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Besov Spaces and Hausdorff Dimension For Some Carnot-Carathéodory Metric Spaces

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Abstract. We regard a system of left invariant vector fields $\mathcal{X} = \{X_1, \dots, X_k\}$ satisfying the Hörmander condition and the related Carnot-Carathéodory metric on a unimodular Lie group G. We define Besov spaces corresponding to the sub-Laplacian $\Delta = \sum X_i^2$ both with positive and negative smoothness. The atomic decomposition of the spaces is given. In consequence we get the distributional characterization of the Hausdorff dimension of Borel subsets with the Haar measure zero.

The theory of Hausdorff dimension of subsets of metric spaces has come to play an important role in many different areas of mathematics. One encounters the Hausdorff dimension in geometric measure theory, calculus of variations, fractal geometry, dynamical systems theory and others fields of mathematics. On the other hand in recent years there has been great interest in the study of Carnot-Carathéodory spaces. These are the metric spaces whose distance is generated by the sub-unit curves related to a family of vector fields of Hörmander type. There exists also the growing literature in the corresponding sub-elliptic analysis in particular the relevant PDEs, *cf. e.g.* [4] and [28]. The Carnot-Carathéodory Hausdorff measures and the Hausdorff dimensions seems to be still not well understood. The paper is a step in shedding some light on the problem.

We regard Carnot-Carathéodory metric ρ on a unimodular Lie group G defined by system of left invariant vector fields $\mathcal{X} = \{X_1, \dots, X_k\}$. It is assumed that the vector fields are linearly independent and satisfy the Hörmander condition. Our aim is to give distributional characterization of the Hausdorff dimension of Borel subsets of G with the Haar measure zero. To make it possible we should classify the distribution with respect to some "smoothness". For this purpose we used Besov spaces $B^s_{p,q}(G, \mathcal{X})$, $s \in \mathbb{R}$, $1 \le p \le \infty$, corresponding to the system $\mathcal{X}$ of the vector fields. The spaces are defined in terms of a heat semi-group related to the sub-Laplacian $\Delta = \sum_i X_i^2$. Function spaces of Sobolev-Besov type for subelliptic operators were studied by many authors, *cf.* [2], [3], [5], [7], [16]. The authors regard the spaces of positive smoothness. We need also the spaces with negative smoothness therefore we develop the theory in this direction. The definitions and facts concerning the systems of vector fields, the Carnot-Carathéodory metrics and Besov spaces are given in Section 1.

The main tool use in the paper is the technic of atomic decompositions. We extend the approach developed for the function spaces related to Beltrami-Laplace operator

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of a Riemannian manifold, *cf.* [19]. This method is a somewhat modified version of the Frazier-Jawerth approach, *cf.* [9], [10]. But it should be mentioned that atomic decompositions in non-Euclidean setting has been developed since the 70's. Coifman and Weiss book [1] on harmonic analysis on homogeneous spaces can be regard as a source of the theory, *cf.* also [12]. The atomic decomposition theorem is proved in Section 2, *cf.* Theorem 1. Using the decomposition one can easily prove elementary embeddings for the function spaces, *cf.* Corollary 1, and can compare the elliptic and subelliptic Besov spaces, *cf.* Corollary 2.

Let F be a subset of G. The Hausdorff s-dimensional measure $\mathcal{H}^s(F)$ of F is defined in the following way

$$\mathcal{H}^s(F) = \sup_{\varepsilon>0} \mathcal{H}^s_\varepsilon(F), \quad \mathcal{H}^s_\varepsilon(F) = \inf \sum_{U_i \in \mathcal{U}} \operatorname{diam}(U_i)^s,$$

where the infimum is over all countable covers $\mathcal{U}$ of F such that $\operatorname{diam}(U_i) \leq \varepsilon$ for each i. If $\sigma > s$ and $\mathcal{H}^s(F)$ is finite, then $\mathcal{H}^\sigma(F) = 0$. The last property allows to define the Hausdorff dimension of the set F with respect to the metric ρ

$$\dim_H^\rho F = \inf\{s : \mathcal{H}^s(F) = 0\}.$$

On the other hand a distributional dimension of a closed subset F of G of the Haar measure 0 can be defined in terms of the Besov spaces $B^s_{\infty,\infty}(G, \mathcal{X})$. We put

$$B^{s,F}_{\infty,\infty}(G, \mathcal{X}) = \{f \in B^s_{\infty,\infty}(G, \mathcal{X}) : f(\varphi) = 0 \text{ if } \varphi \in C_o^\infty(G) \text{ and } \varphi|F = 0\}.$$

Definition The distributional dimension $\dim_D^{\mathcal{X}} F$ of F is

$$\dim_D^{\mathcal{X}} F = \sup\{\delta \in \mathbb{R} : B^{-d+\delta,E}_{\infty,\infty}(G, \mathcal{X}) \neq \{0\} \text{ for some compact } E \subset F\}.$$

The distributional dimension $\dim_D^{\mathcal{X}} F$ of a set F with Haar measure zero describes the ability of the set F to carry non-trivial singular distributions as smooth as possible whereas the Hausdorff dimension $\dim_H^\rho F$ describes the massiveness of the set F. The distributional dimension in Euclidean setting was introduced in [27].

The main result of the paper reads as follows.

Theorem *Let F be a Borel subset of G with the Haar measure* 0*. Then*

$$\dim_H^\rho F = \dim_D^{\mathcal{X}} F. \tag{1}$$

The measure theoretical methods usually allow to prove inequalities $\dim_H^\rho F \leq s$, whereas the distributional dimension can be easier estimated from the below. Thus the above theorem is helpful in calculating the Hausdorff dimension. A proof of the theorem can be found in Section 3.

1 Preliminaries

1.1 Subelliptic Operators on Lie groups

Let G be a connected n-dimensional unimodular Lie group endowed with a Haar measure dx and a system $\mathcal{X} = \{X_1, \dots, X_k\}$ of left invariant vector fields, satisfying the Hörmander condition. We assume that the vectors $X_1(e), \dots, X_k(e)$ are linearly independent.

The Carnot-Carathéodory distance corresponding to $\mathcal{X}$ is defined in the following way. Let $\mathcal{A}_{\mathcal{X}}$ be the family of all absolutely continuous paths $\gamma\colon [0,1] \to G$ such that $\dot\gamma(t) = \sum a_i(t) X_i\big(\gamma(t)\big)$, for almost every $t \in [0,1]$. Put

$$|\gamma| = \int_0^1 \Big(\sum_{i=0}^{k} a_i^2(t)\Big)^{1/2} dt.$$

Since $\mathcal{X}$ satisfies the Hörmander condition any two points of G can be joined by such a path. So we can put

$$\rho_{\mathcal{X}}(x,y) = \rho(x,y) = \inf\{|\gamma| : \gamma \in \mathcal{A}_{\mathcal{X}}, \gamma(0) = x, \gamma(1) = y\}.$$

The function ρ is a left invariant distance on G which induces the topology of G. To simplify the future notation we put also $\rho(x) = \rho(e,x)$.

If the vectors $X_1(e), \dots, X_k(e)$ span T_eG than the distance ρ coincide with Riemannian distance given by the left invariant Riemannian metric with $X_1(e), \dots, X_k(e)$, $n = k$, as an orthogonal basis. Otherwise we deal with sub-Riemannian metric on G, *cf.* [24] for the basic facts of the sub-Riemannian geometry.

If $\mathcal{G}$ is a left-invariant Riemannian metric on G such that the vectors $X_1(e), \dots, X_k(e)$ form an orthonormal system then the sub-Riemannian structure is given by the restriction of $\mathcal{G}_x$ to $S_x = \operatorname{span}\big(X_1(x), \dots, X_k(x)\big)$. The Riemannian distance $\rho_{\mathcal{G}} = \tilde\rho$ and the sub-Riemannian distance always satisfy the following inequality $\tilde\rho(x,y) \le \rho(x,y)$.

The metric space (G,ρ) is complete since the Riemannian manifold $(G,\mathcal{G})$ is complete, *cf.* [24, Theorem 7.4]. In that case any two point of G can be joined by length minimizing sub-Riemannian geodesic, *cf.* Theorem 7.1 in [24].

For a multi-index $I = (i_1, \dots, i_m)$ with $i_j \in \{1, \dots, k\}$ we put

$$X_I = \Big[X_{i_1}\big[X_{i_2}, \dots, [X_{i_{m-1}}, X_{i_m}]\cdots\big]\Big] \quad \text{and} \quad X^I = X_{i_1}X_{i_2}, \dots, X_{i_m}.$$

The Hörmander condition implies the existence of $N \in \mathbb{N}$ such that $\{0\} = K_0 \subset K_1 \subset \cdots \subset K_N = T_eG$, where $K_i = \operatorname{span}\{X_I(e) : |I| \le i\}$. The integer

$$d = \dim K_1 + 2(\dim K_2 - \dim K_1) + \cdots + N(\dim K_N - \dim K_{N-1}) \tag{2}$$

is called a *local dimension* of $(G,\mathcal{X})$, *cf.* [28, Chapter V]. Since both, the Carnot-Carathéodory distance and the Haar measure are left invariant, a volume of balls is independent of centers and depend only on the radius. We will denote the volume of the balls of radius t by $V(t)$. For balls with small radius we have the following estimates:

$$\exists d \in \mathbb{N}\, \exists C > 0\, \forall t \in (0,1] \quad C^{-1}t^d \le V(t) \le Ct^d, \tag{3}$$

cf. [28, Theorem V.4.1].

A stratified group is a nilpotent Lie group G, with the Lie algebra $\mathfrak{g}$ admitting a vector space decomposition $\mathfrak{g} = \bigoplus_1^N V_j$ such that $[V_1, V_j] = V_{j+1}$ for $j < m$ and $[V_1, V_N] = \{0\}$. We define a one-parameter family δ_s of automorphisms of $\mathfrak{g}$, called *dilations*, by the formula $\delta_s(\sum_1^N Y_j) = \sum_r^j Y_j$. The dilations induce automorphisms of G, still called dilations. Let $\mathfrak{X}$ be a basis of V_1. The Carnot-Carathèodory metric generated by the family $\mathfrak{X}$ is equivalent to the metric defined by homogeneous norm $x \to |x|$ on G given by

$$\Big|\exp\Big(\sum_1^N Y_j\Big)\Big| = \Big(\sum_r^j |Y_j|_e^{2N!/j}\Big)^{1/(2N!)}.$$

Here $|\cdot|_e$ denotes the Euclidean norm.

The sub-Laplacian

$$\Delta = \sum_{i=1}^d X_i^2 \tag{4}$$

is a hypoelliptic, symmetric operator. The operator $-\Delta$ is a positively defined, essentially self-adjoint with domain $C_o^\infty(G)$. Its Friedrichs extension is a infinitesimal generator of the symmetric markovian semigroup $H_t = e^{t\Delta}$ that is called the *heat semigroup* associated with Δ. The semigroup acts on the L_p spaces, $1 \le p \le \infty$. Thanks to the left invariance of Δ, H_t admits a right convolution kernel h_t

$$H_t f(x) = f * h_t(x) = \int_G h_t(y^{-1}x) f(y)\, dy.$$

The function $\mathbb{R}_+ \times G \ni (t,x) \mapsto h_t(x)$ is a positive smooth solution of $(\frac{d}{dt} - \Delta)u = 0$ and $\|h_t\|_1 = 1$. Thus H_t is a semigroup of contractions in any L_p spaces.

The following estimates of the heat kernel are important for us:

1. *the Harnack inequality*: $\forall 0 < t_1 < t_2 < \infty$, $\forall I \in \mathfrak{I}(k)$ and $\forall m \in \mathbb{N}$, there exists $C > 0$ such that $\forall x \in G$ and $\forall s \in (0,1]$

$$\sup_{y \in B(x,\sqrt{s})} \Big| X^I \Big(\frac{\partial}{\partial t}\Big)^m h_{st_1}(y)\Big| \le Cs^{-m-|I|/2} \inf_{y \in B(x,\sqrt{s})} |h_{st_2}(y)|, \tag{5}$$

cf. [28, Theorem V.4.2],

2. *the gaussian bounds*: there exists $C, c > 0$ such that, for all $t \in (0,1)$ and all $x \in G$

$$C^{-1}V(\sqrt{t})^{-1}\exp\big(-c^{-1}\rho(x)^2/t\big) \le h_t(x) \le CV(\sqrt{t})^{-1}\exp\big(-c\rho(x)^2/t\big), \tag{6}$$

cf. [28, Theorem V.4.3].

To deal with function spaces of negative smoothness we need a counterpart of Schwartz space of distribution. Let $\tilde{V}(r)$ denote a volume of a geodesic ball of radius r with respect the left invariant Riemannian metric $\mathcal{G}$ on G. The volume $\tilde{V}(r)$ can be estimates in the following way

$$\tilde{V}(r) \le C r^n e^{\kappa r}, \quad n = \dim G,$$

for suitable constants $C > 0$ and $\kappa \ge 0$. We define the seminorms $|\psi|_{m,k}$, $m, k \in \mathbb{N}$ by

$$|\psi|_{m,k} = \sup_{x \in G, j \le k} |\nabla^j \psi(x)| \big(1 + \tilde{\rho}(e, x)\big)^{-m} e^{-\kappa \tilde{\rho}(e,x)}. \tag{7}$$

A space of rapidly decreasing functions $\mathcal{S}(G)$ is a vector space of functions $\psi \in C^\infty(G)$ such that $|\psi|_{m,k} < \infty$ for any m and k. The space $\mathcal{S}(G)$ is a Fréchet space and $C_0^\infty(G)$ is a dense subspace of $\mathcal{S}(G)$. In consequence the space $\mathcal{S}'(G)$ dual to $\mathcal{S}(G)$ can be identified with the subspace of the space of distributions. The functions belonging to $\mathcal{S}(G)$ are p-integrable, $1 \le p < \infty$. Moreover, it follows from the above estimates of the heat kernels that any regarded heat kernel $h_t(x)$ is an element of $\mathcal{S}(G)$. The standard argument with the heat semigroup give as the following decomposition of integrable function f

(8)
$$f(x) = f * h_{m,0} + \frac{1}{(m-1)!} \int_0^1 t^m f * h_t^m \frac{dt}{t} \text{ with } h_t^k = \frac{d^k}{dt^k} h_t \text{ and } h_{m,0} = \sum_{\ell=0}^{m-1} \frac{1}{\ell!} h_1^\ell.$$

This formula can be extended to $f \in \mathcal{S}'$ if the convergence of the integral in (8) is understood in the weak sense.

If $\alpha > 0$ then the integral

$$J_\alpha(x) = \Gamma\Big(\frac{\alpha}{2}\Big)^{-1} \int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} h_t(x)\, dt$$

converges absolutely for any $x \neq e$ to a positive function such that $\int J_\alpha(x)\, dx = 1$. So we can define the Bessel potential $(I - \Delta)^{-\alpha/2}$ by

$$(I - \Delta)^{-\alpha/2} f(x) = \int_G f(y) J_\alpha(y^{-1}x)\, dy. \tag{9}$$

It is straightforwards to verify that:

$$\|(I - \Delta)^{-\alpha/2} f\|_p \le \|f\|_p, \quad \text{if } 1 \le p \le \infty \text{ and } \alpha > 0, \tag{10}$$

$$(I - \Delta)^{-\alpha/2}(I - \Delta)^{-\beta/2} = (I - \Delta)^{-(\alpha+\beta)/2}, \quad \alpha, \beta > 0, \tag{11}$$

$$(I - \Delta)^k (I - \Delta)^{-\alpha/2} = (I - \Delta)^{-(\alpha-2k)/2}, \quad \alpha > 2k. \tag{12}$$

The definition coincides with the definition via spectral theorem for L_2.

1.2 Besov spaces with positive smoothness.

Besov spaces with positive smoothness defined on Lie Groups have been investigated by several authors. The spaces were first treated on stratify groups, *cf.* [8] and [16], then in more general setting for semi-groups generated by sub-Laplacian in [2] and [5]. In contrast to the above mentioned papers we are interested in the spaces with negative smoothness. But in this section we recall the basic facts about spaces with positive smoothness. We adapt the approach via the heat semi-group presented in [5]. This approach can be easily extended to negative smoothness, therefore we formulate a general definition from the very beginning.

Definition 1 Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ and $m > \frac{|s|}{2}$.

$$B^s_{p,q}(G, \mathcal{X}) = \Big\{ f \in \mathcal{S}' : \|f|B^s_{p,q}(G, \mathcal{X})\| = \|f * h_{m,0}\|_p + \Big(\int_0^1 t^{(m-s/2)q} \|f * h^m_t\|^q_p \frac{dt}{t} \Big)^{1/q} < \infty \Big\}$$

Remark 1 1. The norms depend on the chosen m, but the definition of the Besov space is independent of m up to norm equivalence. This is the direct consequence of Proposition 1 if $s > 0$, and the atomic decomposition theorem if $s \leq 0$. The atomic decomposition theorem is proved in Section 2.

2. If $s > 0$ then one can use $\|f\|_p$ instead of $\|f * h_{m,0}\|_p$ in the definition of the norm. Indeed, the inequality $\|f*h_{m,0}\|_\infty \leq C\|f\|_\infty$ is clear. On the other hand using the formula (8), the Minkowski inequality for integrals and the Hölder inequality we get

$$\|f\|_p \leq \|f * h_{m,0}\|_p + \frac{1}{(m-1)!} \Big(\int_0^1 t^{q's/2} \frac{dt}{t} \Big)^{1/q'} \Big(\int_0^1 t^{q(m-s)/2} \|f * h^m_t\|^q_p \frac{dt}{t} \Big)^{1/q}$$
$$\leq C \bigg(\|f * h_{m,0}\|_p + \Big(\int_0^1 t^{q(m-s)/2} \|f * h^m_t\|^q_p \frac{dt}{t} \Big)^{1/q} \bigg).$$

3. If the vectors $X_1(e), \ldots, X_k(e)$ span T_eG, *i.e.* if $k = n$, then the spaces $B^s_{p,q}(G, \mathcal{X})$ coincides with the spaces $B^s_{p,q}(G)$ defined on G by H. Triebel in terms of the left invariant Riemannian metric, *cf.* [26]. In that case the spaces $B^s_{p,q}(G, \mathcal{X})$ are independent of the given system $\mathcal{X}$ such that $k = n$, so they will be denoted by $B^s_{p,q}(G)$.

4. To simplify the notation we will write $\|f|B^s_{p,q}\|$ instead of $\|f|B^s_{p,q}(G, \mathcal{X})\|$ if it does not lead to misunderstanding.

By $C(G)$ we denote the set of all complex-valued bounded and uniformly-continuous functions on G equipped with the sup-norm. Furthermore, if $j \in \mathbb{N}$, we define a space

$$C^j(G, \mathcal{X}) = \{ f \in C(G) : X^I f \in C(G) \text{ for all } |I| \leq j \}$$

endowed with the norm

$$\|f|C^j(G, \mathcal{X})\| = \sum_{I:|I| \leq j} \|X^I f\|_\infty.$$

Let $T(g)$ denote a left-translation with respect to element $g \in G$, *i.e.* $T(g)f(x) = f(gx)$. The left regular representation $G \ni g \mapsto T(g)$ is strong continuous in $L_p(G)$ if $1 \leq p < \infty$, and weak* continuous if $p = \infty$. The heat-semigroup H_t is a strong continuous (weak* continuous if $p = \infty$) semigroup generated by the closure of the subelliptic operator (4). The semigroup is exponentially decreasing, *cf.* (6), and holomorphic, *cf.* Corollary 4.17 in [15].

We put

$$\omega_t^{(m)} f = \sup\left\{ \left\| \big(I - T(g_1)\big) \cdots \big(I - T(g_m)\big) f \right\|_p : g_1, \dots, g_m \in G \rho(g_i) \leq t \right\},$$

and regard the following norms

$$\|f\|_{(1)} = \|f\|_p + \left(\int_0^1 t^{(m-s/2)q} \|(I - H_t)^m f\|_p^q \frac{dt}{t} \right)^{1/q}, \tag{13}$$

$$\|f\|_{(2)} = \|f\|_p + \left(\int_0^1 t^{(m-s/2)q} \|\Delta^m H_t f\|_p^q \frac{dt}{t} \right)^{1/q}, \tag{14}$$

$$\|f\|_{(3)} = \|f\|_p + \left(\int_0^1 (t^{-s} \omega_t^{(m)} f)^q \frac{dt}{t} \right)^{1/q}. \tag{15}$$

The following proposition is a special case of the results proved in [5], *cf.* Theorem 3.1, Theorem 3.2 and Corollary 3.4.

Proposition 1 *Let $s > 0$, $1 \leq p, q \leq \infty$ and $m > \frac{s}{2}$. Then*

$$\begin{aligned} B^s_{p,q}(G) &= \{f \in L_p(G) : \|f\|_{(1)} < \infty\} = \{f \in L_p(G) : \|f\|_{(2)} < \infty\} \\ &= \{f \in L_p(G) : \|f\|_{(3)} < \infty\}. \end{aligned}$$

Moreover, the norms $\left\| \cdot | B^s_{p,q}(G, \mathcal{X}) \right\|$, $\|\cdot\|_{(1)}$, $\|\cdot\|_{(2)}$, $\|\cdot\|_{(3)}$ are equivalent.

Remark 2 If $p = q = \infty$ we get the Hölder-Zygmund spaces $\mathcal{C}^s(G, \mathcal{X}) = B^s_{\infty,\infty}(G, \mathcal{X})$. If $j < s \leq j + 1$ then $C^{j+1}(G, \mathcal{X}) \subset \mathcal{C}^s(G, \mathcal{X}) \subset C^j(G, \mathcal{X})$.

2 Atomic Decomposition and Besov Spaces

In this section we prove the atomic decomposition theorem. To formulate the decomposition we need some coverings of the group G by Carnot-Carathéodory balls. We recall that a covering is called *uniformly locally finite* if any element of the group belongs to at most C balls of the coverings. The smallest possible constant C is called a *multiplicity of the covering*.

Let r_j, $j = 0, 1, 2 \dots$ be a sequence of positive numbers decreasing to zero. Let $\left(\mathcal{B}_j = \{B(x_{j,i}, r_j)\}_{i=0}^\infty\right)_{j=0}^\infty$ be a sequence of uniformly locally finite coverings of G by balls of radius r_j. The supremum of multiplicities of coverings $\mathcal{B}_j$, $j = 0, 1, \dots$, is called the *multiplicity of the sequence* $\mathcal{B}_j$. The sequence $\mathcal{B}_j$ is called *uniformly locally finite* if its multiplicity is finite and the balls $B(x_{j,i}, r_j/2)$ and $B(x_{j,k}, r_j/2)$ have empty intersection for any possible j, i, k, $i \neq k$.

Lemma 1 *Let G be a connected unimodular Lie group equipped with the Carnot-Carathéodory metric. For every sufficiently small $r_0 > 0$ there exists a uniformly locally finite sequence $(\mathcal{B}_j)$ of coverings of G by balls of radius $r_j = 2^{-j}r_0$, $\mathcal{B}_j = \{B(x_{j,i}, r_j)\}_{i\in\mathbb{N}}$, $j = 0, 1, \ldots$. Moreover, if $l \in \mathbb{N}$ and $l \cdot r < r_0$ then the multiplicity of the sequence $(\mathcal{B}_j^{(l)})_{j=0,1,\ldots}$, $\mathcal{B}_j^{(l)} = \{B(x_{j,i}, lr_j)\}_{i\in\mathbb{N}}$, is also finite.*

The proof of the above lemma is the same as the proof of Lemma 4 in [19] therefore it is omitted here. To simplify the notation we will assume that $r_0 = 1$.

Definition 2 Let $s \in \mathbb{R}$ and $0 < p \le \infty$. Let L and M be integers such that $L \ge 0$ and $M \ge -1$.

(a) A smooth function $a(x)$ is called an 1_L-*atom centered* at $B(x, r)$ if

$$\operatorname{supp} a \subset B(x, 2r), \tag{16}$$

$$\sup_{y\in X} |X^I a(y)| \le C \quad \text{for any } |I| \le L. \tag{17}$$

(b) A smooth function $a(x)$ is called an $(s, p)_{L,M}$-*atom centered* at $B(x, r)$ if

$$\operatorname{supp} a \subset B(x, 2r), \tag{18}$$

$$\sup_{y\in X} |X^I a(y)| \le r^{s-|I|-\frac{d}{p}}, \quad \text{for any } |I| \le L, \tag{19}$$

$$\Big| \int_G a(y)\psi(y)\, dy \Big| \le r^{s+M+1+d/p'} \big\| \psi | C^{M+1}\big(\overline{B(x, 2r)}\big) \big\| \tag{20}$$

holds for any $\psi \in C_0^\infty\big(B(x, 3r)\big)$.

If $M = -1$ then (20) means that no moment conditions are required.

Theorem 1 *Let G be a connected unimodular Lie group equipped with the Carnot-Carathéodory metric. Let $\big(\mathcal{B}_j = \{B(x_{j,i}, 2^j)\}_{i\in\mathbb{N}}\big)$ be a uniformly locally finite sequence of coverings of G.*

Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$. Let L and M be fixed integers satisfying the following condition

$$L \ge ([s] + 1)_+ \quad \text{and} \quad M \ge \max([-s], -1). \tag{21}$$

(a) *each $f \in B^s_{p,q}(G, \mathcal{X})$ can be decomposed as follows*

$$f = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} s_{j,i} a_{j,i} \quad (\textit{convergent in } \mathcal{S}') \tag{22}$$

with

$$\Big(\sum_{j=0}^{\infty} \Big(\sum_{i=0}^{\infty} |s_{j,i}|^p \Big)^{q/p} \Big)^{1/q} < \infty. \tag{23}$$

where $a_{0,i}$ is a 1_L-atom centered in $B(x_{0,i}, 1)$ and $a_{j,i}$, $j > 0$, is a $(s, p)_{L,M}$-atom centered in $B(x_{j,i}, 2^{-J})$.

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(b) *Conversely, suppose that* $f \in \mathcal{S}'$ *can be represented as in (22) and (23). Then* $f \in B^s_{p,q}(G, \mathcal{X})$.

Furthermore, the infimum of (23) with respect to all admissible representations (for fixed sequence of coverings and fixed integers L, M) is an equivalent norm in $B^s_{p,q}(G, \mathcal{X})$.

Proof

Step 1 First we prove some auxiliary inequalities. Let M_o denote a local maximal function *i.e.* $M_o f(x) = \sup_{r \le c} V(r)^{-1} \int_{B(x,r)} f(y)\, dy$. M_o is a bounded operator in $L_p(G)$, $1 < p \le \infty$. Let Φ be a nonnegative radial function defined on G supported in $B(e, 1)$. If Φ is decreasing in radial directions then there is a positive constant C such that the inequality

$$|\Phi * f(x)| \le C \int_X \Phi(y)\, dy (M_o|f|)(x) \tag{24}$$

holds for any locally integrable function f. The last inequality can be proved in the same way as the similar inequality in [20, page 57], confer also [18, Proof of Theorem 2].

We choose $\varepsilon > 1$. Let $\tilde{\chi}_{j,i}$ denote the characteristic function of the ball $\Omega(x_{j,i}, \varepsilon 2^{-j})$. If we put $\tilde{\chi}^{(p)}_{j,i} = 2^{\frac{jd}{p}} \tilde{\chi}_{j,i}$, then the following elementary inequality

$$M_o(\tilde{\chi}^{(p)}_{j,i})(x) \le C \big(M_o(\tilde{\chi}^{(p)w}_{j,i}) \big)^{1/w}(x) \tag{25}$$

holds for any $0 < w < 1$ with the constant C independent of j. The inequalities (24)–(25) and the estimates of the heat kernel, *cf.* (5)–(6) imply

$$\begin{aligned} t^m \Big| \sum_{i=0}^{\infty} s_{j,i} \tilde{\chi}^{(p)}_{j,i} \Big| * |h^m_t|(x) &\le \int_{|y| \le \sqrt{t}} + \int_{\sqrt{t} \le |y| \le 1} + \int_{|y| \ge 1} \\ &\le C M_o \Big(\sum_{i=0}^{\infty} |s_{j,i}| \tilde{\chi}^{(p)}_{j,i} \Big)(x) + Ch * \Big(\sum_{i=0}^{\infty} |s_{j,i}| \tilde{\chi}^{(p)}_{j,i} \Big)(x), \end{aligned} \tag{26}$$

where h is a fixed integrable function.

For future use we need two positive constants b and δ. We choose these constants in such a way that the following identities are satisfied $b - \delta^2 = \frac{b}{16}$ and $b - \delta = \frac{b}{4}$. Such constants exist and both b and δ are greater then 1. Let $Q_{j,i} = (b4^{-j-1}, b4^{-j}) \times \Omega(x_{j,i}, 2^{-j})$. Then the Harnack-Moser inequality for subsolutions of parabolic equations implies

$$\sup_{(t,x) \in Q_{j,i}} |h^m_t * f(x)| \le C 2^{jd/w} \Big(\int_{\Omega(x_{j,i}, \delta 2^{-j})} \int_{b4^{-j-2}}^{b4^{-j}} |h^m_t * f(x)|^w \frac{dt}{t}\, dx \Big)^{1/w}, \tag{27}$$

where C is the constant depending only on d, b, δ and w, $0 < w < \infty$, *cf.* Theorem 5.1 in [17].

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Let $\{\psi_{j,i}\}$ be the smooth resolution of unity corresponding to the covering $\{\Omega(x_{j,i}, \epsilon 2^{-j})\}$. We may assume that for every positive m there is a constant b_m such that the inequality

$$\Big| \frac{\partial^{|\gamma|}}{\partial H^{\gamma}} \psi_{j,i} \circ \exp_{x_{j,i}}(H) \Big| \le b_m 2^{-j|\gamma|} \tag{28}$$

holds for every j, i and every $H \in T_{x_{j,i}}X$ and every multi-index γ such that $|\gamma| \le m$. The Theorem III.1.5 in [28] the scaling method, *cf.* Section V.3 ibidem, imply that the inequality

$$|\nabla^k h_t^m * f(x)| \le C 2^{j(k+d)} \int_{Q_{j,i}} |h_t^m * f(y)| \frac{dt}{t}\, dy. \tag{29}$$

holds for any $(t, x) \in [\epsilon b 4^{-j-1}, \epsilon b 4^{-j}] \times \Omega(x_{j,i}, \epsilon 2^{-j})$.

Step 2 We assume that $s > 0$. Let

$$f = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} s_{j,i} a_{j,i} \quad \text{with} \quad \Big(\sum_{j=0}^{\infty} \Big(\sum_{i=0}^{\infty} |s_{j,i}|^p \Big)^{q/p} \Big)^{1/q} < \infty.$$

We prove that $\|f|B^s_{p,q}\|$ can be estimated from above by the atomic norm. The estimate of $\|f * h_{0,m}\|_p$ is almost immediate,

$$\begin{aligned} \Big\| \sum_{j,i=0}^{\infty} s_{j,i} a_{j,i} * h_{m,0} \Big\|_p &\le \|h_{m,0}\|_1 \sum_{j=0}^{\infty} \Big\| \sum_{i=0}^{\infty} s_{j,i} a_{j,i} \Big\|_p \\ &\le C \sum_{j=0}^{\infty} 2^{-j(s-\frac{d}{p})} \Big\| \sum_{i=0}^{\infty} s_{j,i} \chi_{j,i} \Big\|_p \\ &\le C \Big(\sum_{j=0}^{\infty} \Big(\sum_{i=0}^{\infty} |s_{j,i}|^p \Big)^{q/p} \Big)^{1/q}. \end{aligned} \tag{30}$$

To estimate the second part of the norm we divide it into two sums

$$\Big(\int_0^1 t^{(m-s/2)q} \Big\| \sum_{j,i=0}^{\infty} s_{j,i} a_{j,i}(x) * h_t^m \Big\|_p^q \frac{dt}{t} \Big)^{1/q}$$

$$\le \Big(\sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} t^{(m-s/2)q} \Big(\sum_{j=0}^{[k/2]} \Big\| \sum_{i=0}^{\infty} s_{j,i} a_{j,i} * h_t^m \Big\|_p \Big)^q \frac{dt}{t} \Big)^{1/q} \tag{31}$$

$$+ \Big(\sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} t^{(m-s/2)q} \Big(\sum_{j=[k/2]}^{\infty} \Big\| \sum_{i=0}^{\infty} s_{j,i} a_{j,i} * h_t^m \Big\|_p \Big)^q \frac{dt}{t} \Big)^{1/q}. \tag{32}$$

We put $J = \min(m, [\frac{L}{2}])$. If $j \leq [k/2]$ then $(2j-k)(2J-s)$ is a nonpositive number. We choose also $w < 1$ such that $\frac{p}{w} > 1$. By the definition of atoms the sum (31) is less or equal to

$$\Big(\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{[k/2]}\sqrt{2}^{(2j-k)(2J-s)}\Big\|t^{m-J}\sum_{i=0}^{\infty}|s_{j,i}|\tilde{\chi}_{j,i}^{(p)} * |h_t^{m-J}|\Big\|_p\Big)^q\Big)^{1/q}$$

$$\leq C\Big(\sum_{j=0}^{\infty}\Big\|\Big(M_o\Big(\sum_{i=0}^{\infty}|s_{j,i}|^w\tilde{\chi}_{j,i}^{(p)w}\Big)\Big)^{1/w}\Big\|_p^q\Big)^{1/q}$$

$$+C\Big(\sum_{j=0}^{\infty}\Big\|\Big(\sum_{i=0}^{\infty}|s_{j,i}|\tilde{\chi}_{j,i}^{(p)}\Big) * h\Big\|_p^q\Big)^{1/q} \tag{33}$$

$$\leq C\Big(\sum_{j=0}^{\infty}\Big\|\Big(\sum_{i=0}^{\infty}|s_{j,i}|\tilde{\chi}_{j,i}^{(p)}\Big)\Big\|_p^q\Big)^{1/q} \leq C\Big(\sum_{j=0}^{\infty}\Big(\sum_{i=0}^{\infty}|s_{j,i}|^p\Big)^{q/p}\Big)^{1/q}, \tag{34}$$

where the first inequality follows from (25)–(26) and the second by boundedness of the local maximal operator.

Now let $k - 2j \leq 0$. In this case we have,

$$\Big(\sum_{k=0}^{\infty}\int_{2^{-k-1}}^{2^k} t^{(m-s/2)q}\Big(\sum_{j=[k/2]}^{\infty}\Big\|\sum_{i=0}^{\infty}s_{j,i}a_{j,i} * h_t^m\Big\|_p\Big)^q\frac{dt}{t}\Big)^{1/q}$$

$$\leq C\Big(\sum_{j=0}^{\infty}\Big\|\Big(M_o\Big(\sum_{i=0}^{\infty}|s_{j,i}|^w\tilde{\chi}_{j,i}^{(p)w}\Big)\Big)^{1/w}\Big\|_p^q\Big)^{1/q}$$

$$+C\Big(\sum_{j=0}^{\infty}\Big\|\Big(\sum_{i=0}^{\infty}|s_{j,i}|\tilde{\chi}_{j,i}^{(p)}\Big) * h\Big\|^q\Big)^{1/q}$$

$$\leq C\Big(\sum_{j=0}^{\infty}\Big(\sum_{i=1}^{\infty}|s_{j,i}|^p\Big)^{q/p}\Big)^{1/q}.$$

Thus we have proved that the following inequality

$$\|f|B^s_{p,q}(G,\mathfrak{X})\| \leq C\Big(\sum_{j=0}^{\infty}\Big(\sum_{i=0}^{\infty}|s_{j,i}|^p\Big)^{q/p}\Big)^{1/q} \tag{35}$$

holds if $s > 0$.

Step 3 We prove that the inequality (35) holds also for $s \leq 0$. Now the moment condition is of the great importance. In particular, the moment condition and the

Harnack inequality imply

$$t^m|a_{j,i} * h_t^m(x)| = t^m\Big|\int_G a_{j,i}(y)h_t^m(y^{-1}x)\,dy\Big|$$
$$\leq t^m 2^{-j(s+M+1+d/p')}\big\|\, h_t^m(x^{-1}\cdot)|C^{M+1}\big(\overline{B(x_{j,i},2^{-j+2})}\big)\,\big\|$$
$$\leq C2^{-j(s+M+1+d/p')}\sqrt{t}^{-M-1}\inf_{y\in B(x_{j,i},2^{-j+2})} h_{2t}(x^{-1}\cdot).$$

Thus

$$t^m\Big\|\sum_{i=0}^{\infty} s_{j,i}a_{j,i} * h_t^m\Big\|_1 \leq C2^{-j(s+M+1)}\sqrt{t}^{-M-1}\sum_{i=0}^{\infty}|s_{j,i}|\int_G h_{2t}(x^{-1})x_{j,i}\,dx$$
$$\leq C2^{-j(s+M+1)}t^{-M-1}\sum_{i=0}^{\infty}|s_{j,i}|$$

and

$$t^m\Big\|\sum_{i=0}^{\infty} s_{j,i}a_{j,i} * h_t^m(x)\Big\|_\infty \leq C2^{-j(s+M+1+d)}\sqrt{t}^{-M-1}\sup_i|s_{j,i}|$$
$$\times\sup_x\sum_{i=0}^{\infty}2^{jd}\int_{\overline{B(x_{j,i},2^{-j+2})}} h_{2t}(x^{-1}y)\,dy$$
$$\leq C2^{-j(s+M+1)}t^{-M-1}\sup_i|s_{j,i}|.$$

By interpolation we get

$$t^m\Big\|\sum_{i=0}^{\infty} s_{j,i}a_{j,i} * h_t^m(x)\Big\|_p \leq C2^{-j(s+M+1)}\sqrt{t}^{-M-1}\Big(\sum_{i=0}^{\infty}|s_{j,i}|^p\Big)^{1/p}. \tag{36}$$

We divide the integral $\int_0^1$ into two parts as in (31)–(32). The first part can be estimates in the same way as above. To estimates the second one can used (36). From the moment condition and the Harnack inequality we conclude also that

$$\Big\|\sum_{j,i=0}^{\infty} s_{j,i}a_{j,i} * h_{m,0}\Big\|_p \leq \sum_{j=0}^{\infty}\Big\|\sum_{i=0}^{\infty}|s_{j,i}|\,|a_{j,i} * h_{m,0}|\,\Big\|_p$$
$$\leq C\sum_{j=0}^{\infty}2^{-j(s+M+1+d/p')}\Big\|\sum_{i=0}^{\infty}|s_{j,i}|\inf_{y\in B(x_{j,i},2)} h_2(x^{-1}y)\Big\|_p$$
$$\leq C\sum_{j=0}^{\infty}2^{-j(s+M+1)}\Big(\sum_{i=0}^{\infty}|s_{j,i}|^p\Big)^{1/p}$$
$$\leq C\Big(\sum_{j=0}^{\infty}\Big(\sum_{i=0}^{\infty}|s_{j,i}|^p\Big)^{q/p}\Big)^{1/q}$$

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Thus the inequality (35) holds also for nonpositive s.

Step 4 Now we decompose any distribution from $B^s_{p,q}(G,\mathcal{X})$, $s>0$, into a sum of atoms. For this part of proof it is convenient to change the formula (8) a bit and to write it down in the following form

$$f = f * \tilde{h}_{m,0} + C\int_0^{\epsilon b} t^m f * h_t^m \frac{dt}{t}, \tag{37}$$

where b is the positive constant used in inequality (29) and $\tilde{h}_{m,0} = \sum_{\ell=0}^{m-1} \frac{(\epsilon b)^\ell}{\ell!} h^l_{\epsilon b}$.

Using the above resolutions of unity and (37) we get the following decomposition of f

$$\begin{aligned} f(x) &= f * \tilde{h}_{m,0} + C\int_0^{\epsilon b} t^m f * h_t^m \frac{dt}{t} \\ &= f * \tilde{h}_{m,0} + \sum_{j=1,i=0}^{\infty} \psi_{j,i} \int_{\epsilon b 4^{-j-1}}^{\epsilon b 2^{-j}} t^m f * h_t^m \frac{dt}{t} = \sum_{j,i=0}^{\infty} s_{j,i} a_{j,i}, \end{aligned}$$

where

$$a_{j,i}(x) = C 2^{-2jm} s_{j,i}^{-1} \psi_{j,i}(x) \int_{\epsilon b 4^{-j-1}}^{\epsilon b 4^{-j}} t^m f * h_t^m(x) \frac{dt}{t} \quad \text{for } j \geq 1, \tag{38}$$

$$a_{0,i}(x) = s_i^{-1} \psi_{0,i}(x) f * \tilde{h}_{m,0}(x), \tag{39}$$

$$s_{j,i} = 2^{j(s-\frac{n}{p}-2m)} \sum_{\ell \in I_{j,i}} \sup_{x \in Q_{j,\ell}} |h_t * \Delta^m f|(x) \quad \text{for } j \geq 1, \tag{40}$$

$$s_{0,i} = \Big(\int_{B(x_{0,i},2)} |f * \tilde{h}_{m,0}(x)|^p \, dx \Big)^{1/p}, \tag{41}$$

and

$$I_{j,i} = \{\ell \in \mathbb{N} : B(x_{j,\ell}, 2^{-j}) \cap B(x_{j,i}, 2^{-j}) \neq \varnothing\}.$$

It follows from the inequalities proved in the first step that $a_{j,i}$ are (s,p)-atoms *cf.* (27)–(29). On the other hand one can use the standard estimates for hypoelliptic operators, *cf.* [28, Corollary III.1.3], and invariance of the vector fields with respect to the left translations to prove that the functions $a_{0,i}$ are 1_L atoms. Using the same inequalities one can proved that the atomic norm (23) can be estimated from above by $C\|f|B^s_{p,q}(G,\mathcal{X}\|$ with the constant C independent of f. We recall that one can use $\|f\|_p$ instead of $\|f * h_{m,0}\|_p$ since $s>0$.

Step 5 Let $s \leq 0$. Let $k \in \mathbb{N}$ be such that $2k+s>0$. We prove that $(I-\Delta)^k$ is a

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isomorphism of $B^{s+2k}_{p,q}(G,\mathcal{X})$ onto $B^{s}_{p,q}(G,\mathcal{X})$. Let $f \in B^{s+2k}_{p,q}(G,\mathcal{X})$. Then

$$\|(I-\Delta)^k f * h_{k,0}\|_p + \Big(\int_0^1 t^{(k-s/2)q}\|(I-\Delta)^k f * h_t^k\|_p^q \frac{dt}{t}\Big)^{1/q} \\ \leq C\|f\|_p + C\sum_{l=0}^{k}\Big(\int_0^1 t^{(k+l-(s+2l)/2)q}\|f * h_t^{k+l}\|_p^q \frac{dt}{t}\Big)^{1/q}.$$

If $s<0$ then $k+l>k+\frac{s}{2}$ for any possible l so every summand in the last sum is less than or equal to $\|f|B^{s+2k}_{p,q}\|$. If $s=0$ then $k+l>k+\frac{s}{2}$ for $l=1,\dots,k$ so there same argument does not work only for $l=0$. In this case we have

$$\Big(\int_0^1 t^{kq}\|f * h_t^k\|_p^q \frac{dt}{t}\Big)^{1/q} \leq \Big(\int_0^1 t^{(k-\sigma/2)q}\|f * h_t^k\|_p^q \frac{dt}{t}\Big)^{1/q} \\ \leq C\|f|B^{\sigma}_{p,q}\| \leq C\|f|B^{2k}_{p,q}\|,$$

where $0<\sigma<2k$. In consequence

$$\|(I-\Delta)^k f|B^{s}_{p,q}(G,\mathcal{X})\| \leq C\|f|B^{s+2k}_{p,q}(G,\mathcal{X})\|. \tag{42}$$

Now we assume that $f \in B^{s}_{p,q}(G,\mathcal{X})$. Using the method due to E. Stein, one can prove that

$$(-\Delta)^k(I-\Delta)^{-k} f * h_t^k = f * h_t^k + \sum c_m (I-\Delta)^{-m} f * h_t^k,$$

with $\sum|c_m|<\infty$, *cf.* [21, p. 133]. Since $2k>\frac{s+2k}{2}$ we have

$$\Big(\int_0^1 t^{(2k-(s+2k)/2)q}\|(I-\Delta)^{-k} f * h_t^{2k}\|_p^q \frac{dt}{t}\Big)^{1/q} \\ \leq C\|f|B^{s}_{p,q}\| + C\Big(\int_0^1 t^{(k-s/2)q}\Big\|\sum c_m (I-\Delta)^{-m} f * h_t^k\Big\|_p^q \frac{dt}{t}\Big)^{1/q} \leq C\|f|B^{s}_{p,q}\|.$$

Let $f \in B^{s}_{p,q}(G,\mathcal{X})$. We choose $k\in\mathbb{N}$ such that $2k+s\geq 1$. Then there exists $g \in B^{s+2k}_{p,q}(G,\mathcal{X})$ such that $f=(I-\Delta)^k g$. Let

$$g=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} s_{j,i}a_{j,i} \in B^{s+2k}_{p,q}(G,\mathcal{X}).$$

be the atomic decomposition with of g with $(s+2k,p)_{L,-1}$-atoms, $L>2k$. Then

$$f=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} s_{j,i}b_{j,i} \text{ with } b_{j,i}=(I-\Delta)^k a_{j,i}.$$

We prove that $b_{j,i}$ are $(s,p)_{L-2k,2k-1}$-atoms after the suitable normalization. We have

$$|X^I b_{j,i}(x)| \le |X^I(I-\Delta)^m a_{j,i}(x)| \le C2^{-j(s-|I|-\frac{d}{p})},$$

and by the self-adjointness of Δ

$$\Big|\int_G b_{j,i}(y)\psi(y)\,dy\Big| \le C2^{-jd}2^{-j(s+2k-\frac{d}{p})}\big\|\psi|C^{2k}\big(\overline{B(x,2r)}\big)\big\|.$$

This finishes the proof of the theorem. ■

Corollary 1

(1) *The definition of the space $B^s_{p,q}(G,\mathcal{X})$ is independent of m.*
(2) *Let $1\le p_1\le\infty$, $1\le q, q_1\le\infty$ and $s,\sigma\in\mathbb{R}$ then*

$$B^s_{p,q}(G,\mathcal{X})\subset B^s_{p,q_1}(G,\mathcal{X}) \quad \text{if } q\le q_1, \tag{43}$$

$$B^s_{p,q}(G,\mathcal{X})\subset B^\sigma_{p,q_1}(G,\mathcal{X}) \quad \text{if } s>\sigma, \tag{44}$$

$$B^s_{p,q}(G,\mathcal{X})\subset B^\sigma_{p_1,q}(G,\mathcal{X}) \quad \text{if } s-\frac{d}{p}=\sigma-\frac{d}{p_1}, \tag{45}$$

$$\mathcal{S}(G)\subset B^s_{p,q}(G,\mathcal{X}). \tag{46}$$

Moreover $C_0^\infty(G)$ is a dense subspace of $B^s_{p,q}(G,\mathcal{X})$ if $p,q<\infty$.
(3) *The operator $(I-\Delta)^k$ defines an isomorphism of $B^s_{p,q}(G,\mathcal{X})$ onto $B^{s-2k}_{p,q}(G,\mathcal{X})$.*

Proof The proof is standard. The first point is a direct consequence of the last theorem. The first embedding of the point (2) follows from the monotonicity of the ℓ_q-spaces, the second embedding from definition of atoms, the above mention monotonicity if $q\le q_1$ or Hölder inequalities if $q>q_1$. If $s-\frac{d}{p}=\sigma-\frac{d}{p_1}$ then any (s,p)-atom is also (σ,p_1)-atom. This implies the third embedding. If $f\in\mathcal{S}(G)$ then we put

$$s_{0,i}=\sup\{x\in\overline{B(x_{0,i},1)}:|f(x)|\}$$
$$a_{0,i}=s_{0,i}^{-1}\psi_{0,i}f \quad \text{if } s_{0,i}\ne 0,$$
$$f=\sum_i\psi_{0,i}f=\sum_i s_{0,i}a_{0,i},$$

The last formula is a decomposition of f onto the sum of 1_L atoms. Moreover

$$\Big(\sum_i|s_{0,i}|^p\Big)^{1/p}\le C\|f|L_p\|\le C|f|_{m,0}$$

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for sufficiently large m, *cf.* (7).

If $a_{j,i}$ is an (s,p)-atom then

$$|X^J(I-\Delta)^k a_{j,i}| \le C \quad 2^{-j(s-2k-|J|-\frac{d}{p})}$$

and the constant C is independent of j, p, s, J and the given atom $a_{j,i}$. So the functions $b_{j,i} = (I-\Delta)^k a_{j,i}$ are $(s-2k,p)$ atoms up to the new normalization constant C. This proves the corollary. ■

Corollary 2 *If $s>0$ $1\le p\le\infty$ and $1\le q\le\infty$ then*

$$B^s_{p,q}(G) \subset B^s_{p,q}(G,\mathfrak{X}) \subset B^{s/N}_{p,q}(G).$$

Here N is the constant defined in (2).

Proof

Step 1 It is well known that the left invariant Riemannian distance $\tilde\rho$ and the Carnot-Carathéodory distance ρ satisfy the inequalities

$$\tilde\rho(x,y) \le \rho(x,y) \le C\tilde\rho^{1/N}(x,y) \tag{47}$$

if $\tilde\rho(x,y) \le C$, *cf.* [28, III.4]. We will denote by $B(x,r)$ a ball corresponding to the metric ρ and by $\tilde B(x,r)$ a ball corresponding to the metric $\tilde\rho$. The last inequalities implies

$$B(x,r) \subset \tilde B(x,r) \subset B(x, Cr^{1/N}). \tag{48}$$

Let $\{\tilde B(y_{j,k}, 2^{-j})\}$ be a locally uniformly finite sequence of coverings corresponding to the left invariant Riemannian metric ρ. Let $\{B(x_{j,i}, 2^{-j})\}$ be similar sequence for the Carnot-Carathéodory metric ρ. We put

$$K_{j,i} = \{k : \tilde B(y_{j,k}, 2^{-j}) \cap B(x_{j,i}, 2^{-j}) \ne \varnothing\},$$
$$I_{j,k} = \{i : B(x_{j,i}, 2^{-j}) \cap \tilde B(y_{j,k}, 2^{-j}) \ne \varnothing\}.$$

Using the inequalities (47) and the properties of uniformly locally finite coverings one can easily prove that

$$|K_{j,i}| \le C \quad \text{and} \quad |I_{j,k}| \le C2^{j(d-\frac{d}{N})} \le C2^{j(d-n)}. \tag{49}$$

Step 2 Let $f = \sum_{j,k=0}^{\infty} \lambda_{j,k} a_{j,k} \in B^s_{p,q}(G)$ be an atomic decomposition corresponding to the full Laplacian. We put

$$s_{j,i} = 2^{j\frac{n-d}{p}} \sup_{k\in K_{j,i}} |\lambda_{j,k}|,$$

$$b_{j,i}(x) = s_{j,i}^{-1} \psi_{j,i}(x) \sum_{k=0}^{\infty} \lambda_{j,k} a_{j,k}(x).$$

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The direct calculation shows that $f = \sum_{j,i=0}^{\infty} s_{j,i} b_{j,i}$ is an atomic decomposition of f into atoms corresponding to the Carnot-Carathéodory metric. Moreover

$$\Big(\sum_{j=0}^{\infty}\Big(\sum_{i=0}^{\infty}|s_{j,i}|^p\Big)^{q/p}\Big)^{1/q} \le C\Big(\sum_{j=0}^{\infty}\Big(\sum_{k=0}^{\infty}|\lambda_{j,k}|^p\Big)^{q/p}\Big)^{1/q}.$$

since every $|\lambda_{j,k}|$ appear in the sum $\sum_{i=0}^{\infty} \sup_{k\in K_{j,i}} |\lambda_{j,k}|^p$ at most $2^{j(d-n)}$ times.

Step 3 Now let $f = \sum_{j,i=0}^{\infty} \lambda_{j,i} a_{j,i} \in B^s_{p,q}(G, \mathcal{X})$ be an atomic decomposition corresponding to the $\mathcal{X}$ of vector fields. Let $\mathcal{Y} = \{Y_1, \dots, Y_n\}$ be a family of left invariant vector fields such that:

– $Y_i = X_i$ if $1 \le i \le k$,

– the vectors $Y_1(e), \dots, Y_n(e)$ form and orthonormal basis in the space T_eG equipped with the Riemannian scalar product. Then for any multi-index I we have

$$Y^I = \sum_{J:|J|\le N|I|} c_J X^J, \tag{50}$$

cf. (2). Moreover, the formula (2) implies that $\frac{d}{N} < n$. Thus by the definition of (s, p)-atom and (50)–(51) we get

$$\sup_{y\in X} |Y^I a_{j,i}(y)| \le 2^{-j\frac{N}{p}(n-\frac{d}{N})} 2^{-Nj(\frac{s}{N}-|I|-\frac{n}{p})}, \tag{51}$$

for $|I| \le L/N$. We put

$$F_{j,k} = \{i : \tilde{B}(y_{jN,k}, 2^{-jN}) \cap B(x_{j,i}, 2^{-j+1}) \ne \varnothing\},$$
$$E_{j,i} = \{k : \tilde{B}(y_{jN,k}, 2^{-jN}) \cap B(x_{j,i}, 2^{-j+1}) \ne \varnothing\}.$$

Since $\tilde{B}(y_{jN,k}, 2^{-jN}) \subset B(y_{jN,k}, 2^{-j})$, *cf.* (47), the cardinalities of the sets $F_{j,k}$ are uniformly bounded. We define

$$s_{jN,k} = 2^{-j\frac{N}{p}(n-\frac{d}{N})} \sup_{i\in F_{j,k}} |\lambda_{j,i}|,$$

$$b_{jN,k}(x) = s^{-1}_{jN,k} \tilde{\psi}_{jN,k}(x) \sum_{i=0}^{\infty} \lambda_{j,i} a_{j,i}(x),$$

where $\{\tilde{\psi}_{jN,k}\}$ is a resolution of unity corresponding to the covering $\{\tilde{B}(y_{jN,k}, 2^{-jN})\}$. Now the inequality (51) implies that $b_{jN,k}$ is an $(\frac{s}{N}, p)$-atom centered at $\tilde{B}(y_{jN,k}, 2^{-jN})$. Direct calculations show that $f = \sum_{j,k=0}^{\infty} s_{jN,k} b_{jN,k}$ is an atomic decomposition of f in the Besov space corresponding to the full Laplacian. We recall that the constant L is at our disposal. By the properties of coverings

$|E_{j,i}| \le C2^{jN}$. Therefore any $\lambda_{j,i}$ can appear at most $C2^{jN}$ times in the sum $\sum_{k=0}^{\infty} \sup_{i\in F_{j,k}} |\lambda_{j,i}|^p$. Thus

$$\Big(\sum_{j=0}^{\infty}\Big(\sum_{k=0}^{\infty}|s_{jN,k}|^p\Big)^{q/p}\Big)^{1/q} \le C\Big(\sum_{j=0}^{\infty}\Big(\sum_{k=0}^{\infty}2^{-jN(n-\frac{d}{N})}\sup_{i\in F_{j,k}}|\lambda_{j,i}|^p\Big)^{q/p}\Big)^{1/q}$$
$$\le C\Big(\sum_{j=0}^{\infty}\Big(\sum_{i=0}^{\infty}2^{-jN(n-\frac{d}{N}-1)}|\lambda_{j,i}|^p\Big)^{q/p}\Big)^{1/q}.$$

But the formula (2) implies

$$N\Big(n-\frac{d}{N}\Big) = \sum_{i=1}^{N-1}\dim K_i \ge N.$$

In consequence $2^{-jN(n-\frac{d}{N}-1)} \le 1$. This proves the corollary. ∎

3 Hausdorff and Distributional Dimensions

We refer to [6] for basic properties of Hausdorff measures and the Hausdorff dimension. We recall that the Hausdorff dimension of the metric space (G,ρ) is equal to the local dimension d, *cf.* [14, Theorem 2]. In this section we want to give the distributional characterization of the Hausdorff dimension of Borel subsets of (G,ρ). In the Euclidean case such characterization was given by Triebel and Winkelvoß, *cf.* [25, Section 17] and [27].

Let F be a closed subset of G of the Haar measure 0. We put

$$B^{s,F}_{\infty,\infty}(G,\mathcal{X}) = \{f \in B^s_{\infty,\infty}(G,\mathcal{X}) : f(\varphi)=0 \text{ if } \varphi \in C^\infty_o(G) \text{ and } \varphi|F=0\}.$$

Definition 3 Let F be a non-empty Borel subset of G with the Haar measure 0. The distributional dimension $\dim^{\mathcal{X}}_D F$ of F is

$$\dim^{\mathcal{X}}_D F = \sup\{\delta\in\mathbb{R} : B^{-d+\delta,E}_{\infty,\infty}(G,\mathcal{X}) \ne \{0\} \text{ for some compact } E\subset F\}.$$

Remark 3 1. The definition becomes obvious if we note that the Dirac distribution at identity δ_e is an element of $B^{-d}_{\infty,\infty}(G,\mathcal{X})$. This is the direct consequence of the definition of the Besov spaces and the upper Gauss estimates. On the other hand the lower gaussian estimates imply that δ_e is not an element of $B^s_{\infty,\infty}(G,\mathcal{X})$ if $s > -d$. By left translations invariance of the spaces the same is true for the Dirac distribution at any point of G. Thus the distributional dimension of a one point set is zero. In consequence

$$0 \le \dim^{\mathcal{X}}_D F \le d. \tag{52}$$

2. The distributional dimension $\dim^{\mathcal{X}}_D F$ is monotone with respect to F. So, if F is compact then

$$\dim^{\mathcal{X}}_D F = \sup\big\{\delta\in\mathbb{R} : B^{-d+\delta,F}_{\infty,\infty}(G,\mathcal{X}) \ne \{0\}\big\}.$$

The above characterization holds also if F is a closed set, *cf.* [27].

3. The distributional dimension $\dim_D^{\mathcal{X}} F$ of a set F with Haar measure zero describe the ability of the set F to carry non-trivial singular distributions as smooth as possible whereas the Hausdorff dimension $\dim_H^{\rho} F$ describe the massiveness of the set F.

4. Suppose that there is a sequence of covers of the set F $\{\mathcal{U}_j : j = 1, 2, \dots\}$ such that the diameters of the covers go to zero and $\lim_{j\to\infty} \sum_{U_i \in \mathcal{U}_j} \operatorname{diam}(U_i)^s = 0$. Then from the definition of the Hausdorff dimension $\dim_H^{\rho} F \le s$. On the other hand it is sufficient to find a nonzero distribution $f \in B^{-d+s,F}_{\infty,\infty}(G, \mathcal{X})$ in order to prove that $\dim_D^{\mathcal{X}} F \ge s$. Thus if $\dim_D^{\mathcal{X}} F = \dim_H^{\rho} F$, then one can use the both characterizations to calculate the Hausdorff dimension, *cf.* Example 1.

The following lemma will be needed later.

Lemma 2

(a) *Let $s \in \mathbb{R}$ then $\left(B^s_{1,1}(G, \mathcal{X})\right)' = B^{-s}_{\infty,\infty}(G, \mathcal{X})$.*

(b) *Let $F \subset G$ be a nonempty closed set of the Haar measure zero. Let $B^{s,F}_{1,1}(G, \mathcal{X})$ denote the closure of $\{\psi \in \mathcal{S}(G) : \psi|F = 0\}$ in $B^s_{1,1}(G, \mathcal{X})$. If $s \le 0$ then*

$$B^{s,F}_{\infty,\infty}(G, \mathcal{X}) \ne \{0\} \quad \textit{if and only if } B^{-s,F}_{1,1}(G, \mathcal{X}) \ne B^{-s}_{1,1}(G, \mathcal{X}).$$

Proof If $f \in B^{-s}_{\infty,\infty}(G)$ then it can be decomposed into the sum of (∞, L, M)-atoms, $f = \sum_{j,i=0}^{\infty} \lambda_{j,i} a_{j,i}$. We assume that $s > 0$. Let $b_{k,\ell}$ be a $(1, M+1, -1)$ atom. Let $I_{j,k,\ell} = \{i : B(x_{j,i}, 2^{-j+1}) \cap B(x_{k,\ell}, 2^{-k+1}) \ne \varnothing\}$. It follows from the properties of the covering that there is a constant C independent of j, k and ℓ such that $|I_{j,k,\ell}| \le C \max(1, 2^{(j-k)d})$. The properties of the atoms imply

$$\begin{aligned}
|f(b_{k,\ell})| &\le \sum_{j,i=0}^{\infty} |\lambda_{j,i}|\, |a_{j,i}(b_{k,\ell})| \\
&\le \sum_{j\le k}^{\infty} \sum_{i\in I_{j,k,\ell}} |\lambda_{j,i}|\, |a_{j,i}(b_{k,\ell})| + \sum_{j\ge k}^{\infty} \sum_{i\in I_{j,k,\ell}} |\lambda_{j,i}|\, |a_{j,i}(b_{k,\ell})| \\
&\le C \sum_{j\le k}^{\infty} 2^{-(k-j)s} |\lambda_{j,i}| + \sum_{j\ge k}^{\infty} 2^{-(j-k)(M+1-s)} |\lambda_{j,i}| \le C \sup_{j,i} |\lambda_{j,i}|.
\end{aligned}$$

In particular, if $j \ge k$ then the last inequality follows by the moment condition. The similar argument works also for $s \le 0$. Thus

$$|f(\varphi)| \le \|\varphi | B^s_{1,1}(G)\|.$$

To prove that every continuous functional is an element of $B^{-s}_{\infty,\infty}(G)$ we first assume that $s > 0$. Let us put $\tilde{h}^m_t = t^m h^m_t$. We prove that

$$\|\tilde{h}^m_t | B^s_{1,1}(G)\| \le C t^{-s/2}. \tag{53}$$

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To prove this inequality we decompose $\tilde{h}_t^m$ into atoms. Let us choose j such that $2^{-j} \le \sqrt{t} < 2^{-j+1}$. We put

$$s_{j,i} = 2^{js} \inf_{B(x_{j,i},2^{-j+1})} h_{\beta t}(x) \quad \text{and} \quad a_{j,i} = 2^{-js} s_{j,i}^{-1} \varphi_{j,i} \tilde{h}_t^m(x).$$

Then, it follows from the Harnack inequality that $\tilde{h}_t^m = \sum_{i=0}^{\infty} s_{j,i} a_{j,i}$ is the atomic decomposition of $\tilde{h}_t^m$ and $\sum_{i=0}^{\infty} s_{j,i} \le 2^{js}$. So, the inequality (53) holds.

Now let $f \in \bigl(B_{1,1}^s(G)\bigr)'$. Then the inequality (53) implies

$$|f * \tilde{h}_t^m(x)| = \Bigl| f\bigl(\tilde{h}_t^m(\cdot x)\bigr) \Bigr| \le C t^{-s/2}.$$

In the similar way

$$|f * h_{m,0}(x)| = \Bigl| f\bigl(h_{m,0}(\cdot x)\bigr) \Bigr| \le C.$$

Thus $f \in B_{\infty,\infty}^{-s}(G)$.

If $s \le 0$ and $s + m > 0$ then $f \circ (I - \Delta)^m \in \bigl(B_{1,1}^{s+2m}(G)\bigr)'$. So $f \in B_{\infty,\infty}^{-s}(G)$ by the above result and the lift property.

The point (b) follows from (a) and the Hahn-Banach theorem. ■

Theorem 2 *Let F be a Borel subset of G of the Haar measure* 0*. Then*

$$\dim_H^{\rho} F = \dim_D^{\mathcal{X}} F. \tag{54}$$

Proof

Step 1 The set F is a Borel subset of a separable complete metric space therefore

$$\dim_H^{\rho} F = \sup\{\dim_H^{\rho} E : E \text{ compact}, E \subset F\},$$

cf. [13, Section 8.13]. Thus it is sufficient to prove (54) for a non-empty compact set F.

Step 2 First we prove that $\dim_H^{\rho} F \le \dim_D^{\mathcal{X}} F$. We may assume that $\dim_H^{\rho} F > 0$, otherwise all is obvious. If $0 \le \gamma < \dim_H^{\rho} F$ then $\mathcal{H}^{\gamma} F = \infty$.

By the Frostman lemma there are a Radon measure μ on F and a positive number δ such that $\mu(F) > 0$ and

$$\mu(E) \le d(E)^{\gamma} \quad \text{for all } E \subset F \text{ with } d(E) < \delta, \tag{55}$$

cf. [13, Theorem 8.17]. Here $d(E)$ denotes the diameter of the set E.

We define $f \in \mathcal{D}'(G)$ by

$$f(\varphi) = \int_F \varphi(x)\, d\mu(x).$$

It should be clear that f is a distribution on G and that $f(\varphi) = 0$ if $\varphi|F = 0$.

Let $\varphi = \sum_{i,j=0}^{\infty} s_{j,i} a_{j,i} \in B^{d-\gamma}_{1,1}(G, \mathfrak{X})$. Then

$$|f(\varphi)| \le C \sum_{i,j=0}^{\infty} |s_{j,i}| \int_F |a_{j,i}(x)|\, d\mu(x) \le C \sum_{i,j=0}^{\infty} |s_{j,i}|.$$

Thus $f \in \left(B^{d-\gamma}_{1,1}(G, \mathfrak{X})\right)' = B^{\gamma-d}_{\infty,\infty}(G, \mathfrak{X})$, *cf.* Lemma 2.

Step 3 Now we prove that $\dim^{\mathfrak{X}}_D F \le \dim^{\rho}_H F$. We may assume that $\dim^{\mathfrak{X}}_D F < d$, see (52). If $\dim^{\rho}_H F < \gamma < d$ then $\mathcal{H}^{\gamma}(F) = 0$. We want to show that $B^{-d+\gamma,F}_{\infty,\infty}(G, \mathfrak{X}) = \{0\}$. But it follows from Lemma 2 that it is sufficient to show that $B^{d-\gamma,F}_{1,1}(G, \mathfrak{X}) = B^{d-\gamma}_{1,1}(G, \mathfrak{X})$. Using the atomic decomposition one can prove the last identity in the similar way to the Euclidean case, *cf.* [27, 3.3]. We sketch the argument for completeness. For every $\varepsilon > 0$ there exist $\delta > 0$ and a finite cover of F by open balls B_j centered at F with diameters less than δ such that

$$\sum_{j=1}^{K} (\operatorname{diam} B_j)^{\gamma} < \varepsilon.$$

Moreover, there exists $\beta > 0$ such that $\bigcup_{j=1}^{K} B_j$ covers the closure $\bar{F}_\beta$ of the set $F_\beta = \{x \in G : \operatorname{dist}(x, F) < \beta\}$. Let $\{\varphi\}_{j=1}^{K}$ be a smooth resolution of unity on $\bar{F}_\beta$ related to the cover $\{B_j\}$. The functions $(\operatorname{diam} B_j)^{-\gamma}\varphi_j$ are a family of $(d-\gamma, 1)$-atoms, after a suitable normalization. Thus if $\varphi = \sum_{j=1}^{K} \varphi_j$ then

$$\|\varphi | B^{n-\gamma}_{1,1}(G, \mathfrak{X})\| \le c \sum_{j=1}^{N} (\operatorname{diam} B_j)^{\gamma} < c\varepsilon. \tag{56}$$

Let $\psi \in C^{\infty}_o(G)$ be a smooth compactly supported function. We choose $\varphi_1 \in C^{\infty}_o(G)$ such that $\varphi_1(x) = 1$ if $x \in \operatorname{supp} \psi \cup \operatorname{supp} \varphi$. Then $\psi(\varphi_1 - \varphi) \in C^{\infty}_o(G)$ and $\psi(x)\big(\varphi_1(x) - \varphi(x)\big) = 0$ if $x \in F$. Moreover,

$$\|\psi - \psi(\varphi_1 - \varphi) | B^{n-\gamma}_{1,1}(G, \mathfrak{X})\| = \|\psi\varphi | B^{n-\gamma}_{1,1}(G, \mathfrak{X})\| \le c\|\varphi | B^{n-\gamma}_{1,1}(G, \mathfrak{X})\| \le c\varepsilon,$$

where the last but one inequality follows from the fact that ψ is a pointwise multiplier for $B^{n-\gamma}_{1,1}(G, \mathfrak{X})$ and the last one follows by (56). The space of test functions $C^{\infty}_o(G)$ is dense in $B^{n-\gamma}_{1,1}(G, \mathfrak{X})$ therefore $B^{n-\gamma}_{1,1}(G, \mathfrak{X}) \subset B^{n-\gamma,F}_{1,1}(G, \mathfrak{X})$. The opposite inclusion is obvious. ■

As a easy consequence of the above theorem we get the following inequalities.

Corollary 3 *Let $\dim^{\tilde{\rho}}_H F$ denote the Hausdorff dimension of the set F in the metric space $(G, \tilde{\rho})$. If F is a set of Haar measure zero then*

$$\max\{\dim^{\tilde{\rho}}_H F, d - N(n - \dim^{\tilde{\rho}}_H F)\} \le \dim^{\rho}_H F \le \min\{N \dim^{\tilde{\rho}}_H F, d - n + \dim^{\tilde{\rho}}_H F\}$$

Proof We may assume that F is a compact set. The inequalities $\tilde{\rho}(x,y) \le \rho(x,y) \le C\tilde{\rho}^{1/N}(x,y)$, $\tilde{\rho} \le c$, and the definition of Hausdorff dimension imply

$$\dim_H^{\tilde{\rho}} F \le \dim_H^{\rho} F \le N \dim_H^{\tilde{\rho}} F$$

On the other hand by the last theorem we have

$$\dim_H^{\rho} F = \sup\left\{\delta \in \mathbb{R} : B^{-d+\delta,F}_{\infty,\infty}(G,\mathcal{X}) \ne \{0\}\right\}$$
$$\dim_H^{\tilde{\rho}} F = \sup\left\{\delta \in \mathbb{R} : B^{-n+\delta,F}_{\infty,\infty}(G) \ne \{0\}\right\}$$

Moreover Corollary 2 and Lemma 2 imply

$$B^{-s/N}_{\infty,\infty}(G) \subset B^{-s}_{\infty,\infty}(G,\mathcal{X}) \subset B^{-s}_{\infty,\infty}(G). \tag{57}$$

Let $\delta < \dim_H^{\tilde{\rho}} F$. Then $B^{-n+\delta,F}_{\infty,\infty}(G) \ne \{0\}$. But, it follows from (57) that

$$B^{-n+\delta,F}_{\infty,\infty}(G) \subset B^{N(\delta-n),F}_{\infty,\infty}(G,\mathcal{X}).$$

So, $B^{-d+d+N(\delta-n),F}_{\infty,\infty}(G,\mathcal{X}) \ne \{0\}$ and

$$\dim_H^{\rho} F \ge d + N(\dim_H^{\tilde{\rho}} F - n).$$

On the other hand, if $\delta < \dim_H^{\rho} F$ then $B^{-d+\delta,F}_{\infty,\infty}(G,\mathcal{X}) \ne \{0\}$. But, it follows from (57) that

$$\{0\} \ne B^{-d+\delta,F}_{\infty,\infty}(G,\mathcal{X}) \subset B^{-n+n-d+\delta,F}_{\infty,\infty}(G).$$

In consequence

$$\dim_H^{\tilde{\rho}} F \ge n - d + \dim_H^{\rho} F. \qquad \blacksquare$$

At the end we give two simple examples.

Example 1 Let G be a unimodular Lie group and $\mathcal{X} = \{X_1, \dots, X_k\}$ a system of left invariant vector fields satisfying the Hörmander condition. We use the notation described in Section 1.1. Let $\gamma\colon \langle -1,1\rangle \to G$ be an integral curve of a vector field X_I, $X_I \in K_\ell \setminus K_{\ell-1}$, $1 \le \ell \le N$. We will calculate the Hausdorff dimension of $F = \gamma(\langle -1,1\rangle)$. Let $V_{j,k} = \gamma\big(\langle k2^{-j},(k+1)2^{-j}\rangle\big)$, $k = -2^j, -2^j+1, \dots, 2^j-1$, $j = 0,1,\dots$. The family $\{V_{j,k}\}_k$ is a cover of F. The Carnot-Carathéodory metric ρ is left invariant therefore diameters of the sets $V_{j,k}$ and $V_{j,i}$ are the same. If we rescale every vector field X_i by parameter τ then the vector X_I is rescale by the parameter τ^ℓ. In consequence $\operatorname{diam} V_{j,k} \sim 2^{-j/\ell}$. Now we can estimate the Hausdorff s-dimensional measure $\mathcal{H}^s$ of F,

$$\mathcal{H}^s(F) \le \lim_{j\to\infty} \sum_{k=-2^j}^{2^j-1} (\operatorname{diam} V_{j,k})^s = c \lim_{j\to\infty} 2^{j(1-s/\ell)}$$

Thus, $\mathcal{H}^s(F) = 0$ if $s > \ell$. In consequence $\dim_H^\rho(F) \le \ell$.

To prove the converse inequality we use Theorem 2. Let μ be the image of the one dimensional Lebesgue measure under the mapping $\gamma\colon \langle -1, 1\rangle \to F$. The measure μ is a non-zero Radon measure on F and the above mention method of rescaling proves that the exist a constant $C > 0$ such that for any $r > 0$ we have $\mu\big(B(x,r) \cap F\big) \le Cr^\ell$. We define a distribution f supported at F by

$$f(\psi) = \int_F \psi|_F \, d\mu, \quad \psi \in C_0^\infty(G). \tag{58}$$

If $\psi = \sum_j \sum_i s_{j,i} a_{j,i} \in B_{1,1}^{d-\ell}(G, \mathcal{X})$ is an atomic decomposition then

$$\begin{aligned} |f(\psi)| &\le \sum_j \sum_i |s_{j,i}| \int_F |a_{j,i}| \, d\mu \le \sum_j \sum_i |s_{j,i}| 2^{j\ell} \int_{B(0,2^{-j+1}) \cap F} d\mu \\ &\le C \sum_j \sum_i |s_{j,i}|, \end{aligned} \tag{59}$$

Thus $f \in B_{\infty,\infty}^{-d+\ell}(G, \mathcal{X})$ and by Theorem 2 $\dim_H^\rho F = \dim_D^{\mathcal{X}} F \ge \ell$.

Example 2 Let G be a Heisenberg group $\mathbb{H}^m = \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$. We will use the standard notations:

$$z = (x, y, t), \quad x, y \in \mathbb{R}^m, \ t \in \mathbb{R},$$
$$z_0 \cdot z_1 = (x_0 + x_1, y_0 + y_1, x_0 y_1 - x_1 y_0 + t_0 + t_1),$$

for $z, z_0, z_1 \in \mathbb{H}^m$, and refer to chapters XII and XIII in [20] for details.

We regard the Carnot-Carathéodory space related to the Heisenberg sublaplacian Δ. In this case $d = 2m + 2$, $n = 2m + 1$ and $N = 2$. The Haar measure on $\mathbb{H}^m$ coincides with the Lebesgue measure on $\mathbb{R}^{2m+1}$. Moreover, the metric spaces related to the sub-elliptic Laplacian is equivalent to the space defined by homogeneous norm

$$|z| = \Big((|x|_e^2 + |y|_e^2)^2 + t^2 \Big)^{1/4},$$

whereas the metric space corresponding to the full Laplacian is locally Euclidean. Here $|\cdot|_e$ denotes the Euclidean norm. The group is a stratified group. The corresponding dilations are given by

$$\delta_s(x, y, t) = (sx, sy, s^2 t), \quad s \in \mathbb{R}_+.$$

The dilation and the homogeneous norm are related by the formula $|\delta_s(z)| = s|z|$.

The homogeneous norm restricted to $W_{2m} = \{(x, y, 0) \in \mathbb{H}^m : x, y \in \mathbb{R}^m\}$ is the euclidean norm so for any subset F of W_{2m} its elliptic Hausdorff dimension $\dim_H^{\tilde\rho} F$ coincides with the sub-elliptic Hausdorff dimension $\dim_H^\rho F$.

Let $e_1, \dots, e_{2m+1}$ be a standard basis in $\mathbb{R}^{2m+1}$. We put

$$V_{2m+1} = \operatorname{span}\{e_{2m+1}\} = \{(0,0,t) : t \in \mathbb{R}\},$$
$$V_k = \operatorname{span}\{e_1, \dots, e_k, e_{2m+1}\}, \quad 1 \le k \le m.$$

By elementary calculations we get

$$(60) \quad \operatorname{vol}\bigl(B(0,r) \cap V_k\bigr) = \int_{-r^2}^{r^2} \int_{\{x \in \mathbb{R}^k : |x|_e^4 \le r^4 - t^2\}} dx\, dt \le Cr^{k+2}, \quad k = 1, \dots, m$$

where $B(0,r)$ is a ball in subelliptic metric space and vol is a $k+1$-dimensional volume. In the similar way $\operatorname{vol}\bigl(B(0,r) \cap V_{2m+1}\bigr) \le Cr^2$.

We define a distribution f supported at V_k given by

$$(61) \quad f(\psi) = \int_{V_k} \psi \, d\operatorname{vol}, \quad \psi \in C_0^\infty(\mathbb{H}^m).$$

If $\psi = \sum_j \sum_i s_{j,i} a_{j,i} \in B_{1,1}^{d-k-2}(G, \mathcal{X})$ is an atomic decomposition then

$$(62) \quad |f(\psi)| \le \sum_j \sum_i |s_{j,i}| \int_{V_k} |a_{j,i}| \, d\operatorname{vol} \le \sum_j \sum_i |s_{j,i}| 2^{j(k+2)} \int_{B(0,2^{-j+1}) \cap V_k} d\operatorname{vol}$$
$$\le C \sum_j \sum_i |s_{j,i}|,$$

where the last inequality follows from (60) and the fact that the Heisenberg translations coincides on V_k with the Euclidean translations. So, (62) proves that the distribution (61) defines a continuous functional on $B_{1,1}^{d-k-2}(G, \mathcal{X})$. That is $f \in B_{\infty,\infty}^{-d+k+2}(G, \mathcal{X})$. But this and the last corollary imply

$$\dim_H^\rho V_k = k + 2 = \dim_H^{\tilde\rho} V_k + 1 = d - n + \dim_H^{\tilde\rho} V_k.$$

In the similar way $\dim_H^\rho V_{2m+1} = 2 = \dim_H^{\tilde\rho} V_{2m+1} + 1$.

Remark 4 1. Obviously, the number $N \dim_H^{\tilde\rho} F$ is a relevant estimate from above of $\dim_H^{\tilde\rho} F$ only near zero, *i.e.* if $\dim_H^{\tilde\rho} F \in < 0, \frac{d-n}{N-1})$. The number $d - N(n - \dim_H^{\tilde\rho} F)$ is a relevant estimate from below only near n, *i.e.* if $\dim + H^{\tilde\rho} F \in < \frac{Nn-d}{N-1}, n)$. On the remaining part of the interval $< 0, n)$ we have the estimates $\dim_H^{\tilde\rho} F \le \dim_H^\rho F \le d - n + \dim_H^{\tilde\rho} F$. The above examples of subsets of the Heisenberg groups show that both inequalities can not be improved in general.

2. The notation of self-similarity can be generalized to stratified nilpotent Lie groups, *cf.* [23]. One can repeat Hutchinson's construction of self-similar fractals for the Carnot-Carathéodory metric appointed by the sub-Laplacian. In that case Theorem 2 implies that the self-similar dimension of the fractal is equal to its distributional dimension.

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