_0y'_1$, x_2x_3, $x'_2x'_3$ are horizontal. We consider as the origin of the coordinate system the intersection of L_0 with the horizontal line through x_2. Then the coordinates of the points of the configuration C can be written in the following way:

$$x_0 = (1, y), \quad y_0 = (1, xy), \quad x'_0 = (1, yu), \quad y'_0 = (1, x \cdot xy),$$
$$x_1 = (x, y), \quad y_1 = (x, xy), \quad x'_1 = (x, yu), \quad y'_1 = (x, x \cdot yu),$$
$$x_2 = (x \cdot xy, 1), \quad x'_2 = (x \cdot xy, u), \quad x_3 = (y, 1), \quad x'_3 = (y, u).$$

Since x'_2 and y'_1 are contained in a transversal line we obtain $(x \cdot xy)u = x(x \cdot yu)$. Since N is a Bol net in the coordinate loop L we have the identity $x \cdot xy = x^2y$. Consequently $x^2y \cdot u = x^2 \cdot yu$ for all $x, y, u \in L$.

Let C' be a Burn configuration such that the lines x_0x_1, y_0y_1, $x'_0x'_1$, $y'_0y'_1$, x_2x_3, $x'_2x'_3$ are transversal. Let the origin of the coodinate system be the intersection of L_0 with the horizontal line through x_2. The coordinates of the points of the configuration C' are:

$$y_0 = (1, x), \quad x_0 = (1, x^2), \quad y'_0 = (1, xu), \quad x'_0 = (1, x \cdot xu),$$
$$y_1 = (x, 1), \quad x_1 = (x, x), \quad y'_1 = (x, u), \quad x'_1 = (x, xu),$$
$$x_2 = (yx^2, 1), \quad x'_2 = (yx^2, u), \quad x_3 = (y, x^2), \quad x'_3 = (y, x \cdot xu).$$

Since x'_3 and x'_2 are contained in a transversal line we obtain $y(x \cdot xu) = yx^2 \cdot u$. Since $x \cdot xu = x^2u$ we obtain the identity $y \cdot x^2u = yx^2 \cdot u$. ∎

COROLLARY 3.2.3. *Every coordinate loop of a* 3*-net satisfying the Burn condition is a Burn loop.*

The assertion follows from Theorem 1.3.1 and from the previous proposition.

PROPOSITION 3.2.4. *If L is a universal Burn loop then the corresponding* 3*-net $N = L \times L$ satisfies the Burn condition.*

PROOF. From the definition of a Burn loop, it follows that all loops isotopic to a Burn loop are Burn loops. Hence N is a Bol net and in any coordinate loop of this net we have the identity $x^2y \cdot u = x^2 \cdot yu$. If we consider a non-closed Burn configuration the points of which have the coordinates of the points of the configuration C used in the proof of the previous proposition then $x^2y \cdot u = x^2 \cdot yu$ implies that the point $x'_3 = (y, u)$ is contained in the vertical line L_3. Using the involutory collineation σ_{L_0} introduced at the beginning of the section any configuration C' constructed in the proof of Proposition 3.2.2 is the image of a configuration C. Hence the assertion is proved. ∎

From the previous results we obtain the following

THEOREM 3.2.5. *For a Bol loop L the following conditions are equivalent:*

(i) L is a universal Burn loop;

(ii) for every Burn loop isotopic to L all squares x^2 ($x \in L$) are contained in the middle nucleus of L;

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(iii) *for every Burn loop isotopic to L all squares x^2 ($x \in L$) are contained in the intersection of the left and middle nucleus of L.*

The class of (left-)Burn loops is defined by the identities $\lambda_x\lambda_y\lambda_x = \lambda_{x\cdot yx}$ and $\lambda_x^{-1}\lambda_y\lambda_x = \lambda_{x\backslash yx}$ in the group of left multiplications. If we use instead of left multiplications $\{\lambda_x\}$ the right multiplications $\{\varrho_x\}$ we will call the corresponding class of loops *right Burn loops*.

PROPOSITION 3.2.6. *A conjugacy closed Bol loop L is an extra loop.*

PROOF. See [10], Theorem 2.1. ■

In the following we call the Burn condition for a 3-net as defined at the begin of this section the *vertical Burn condition*. If we take for the parallel lines L_i ($i = 0, 1, 2, 3$) in the vertical Burn condition horizontal or transversal lines instead of vertical ones we obtain the horizontal or transversal Burn condition, respectively.

THEOREM 3.2.7. *If a 3-net N satisfies one of the three Burn conditions and a Bol condition which does not follow from this Burn condition then any coordinate loop of N is an extra loop.*

PROOF. It follows from our assumption that in N two independent Bol conditions are satisfied. It follows that N is a Moufang 3-net and Proposition 3.2.6 implies the assertion. ■

THEOREM 3.2.8. *In the 3-net N belonging to an extra loop L all three Burn conditions hold.*

PROOF. Since the squares x^2 ($x \in L$) are contained in the nucleus of L the net N satisfies the vertical Burn condition (*cf.* Theorem 3.2.5). Since the stabilizer of any point q in the collineation group of N is transitive on the three lines through q (*cf.* [2], Theorem 10.3, p. 45) also the horizontal and transversal Burn conditions are satisfied. ■

4. Geometry of left conjugacy closed loops.

4.1. *Projectivities.* Let L be a loop and let N be the 3-net associated with L (*cf.* [22], [3], [2], [21]); we use the terminology and notation introduced in [2]. The loops arising from N are all isotopic to L. A perspectivity of a line L_i onto a line L_k in N is given by a pencil Υ of parallel lines, such that $L_i, L_k \notin \Upsilon$; we write $[L_i, \Upsilon, L_k]$. A projectivity from L_0 onto the line L_n is a product of perspectivities and has a representation

$$\pi = \prod_{i=1}^{n}[L_{i-1}, \Upsilon_i, L_i] \tag{6}$$

with $L_{i-1} \neq L_i$ ($i = 1, \ldots, n$) and $\Upsilon_i \neq \Upsilon_{i+1}$ for all $i = 1, \ldots, n-1$. We call the number n the length of the representation (6).

We consider now the sets

$$\mathcal{M}_n = \left\{\prod_{i=1}^{k}[L_{i-1}, \Upsilon_i, L_i]; L_0 = L_k, \Upsilon_1 \neq \Upsilon_k, k \leq n\right\}$$

where $L_0, \ldots, L_k$ are arbitrary parallel lines. It follows that the natural number k is even.

THEOREM 4.1.1. *Let $n \geq 4$. Every $\pi \in \mathcal{M}_n$ which has a fixed point is the identity if and only if every loop associated with the* 3*-net N is a group.*

PROOF. We assume that the condition holds for $n = 4$.

If the projectivity π is given by

$$\pi = \prod_{i=1}^{4}[L_{i-1}, \Upsilon_i, L_i] \quad (L_4 = L_0)$$

with different lines L_0, L_1, L_2, L_3 and satisfies $\pi(x) = x$ for some $x \in L_0$ then $\pi(y) = y$ ($y \in L_0$) shows that the Reidemeister condition holds.

Now, we have to show that for a 3-net N associated with a group G the condition of the theorem holds for any $n \geq 4$. If $\mathcal{V}$ is the family of vertical lines in N then the group $\bigcup_{n\geq 2}^{\infty} \mathcal{M}_n$ with lines $L_i \in \mathcal{V}$ is isomorphic in a natural way to the group of left translations $x \longmapsto ax: G \longrightarrow G, a \in G$ and it is sharply transitive on the line L_0 (*cf.* [2], Theorem 6.1). ∎

THEOREM 4.1.2. *Let $\mathcal{M}_4'$ be the set of projectivities of length 4 contained in $\mathcal{M}_4$ such that $L_1 = L_3$ and all lines L_i are vertical. Every $\pi \in \mathcal{M}_4'$ which has a fixed point is the identity if and only if every loop associated with the* 3*-net N is a Bol loop.*

PROOF. If the projectivity π given by

$$\pi = \prod_{i=1}^{4}[L_{i-1}, \Upsilon_i, L_i]$$

with $L_4 = L_0$ and $L_1 = L_3$ and vertical lines L_i ($i = 0, \ldots, 4$) satisfies $\pi(x) = x$ for some $x \in L_0$, then $\pi(y) = y$ gives the Bol condition. ∎

4.2. *Reidemeister condition.* Now we consider for a projectivity π representations

$$\pi = \prod_{i=1}^{n}[L_{i-1}, \Upsilon_i, L_i], \quad L_0 = L_n, \tag{7}$$

where all L_i ($i = 0, \ldots, n$) are parallel, but we allow $L_i = L_{i+1}$. Any such representation can be reduced to a representation of the previous type (6); the representations of the type (6) are called *irreducible*. The set of all projectivities having representations (7) for a fixed $n \geq 2$ we call $\tilde{\mathcal{M}}_n$.

Let $\tilde{\mathcal{M}}_6^{(i,j)}$ denote the subset of all projectivities from $\tilde{\mathcal{M}}_6$ such that amongst their representations (7) there exists one with $L_i = L_j$ for fixed $i \neq j$, $i, j \in \{0, \ldots, 5\}$. Any set $\tilde{\mathcal{M}}_6^{(i,j)}$ contains $\mathcal{M}_4$; consequently every $\pi \in \tilde{\mathcal{M}}_6^{(i,j)}$ which has a fixed point is the identity if and only if every loop associated with the 3-net N is a group.

Let $\tilde{\mathcal{M}}_6^{(i,j,k)}$ denote the subset of all projectivities from $\tilde{\mathcal{M}}_6$ such that amongst their representations (7) there is one with $L_i = L_j = L_k$ for fixed i, j, k. We want to discuss properties of the set $\tilde{\mathcal{M}}_6^{(i,j,k)}$ with respect to different choices of the indices $i, j, k \in \{0, \ldots, 5\}$.

If we assume that $\min\{|i-j|, |j-k|, |k-i|\} = 1$, then any set $\tilde{\mathcal{M}}_6^{(i,j,k)}$ contains $\mathcal{M}_4$ and the 3-net N is associated with a group if and only if any projectivity π having a representation in $\tilde{\mathcal{M}}_6^{(i,j,k)}$ with a fixed point is the identity.

We consider now the subset of all projectivities from $\tilde{\mathcal{M}}_6^{(i,j,k)}$ such that $L_i = L_j = L_k$ and $|i-j| = |j-k| = |i-k| = 2$. In this case we can assume $L_0 = L_2 = L_4$. From the fact that in $\tilde{\mathcal{M}}_6^{(0,2,4)}$ there are also representations with $L_0 = L_1$ we have $\mathcal{M}_4' \subset \tilde{\mathcal{M}}_6^{(0,2,4)}$. Consequently if any projectivity π having a representation in $\tilde{\mathcal{M}}_6^{(0,2,4)}$ and fixing a point of L_0 is the identity then the coordinate loops associated with N are Bol loops. Moreover we prove that this condition is equivalent even to the Reidemeister condition.

Let $L_5, L_0 = L_2 = L_4, L_3$ and S be four lines from the same family and let p_5, p_5', p_4, p_4', p_3, p_3', x, x', x'' be points such that the following conditions are satisfied:

(i) $p_5, p_5' \in L_5$, $p_4, p_4' \in L_4$, $p_3, p_3' \in L_3$, $x, x', x'' \in S$;

(ii) the pairs of points $\{p_3, p_4\}$, $\{p_3', p_4'\}$, $\{p_5, x\}$, $\{p_5', x''\}$ determine lines from the same family;

(iii) the pairs of points $\{p_4, p_5\}$, $\{p_4', p_5'\}$, $\{p_3, x\}$, $\{p_3', x'\}$ determine lines from the same family;

(iv) the lines in (i), (ii), (iii) belong to different families of lines.

The Reidemeister condition holds if and only if in any such configuration $x' = x''$.

Let the points p_0, p_2, p_2' be the intersections of the line L_4 with the lines $\{p_5, x\}$, $\{p_3, x\}$, $\{p_3', x''\}$. Let p_0' be the intersection of the line L_4 with the line through x'' parallel to the line given by $\{p_5, x\}$. Let Q and Q' respectively be the lines through p_0 and p_0' which are parallel to the line determined by $\{p_4, p_5\}$. Let R and R' respectively be the lines through p_2 and p_2' which are parallel to the line determined by $\{p_3, p_4\}$. We denote $p_1 = Q \cap R$ and $p_1' = Q' \cap R'$. From the Bol condition, it follows that the points p_1 and p_1' determine a line L_1 parallel to L_3, L_4, L_5. The projectivity $\pi = \prod_{i=1}^{6}[L_{i-1}, \Upsilon_i, L_i]$ where $L_0 = L_2 = L_4$ and Υ_i is the parallel pencil of lines determined by the line $\{p_{i-1}, p_i\}$ fixes the point $p_0 \in L_0$ and hence π is the identity. From this, it follows that $x' = x''$.

Conversely, if a 3-net N satisfies the Reidemeister condition then from Theorem 4.1.1 it follows that any projectivity π having a representation in $\tilde{\mathcal{M}}_6^{(0,2,4)}$ and fixing a point of L_0 is the identity.

By this discussion, we have classified all 3-nets in which every projectivity of $\tilde{\mathcal{M}}_6^{(i,j,k)}$ having a fixed point is the identity.

4.3. *Burn condition.* Let $\tilde{\mathcal{M}}_6^{(i,j;k,l)}$ denote the subset of all projectivities from $\tilde{\mathcal{M}}_6$ such that amongst their representations (7) there is one with $L_i = L_j$ and $L_k = L_l$.

If $\min\{|i-j|, |k-l|\} = 1$ then the set of projectivities $\pi \in \tilde{\mathcal{M}}_6^{(i,j;k,l)}$ having a fixed point coincides with the set of projectivities of $\mathcal{M}_4$ or $\mathcal{M}_4'$ respectively having a fixed point. Hence the condition that any such $\pi \in \tilde{\mathcal{M}}_6^{(i,j;k,l)}$ is the identity is equivalent to the Reidemeister or to the Bol condition for 3-nets, respectively.

Now we consider the set $\tilde{\mathcal{M}}_6^{(0,2;1,3)}$ of projectivities. One sees immediately: A projectivity $\pi = \sum_{i=1}^{6}[L_{i-1}, \Upsilon_i, L_i] \in \tilde{\mathcal{M}}_6^{(0,2;1,3)}$ having a fixed point on the line L_0 is the identity if and only if the lines L_i $(i = 0, \ldots, 5)$ and the points $x_k^\alpha = x_0^\alpha(\sum_{i=1}^{k}[L_{i-1}, \Upsilon_i, L_i])$ for any two points $x_0^\alpha \in L_0$ $(\alpha = 1, 2)$ and the connecting lines of the points x_{k-1}^α and x_k^α form a Burn configuration.

Let us consider the set $\tilde{\mathcal{M}}_6^{(0,2;1,4)}$ of projectivities. A projectivity $\pi = \sum_{i=1}^{6}[L_{i-1}, \Upsilon_i, L_i] \in \tilde{\mathcal{M}}_6^{(0,2;1,4)}$ having a fixed point on the line L_0 is the identity if and only if for any two points x_0^α ($\alpha = 1, 2$), the lines L_i, the points x_k^α and the joining lines of x_{k-1}^α and x_k^α form the following configuration:

(i) L_0, L_1, L_3, L_5 are vertical lines;

(ii) $x_0^\alpha, x_2^\alpha \in L_0$, $x_1^\alpha, x_4^\alpha \in L_1$, $x_3^\alpha \in L_3$, $x_5^\alpha \in L_5$;

(iii) the pairs of points $\{x_{k-1}^\alpha, x_k^\alpha\}$ ($k = 1, \ldots, 6$) determine a line if they are different;

(iv) the lines $x_0^1x_1^1$, $x_2^1x_3^1$, $x_4^1x_5^1$, $x_0^2x_1^2$, $x_2^2x_3^2$, $x_4^2x_5^2$ belong to the same family, the same holds for the lines $x_1^1x_2^1$, $x_3^1x_4^1$, $x_5^1x_0^1$, $x_1^2x_2^2$, $x_3^2x_4^2$, $x_5^2x_0^2$, but these two families are different.

This configuration we call a *D-configuration.*

If we replace in the defining properties of a D-configuration the condition (ii) by the weaker condition

(ii') $x_2^\alpha \in L_0$, $x_1^\alpha, x_4^\alpha \in L_1$, $x_3^\alpha \in L_3$, $x_5^\alpha \in L_5$ and $x_0^1 \in L_0$;

then this configuration is called a *non-closed* D-configuration. We say that a non-closed D-configuration D can be completed to a D-configuration if in D the incidence $x_0^2 \in L_0$ holds, too.

THEOREM 4.3.1. *In a* 3*-net N all non-closed D-configurations can be completed to D-configurations if and only if N satisfies the Burn condition.*

PROOF. First, we assume that any non-closed D-configuration can be completed. If we assume in a D-configuration $L_0 = L_5$ (and forget the points x_5^1, x_5^2 and the lines $x_0^1x_5^1$ and $x_0^2x_5^2$) then we have a (closed) Bol configuration. Hence, if every projectivity $\pi \in \tilde{\mathcal{M}}_6^{(0,2;1,4)}$ having a fixed point on the line L_0 is the identity then N is a Bol net. If we now map the points x_3^1, x_3^2 and the line L_3 under the involutory collineation σ_{L_1} introduced at the beginning of this section then the lines $L_0, L_1, \sigma_{L_1}(L_3), L_5$, the points x_0^1, x_0^2, x_2^1, x_2^2, $x_1^1, x_1^2, x_4^{1\prime} = L_1 \cap x_3^1x_2^1$, $x_4^{2\prime} = L_1 \cap x_3^2x_2^2$, $\sigma_{L_1}(x_3^1)$, $\sigma_{L_1}(x_3^2)$ and the corresponding joining lines $x_0^1x_1^1$, $x_0^2x_1^2$, $x_1^1x_2^1$, $x_1^2x_2^2$, $x_2^1x_4^{1\prime}$, $x_2^2x_4^{2\prime}$, $x_4^{1\prime}\sigma_{L_1}(x_3^1)$, $x_4^{2\prime}\sigma_{L_1}(x_3^2)$, $\sigma_{L_1}(x_3^1)x_5^1$, $\sigma_{L_1}(x_3^2)x_5^2$, $x_5^1x_0^1$, $x_5^2x_0^2$ form a Burn configuration. Hence every projectivity $\pi \in \tilde{\mathcal{M}}_6^{(0,2;1,4)}$ having a fixed point is the identity (or equivalently every non-closed D-configuration can be completed) if and only if the 3-net N is a Burn net.

Conversely if a 3-net satisfies the Burn condition then it satisfies the Bol condition and hence the mapping σ_L is a collineation for any vertical line L. If one has a (closed) Burn configuration with respect to the vertical lines L_0, L_1, L_2, L_3 then we obtain a (closed) D-configuration if we take instead of the line L_2 the line $\sigma_{L_1}(L_2)$ and instead of the points x_2 and x_2' the points $\sigma_{L_1}(x_2)$ and $\sigma_{L_1}(x_2')$. ∎

4.4. CC-*conditions.* Now we investigate the set $\tilde{\mathcal{M}}_6^{(0,2;3,5)}$ of projectivities. If we consider the subset $\mathcal{T} \subset \tilde{\mathcal{M}}_6^{(0,2;3,5)}$ of projectivities for which $L_6 = L_0$ then $\mathcal{T}$ contains the set $\mathcal{M}_4'$ of irreducible projectivities. It follows that if every projectivity $\pi \in \tilde{\mathcal{M}}_6^{(0,2;3,5)}$ having a fixed point is the identity then the 3-net N is a Bol net.

Conversely in a Bol net N we consider a projectivity $\pi = \prod_{i=1}^6 [L_{i-1}, \Upsilon_i, L_i] \in \tilde{\mathcal{M}}_6^{(0,2;3,5)}$ having a fixed point x_0^1 on L_0. We denote by L the vertical line through the intersection point y of the lines joining the points x_0^1, x_5^1 and the points x_2^1, x_3^1 where $x_k^\alpha = x_0^\alpha\left(\prod_{j=1}^k [L_{j-1}, \Upsilon_j, L_j]\right)$. It follows from the (closed) Bol configuration that the intersection point y' of the lines joining the points x_0^2, x_5^2 and the points x_2^2, x_3^2 is contained in the line L. Since the reflection σ_{L_0} is a collineation of N we have $\sigma_{L_0}(y) = x_1^1$ and $\sigma_{L_0}(y') = x_1^2$ and the configuration belonging to the projectivity π is closed. Hence every projectivity $\pi \in \tilde{\mathcal{M}}_6^{(0,2;3,5)}$ having a fixed point is the identity if and only if N is a Bol net.

The sets $\tilde{\mathcal{M}}_6^{(i,j;k,l)}$ of projectivities are invariant under the cyclic group $\mathbb{Z}_6$ on the set of indices $(i,j;k,l)$. Hence if $\min\{|i-j|,|k-l|\} \leq 2$ then every projectivity $\pi \in \tilde{\mathcal{M}}_6^{(i,j;k,l)}$ having a fixed point is the identity if and only if the 3-net N is a group net, Bol net or Burn net.

At the end of our discussion we have to consider the sets $\tilde{\mathcal{M}}_6^{(i,j;k,l)}$ where $|i-j| = |k-l| = 3$. Up to a cyclic permutation of indices we have to consider the set $\tilde{\mathcal{M}}_6^{(0,3;1,4)}$ of projectivities. Every projectivity $\pi \in \tilde{\mathcal{M}}_6^{(0,3;1,4)}$ with representation $\pi = \prod_{i=1}^{6}[L_{i-1}, \Upsilon_i, L_i]$ having a fixed point on the line L_0 is the identity if and only if any two points $x_0^\alpha \in L_0$ $(\alpha = 1,2)$, the lines L_i, the points $x_k^\alpha = x_0^\alpha(\prod_{j=1}^{k}[L_{j-1}, \Upsilon_j, L_j])$ and the joining lines $x_{k-1}^\alpha x_k^\alpha$ form the following configuration which we will call the CC-*configuration*:

(i) L_0, L_1, L_2, L_5 are vertical lines;

(ii) $x_0^1, x_0^2, x_3^1, x_3^2 \in L_0$, $x_1^1, x_1^2, x_4^1, x_4^2 \in L_1$, $x_2^1, x_2^2 \in L_2$ and $x_5^1, x_5^2 \in L_5$;

(iii) the pairs of points $\{x_0^1, x_1^1\}$, $\{x_0^2, x_1^2\}$, $\{x_1^1, x_2^1\}$, $\{x_1^2, x_2^2\}$, $\{x_2^1, x_3^1\}$, $\{x_2^2, x_3^2\}$, $\{x_3^1, x_4^1\}$, $\{x_3^2, x_4^2\}$, $\{x_4^1, x_5^1\}$, $\{x_4^2, x_5^2\}$, $\{x_5^1, x_0^1\}$, $\{x_5^2, x_0^2\}$ determine lines whenever the points are different;

(iv) the lines $x_0^1x_1^1$, $x_0^2x_1^2$, $x_2^1x_3^1$, $x_2^2x_3^2$, $x_4^1x_5^1$, $x_4^2x_5^2$ belong to the same family, the same holds for the lines $x_1^1x_2^1$, $x_1^2x_2^2$, $x_3^1x_4^1$, $x_3^2x_4^2$, $x_5^1x_0^1$, $x_5^2x_0^2$, but these two families are different.

If we replace in the defining properties of a CC-configuration the condition (ii) by the weaker condition

(ii)' $x_0^1, x_0^2, x_3^1, x_3^2 \in L_0$, $x_1^1, x_1^2, x_4^1, x_4^2 \in L_1$, $x_2^1, x_2^2 \in L_2$ and $x_5^2 \in L_5$;

then we call this configuration a *non-closed* CC-configuration.

We say that a 3-net N satisfies the CC-condition if all non-closed CC-configurations can be completed to CC-configurations, *i.e.* if in any non-closed CC-configuration the incidence $x_5^1 \in L_5$ is a consequence of all other incidences. Clearly the 3-net associated with a universal left conjugacy closed loop L satisfies the CC-condition (*cf.* proof of Theorem 4.4.1).

A conjugacy closed loop L is a left conjugacy closed loop for which also the set $\{\varrho_x, x \in L\}$ is invariant under the inner automorphisms in the group generated by the set $\{\varrho_x, x \in L\}$ (*cf.* §1). From Theorem 1.1.1, it follows that a loop L is conjugacy closed precisely if in L the following two identities are satisfied:

(i) $x \setminus (yz) = x \setminus (yx) \cdot x \setminus z$;

(ii) $(zy)/x = z/x \cdot (xy)/x$.

The conjugacy closed loops L have the property that all loops which are isotopic to L are already isomorphic to L.

We can call our CC-condition for a 3-net N the *vertical* CC-*condition*. The horizontal or transversal CC-conditions arise from the vertical CC-condition if we replace the vertical lines L_i by horizontal or transversal lines, respectively. We say that the 3-net N satisfies the CC-condition for a given line L and a non-parallel line K if every non-closed CC-configuration with the line L as L_0 and $x_4^1, x_5^1 \in K$ can be completed within N to a CC-configuration.

THEOREM 4.4.1. *If a 3-net N satisfies the vertical CC-condition with respect to the lines L and K and the horizontal CC-condition with respect to the lines L' and K' then N satisfies the horizontal CC-condition and the vertical CC-condition and every coordinate loop associated with N is a conjugacy closed loop.*

PROOF. Since every non-closed vertical CC-configuration determines a projectivity $\pi = \prod_{i=1}^{6}[L_{i-1}, \Upsilon_i, L_i]$ of the line L_0, the inverse projectivity π^{-1} determines the same CC-configuration up to indexing of the lines L_i $(i \neq 0)$. Hence we can assume that the line K containing the points x_4^1 and x_5^1 is a horizontal line. Now we consider in N the coordinate system which is given by the lines K and L. The coordinates of the points of the CC-configurations with $L_0 = L$ and $x_4^1, x_5^1 \in K$ are:

$$x_4^1 = (x, 1), \quad x_3^1 = (1, x), \quad x_2^1 = (z, x), \quad x_5^1 = (x \backslash zx, 1),$$
$$x_0^1 = (1, x \backslash zx), \quad x_1^1 = (x, x \backslash zx), \quad x_3^2 = (1, t), \quad x_2^2 = (z, t),$$
$$x_5^2 = (x \backslash zx, x \backslash t), \quad x_4^2 = (x, x \backslash t), \quad x_1^2 = (x, x \backslash zt), \quad x_0^2 = (1, x \backslash zt).$$

If the points x_5^1 and x_5^2 are contained in the same vertical line L_5 (this is the case for CC-configurations) then we obtain the identity

$$x \backslash zt = (x \backslash zx) \cdot (x \backslash t) \tag{8}$$

since the points x_5^2 and x_0^2 are contained in the same transversal line. Hence the coordinate loop $\hat{L}$ corresponding to the origin $K \cap L$ is a left conjugacy closed loop.

Conversely if we have a non-closed CC-configuration then it follows from the identity (8) that the points x_5^1 and x_5^2 are contained in a vertical line.

Now, we consider the coordinate loop belonging to the axes L' and K', since we can assume that K' is vertical. The coordinates of the points of the horizontal CC-configurations with $L_0 = L$ and $x_4^1, x_5^1 \in K'$ are:

$$x_4^1 = (1, x), \quad x_3^1 = (x, 1), \quad x_2^1 = (x, z), \quad x_5^1 = (1, xz/x),$$
$$x_0^1 = (1, xz/x), \quad x_1^1 = (xz/x, x), \quad x_3^2 = (t, 1), \quad x_2^2 = (t, z),$$
$$x_5^2 = (t/x, xz/x), \quad x_4^2 = (t/x, x), \quad x_1^2 = (tz/x, x), \quad x_0^2 = (tz/x, 1).$$

If the points x_5^1 and x_5^2 are contained in the same horizontal line L_5, then we obtain the identity

$$tz/x = (t/x) \cdot (xz/x), \tag{9}$$

since the points x_5^2 and x_0^2 are contained in the same transversal line. Hence the coordinate loop $\hat{L}'$ corresponding to the origin $L' \cap K'$ is a right conjugacy closed loop. Conversely the identity (9) forces the points x_5^1 and x_5^2 to be contained in a horizontal line.

We consider now the coordinate loop $\tilde{L}$ belonging to the axes L' and L. Since all loops which are left (right) isotopic to a left (right) conjugacy closed loop are left (right) conjugacy closed $\tilde{L}$ is a conjugacy closed loop. Hence all coordinate loops of the net N are conjugacy closed. ∎

Clearly the 3-net associated with a conjugacy closed loop L satisfies the vertical CC-condition and the horizontal CC-condition.

THEOREM 4.4.2. *A* 3*-net* N *satisfies the transversal* CC*-condition with respect to a given transversal line* L_0 *and another non-transversal line* K *if and only if the coordinate loop* L *of* N *belonging to the origin* $L \cap K$ *satisfies the following identity:*

$$x/\big((1/z)\setminus y\big)\cdot(x/z)\setminus y = x/y\cdot x\setminus y \tag{10}$$

for all $x, y, z \in L$.

PROOF. Let C be a transversal CC-configuration in N. We choose as coordinate axes the horizontal and vertical lines through the point x_2^1. The coordinates of the points of C with transversal lines L_i can be written in the following way:

$$x_1^1 = (x,1),\quad x_2^1 = (1,1),\quad x_0^1 = (x, x\setminus y),\quad x_3^1 = (1,y),\quad x_4^1 = (x/y, y),$$
$$x_5^1 = (x/y, x\setminus y),\quad x_2^2 = (1/z, z),\quad x_1^2 = (x/z, z),\quad x_0^2 = \big(x/z, (x/z)\setminus y\big),$$
$$x_3^2 = \big(1/z, (1/z)\setminus y\big),\ x_4^2 = \Big(x/\big((1/z)\setminus y\big), (1/z)\setminus y\Big),\ x_5^2 = \Big(x/\big((1/z)\setminus y\big), (x/z)\setminus y\Big).$$

Then the points x_5^1 and x_5^2 are contained in the same transversal line if and only if the identity (10) is satisfied. ■

It is well-known that in a 3-net N the Bol conditions for two different families of lines imply the Bol condition for the third class. In §3 (*cf.* Theorem 3.2.8) we have proved the same for the Burn conditions. Now we want to show that an analogous statement for CC-conditions does not hold.

We consider the conjugacy closed loop constructed by V.D. Belousov (*cf.* [3], p. 184, and [13], Theorem 3.3): Let F be a field, F^* its multiplicative group and let $G = F^* \times F$. The multiplication in G is defined by

$$(x,\xi)\cdot(y,\eta) = \big(xy, (x^{-1}-1)(y^{-1}-1) + y^{-1}\xi + \eta\big).$$

Then we have

$$(x,\xi)\setminus(z,\zeta) = \big(x^{-1}z, \zeta - xz^{-1}\xi - (x^{-1}-1)(xz^{-1}-1)\big)$$

and

$$(z,\zeta)/(y,\eta) = \big(zy^{-1}, y[\zeta - \eta - (z^{-1}y - 1)(y^{-1}-1)]\big).$$

Since the loop G is conjugacy closed the 3-net N corresponding to G satisfies the vertical and the horizontal CC-condition. It would satisfy the transversal CC-condition if in G the identity (10) of the Theorem 4.4.2 holds. In particular we have

$$(x,0)/(y,0)\cdot(x,0)\setminus(y,0) = (1, xy^{-1} - x^{-1}y + y - y^{-1} + x^{-1} - x) \tag{11}$$

and

$$\begin{aligned}(x,0)/\big((z,0)^{-1}\setminus(y,0)\big)\cdot\big((x,0)/(z,0)\big)\setminus(y,0)\\ = \big(1, z[xy^{-1} - x^{-1}y + y - y^{-1} + x^{-1} - x]\big),\end{aligned} \tag{12}$$

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where $(z,0)^{-1} = (1,0)/(z,0) = \big(z^{-1},(z-1)^2\big)$.

But if the field F has at least 6 elements then we can choose $x, y, z \in F^* \setminus \{1\}$ in such a way that

$$z[xy^{-1} - x^{-1}y + y - y^{-1} + x^{-1} - x] \neq xy^{-1} - x^{-1}y + y - y^{-1} + x^{-1} - x.$$

The validity of all three CC-conditions in a 3-net determines a proper subclass in the class of conjugacy closed loops. This class contains the extra loops but also loops with very weak associativity. To demonstrate this we use a class of examples constructed by Goodaire and Robinson in [13], p. 668.

Let R and S be rings with R commutative and associative and let $\theta\colon R \longrightarrow S$ be a homomorphism of $(R,+)$ into $(S,+)$. For (x,ξ) and (y,η) in $G = R \times S$ define

$$(x,\xi)(y,\eta) = \big(x+y, \xi+\eta+(xy^2)\theta\big).$$

Then $(G,\cdot)$ is a conjugacy closed loop with identity $(0,0)$ whose nucleus N is given by

$$N = \big\{(x,\xi) \in G : 2xyz \in \operatorname{Ker}\theta \text{ for all } y, z \in R\big\}.$$

This implies that if R is an integral domain of characteristic $\neq 2$ and the homomorphism θ is not trivial then G is a proper loop.

The inverse operations of the loop G have the following form:

$$(x,\xi) \setminus (z,\zeta) = \big(z-x, \zeta-\xi-[x(z-x)^2]\theta\big),$$
$$(z,\zeta)/(y,\eta) = \big(z-y, \zeta-\eta-[(z-y)y^2]\theta\big).$$

It follows that $(x,\xi)\backslash(0,0) = (-x, -\xi-x^3\theta) \neq (0,0)/(x,\xi) = (-x,-\xi+x^3\theta)$ if $x^3 \notin \operatorname{Ker}\theta$. Hence such a loop G has neither the left nor the right inverse property.

PROPOSITION 4.4.3. *The* 3*-net associated with a Goodaire-Robinson loop* $(G,\cdot)$ *satisfies all three* CC*-conditions.*

PROOF. Since $(G,\cdot)$ is a conjugacy closed loop we have only to prove that G satisfies the identity

$$(x,\xi)/\{[(0,0)/(z,\zeta)] \setminus (y,\eta)\} \cdot [(x,\xi)/(z,\zeta)] \setminus (y,\eta) = [(x,\xi)/(y,\eta)] \cdot [(x,\xi) \setminus (y,\eta)]$$

(*cf.* Theorem 4.4.2). We have

$$(x,\xi)/\{[(0,0)/(z,\zeta)] \setminus (y,\eta)\} = \big(x-y-z, \xi-\eta-\zeta+[z^3-(y+z)^2(x-y)]\theta\big)$$

and

$$[(x,\xi)/(z,\zeta)] \setminus (y,\eta) = \Big(y-x+z, \eta-\xi+\zeta+\big[(x-z)\big(z^2-(y-x+z)^2\big)\big]\theta\Big).$$

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Since

$$\begin{aligned}[(x,\xi)/(y,\eta)]\cdot[(x,\xi)\setminus(y,\eta)] \\ &= \big(0,[xy(y-x)]\theta\big) \\ &= \big(x-y-z,\xi-\eta-\zeta+[z^3-(y+z)^2(x-y)]\theta\big) \\ &\quad\cdot\Big(-x+y+z,-\xi+\eta+\zeta+\big[(x-z)\big(z^2-(y-x+z)^2\big)\big]\theta\Big)\end{aligned}$$

we obtain the assertion. ∎

In summary, in §4 we have established conditions, in terms of projectivities of length at most 6, which assure that a 3-net satisfies a configurational condition.

THEOREM 4.4.4. *Let N be a 3-net, L_0 be a line in N and n a given number ≤ 6. Let Σ be a set of projectivities of L_0 onto itself such that Σ consists of all projectivities having an irreducible representation*

$$\prod_{i=1}^{n}[L_{i-1},\Upsilon_i,L_i] \tag{13}$$

of length n with the following properties:

1) All lines L_i belong to the same family.

2) There exist subsets I_j of the set $\{0,\dots,n\}$ such that for any I_j the following condition holds: If $k,k'\in I_j$ then $L_k=L_{k'}$ in (13).

If every projectivity of Σ having a fixed point on L_0 is the identity then N satisfies one of the following conditions with respect to L_0: Reidemeister condition, Burn condition, Bol condition or CC*-condition.*

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