A CONVERSE OF AN INEQUALITY OF G. BENNETT

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0. Abstract. We prove that if $n>0$ is an integer and $r>0$ is a real number, then

$$Q_n(r)=\left(\frac{n\sum_{i=1}^{n+1}\left(\frac{n+2-i}{i}\right)^r}{(n+1)\sum_{i=1}^{n}\left(\frac{n+1-i}{i}\right)^r}\right)^{1/r}<\frac{n+1}{n}. \qquad (*)$$

The upper bound is best possible. Inequality (*) is a converse of a result of G. Bennett who proved that $Q_n(r)>1$.

1. Introduction. In three recently published papers [**1, 2, 3**] G. Bennett presented several interesting extensions as well as elegant new proofs of some classical inequalities due to Hardy, Copson, Carleman and others. Furthermore, he established remarkable new inequalities. One of the new results ([**2**]) states: if $r\in(0,1)$ and $x_i\geq 0$ $(i=1,2,\ldots)$ are real numbers, then

$$\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{\infty}x_k\right)^r<\frac{\pi r}{\sin(\pi r)}\sum_{i=1}^{\infty}\sup_{k\geq i}x_k^r, \qquad (1.1)$$

unless $x_1=x_2=\ldots=0$. The constant is best possible.

To prove (1.1) Bennett provided an intriguing inequality for sums.

If $r\in(0,1)$ and $x_i\geq 0$ $(i=1,\ldots,n)$ are real numbers, then

$$\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{k=i}^{n}x_k\right)^r\leq\lambda_n(r)\sum_{i=1}^{n}\max_{i\leq k\leq n}x_k^r, \qquad (1.2)$$

where $\lambda_n(r)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{n+1-i}{i}\right)^r$.

Equality holds in (1.2) if and only if $x_1=\ldots=x_n$.

Since

$$\lim_{n\to\infty}\lambda_n(r)=\frac{\pi r}{\sin(\pi r)},$$

inequality (1.1) follows from (1.2) by letting n tend to ∞. A crucial role in the proof of (1.2) is played by the inequality

$$1<\left(\frac{n\sum_{i=1}^{n+1}\left(\frac{n+2-i}{i}\right)^r}{(n+1)\sum_{i=1}^{n}\left(\frac{n+1-i}{i}\right)^r}\right)^{1/r}=Q_n(r)\quad(n=1,2,\ldots,r>0). \qquad (1.3)$$

Bennett, who emphasized that inequality (1.3) "seems to be genuinely difficult" [**2**, p. 397], presented an interesting—but rather complicated—proof of (1.3) by using the theory of majorization. It is worth mentioning that in (1.3) the lower bound 1 cannot be

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replaced by a greater number (which is independent of r). Indeed, since

$$\lim_{r\to 0}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{n+1-i}{i}\right)^r\right)^{1/r}=\prod_{i=1}^{n}\left(\frac{n+1-i}{i}\right)^{1/n}=1$$

(see [**4**, p. 15]) it follows that $\lim_{r\to 0} Q_n(r)=1$. It is natural to look for an upper bound for the ratio $Q_n(r)$. More precisely we ask: what is the best possible constant c_n, so that $Q_n(r)<c_n$ holds for all $r>0$? It is the aim of this paper to answer this question. In the next section we prove that the best possible constant is given by $c_n=(n+1)/n$.

2. A converse inequality. Our main result is the following converse of inequality (1.3).

THEOREM. *Let $n>0$ be an integer. Then we have, for all real $r>0$:*

$$\left(\frac{n\sum_{i=1}^{n+1}\left(\frac{n+2-i}{i}\right)^r}{(n+1)\sum_{i=1}^{n}\left(\frac{n+1-i}{i}\right)^r}\right)^{1/r}<\frac{n+1}{n}. \tag{2.1}$$

The constant on the right-hand side is best possible.

Proof. The basic tool to establish (2.1) is the following lemma.

LEMMA. *Let $r>0$, a_i and b_i $(i=1,\ldots,m)$ be real numbers satisfying*

$$a_1\geq a_2\geq\ldots\geq a_m>0,\qquad b_1\geq b_2\geq\ldots\geq b_m>0,$$
$$b_m>a_m,\qquad \prod_{i=1}^{k}a_i\leq\prod_{i=1}^{k}b_i\quad \textit{for}\quad k=1,\ldots,m.$$

Then $\sum_{i=1}^{m}a_i^r<\sum_{i=1}^{m}b_i^r$.

A proof can be found in [**5**, p. 35]; see also [**6**, p. 117].
We define

$$a_{pn+1}=a_{pn+2}=\ldots=a_{(p+1)n}=\frac{(n+1-p)n}{p+1},\quad \text{for}\quad p=0,1,\ldots,n,$$

$$b_{q(n+1)+1}=b_{q(n+1)+2}=\ldots=b_{(q+1)(n+1)}=\frac{(n-q)(n+1)}{q+1},$$

for $q=0,1,\ldots,n-1$, and

$$A_k=\prod_{i=1}^{k}a_i,\qquad B_k=\prod_{i=1}^{k}b_i,\quad \text{for}\quad k=1,2,\ldots,n(n+1).$$

Then inequality (2.1) is equivalent to the inequality

$$\sum_{i=1}^{n(n+1)}a_i^r<\sum_{i=1}^{n(n+1)}b_i^r.$$

Since

$$a_1 \geq a_2 \geq \ldots \geq a_{n(n+1)} > 0, \qquad b_1 \geq b_2 \geq \ldots \geq b_{n(n+1)} > 0,$$

and

$$b_{n(n+1)} = (n+1)/n > n/(n+1) = a_{n(n+1)},$$

it remains to prove that

$$A_k \leq B_k, \quad \text{for} \quad k = 1, \ldots, n(n+1), \tag{2.2}$$

where

$$A_k = \prod_{\nu=1}^{i} \left(\frac{(n+2-\nu)n}{\nu}\right)^n \left(\frac{(n+1-i)n}{i+1}\right)^{k-in},$$

for $in+1 \leq k \leq (i+1)n$, $0 \leq i \leq n$, and

$$B_k = \prod_{\nu=1}^{j} \left(\frac{(n+1-\nu)(n+1)}{\nu}\right)^{n+1} \left(\frac{(n-j)(n+1)}{j+1}\right)^{k-j(n+1)},$$

for $j(n+1)+1 \leq k \leq (j+1)(n+1)$ and $0 \leq j \leq n-1$.

Let $k \in \{1, \ldots, n(n+1)\}$; then there exists a uniquely determined integer $i \in \{0, \ldots, n\}$, so that $in+1 \leq k \leq (i+1)n$. To prove inequality (2.2) we consider three cases.

Case 1: $i = 0$. We have $1 \leq k \leq n$, which implies that $A_k = B_k = (n(n+1))^k$.

Case 2: $i = n$. Since $n^2+1 \leq k \leq n(n+1)$, we obtain

$$A_k = n^k (n+1)^{n^2+n-k} \leq (n+1)^k n^{n^2+n-k} = B_k.$$

Case 3: $1 \leq i \leq n-1$. Then we have $in+1 \leq k \leq i(n+1)$ or $i(n+1)+1 \leq k \leq (i+1)n$. First we assume that $in+1 \leq k \leq i(n+1)$. Then

$$A_k = \prod_{\nu=1}^{i} \left(\frac{(n+2-\nu)n}{\nu}\right)^n \left(\frac{(n+1-i)n}{i+1}\right)^{k-in}$$

and

$$B_k = \prod_{\nu=1}^{i-1} \left(\frac{(n+1-\nu)(n+1)}{\nu}\right)^{n+1} \left(\frac{(n+1-i)(n+1)}{i}\right)^{k-(i-1)(n+1)}$$

which yields

$$\begin{aligned}\frac{B_k}{A_k} &= \binom{n}{i-1} \frac{(n+1-i)^{n+1-i}}{(n+1)^n} \frac{i^{i(n+1)-1}}{(i+1)^{in}} \left(\frac{(n+1)(i+1)}{ni}\right)^k \\ &\geq \binom{n}{i-1} (n+1-i)^{n+1-i} (n+1)^{n(i-1)+1} n^{-ni-1} i^{i-2} (i+1) = \alpha_i(n),\end{aligned}$$

say. We show that $\alpha_i(n) > 1$ for $1 \leq i \leq n-1$. Since $(1+1/n)^n$ is strictly increasing we obtain

$$\left(\frac{n-i}{n+1-i}\right)^{n-i} \left(\frac{n+1}{n}\right)^n > 1,$$

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which implies that

$$\frac{\alpha_{i+1}(n)}{\alpha_i(n)}=\left(\frac{n-i}{n+1-i}\right)^{n-i}\left(\frac{n+1}{n}\right)^{n}\left(\frac{i+1}{i}\right)^{i-1}\frac{i+2}{i+1}>1,$$

and inductively we get

$$\alpha_i(n)\geq\alpha_1(n)=\frac{2(n+1)}{n}>1\quad\text{for}\quad 1\leq i\leq n-1.$$

Next we assume that $i(n+1)+1\leq k\leq(i+1)n$. Then we have

$$A_k=\prod_{v=1}^{i}\left(\frac{(n+2-v)n}{v}\right)^{n}\left(\frac{(n+1-i)n}{i+1}\right)^{k-in}$$

and

$$B_k=\prod_{v=1}^{i}\left(\frac{(n+1-v)(n+1)}{v}\right)^{n+1}\left(\frac{(n-i)(n+1)}{i+1}\right)^{k-i(n+1)}.$$

Thus, we obtain

$$\begin{aligned}\frac{B_k}{A_k}&=\binom{n}{i}\frac{(n+1-i)^{n(i+1)}}{(n-i)^{i(n+1)}}\frac{(i+1)^{i}}{(n+1)^{n}}\left(\frac{(n-i)(n+1)}{(n+1-i)n}\right)^{k}\\&\geq\binom{n}{i}(i+1)^{i}(n-i)^{n-i}(n+1)^{ni}n^{-n(i+1)}=\beta_i(n),\end{aligned}$$

say. The monotonicity of $(1+1/n)^n$ implies that

$$\frac{\beta_{i+1}(n)}{\beta_i(n)}=\left(\frac{n-i-1}{n-i}\right)^{n-i-1}\left(\frac{n+1}{n}\right)^{n}\left(\frac{i+2}{i+1}\right)^{i+1}>1.$$

Hence, we get

$$\beta_i(n)\geq\beta_1(n)=\frac{2n}{n-1}(1-n^{-2})^{n}>1$$

for $1\leq i\leq n-1$. This completes the proof of inequality (2.1). Because of

$$\lim_{r\to\infty}\left(\frac{n\sum_{i=1}^{n+1}\left(\frac{n+2-i}{i}\right)^{r}}{(n+1)\sum_{i=1}^{n}\left(\frac{n+1-i}{i}\right)^{r}}\right)^{1/r}=\frac{\max_{1\leq i\leq n+1}\frac{n+2-i}{i}}{\max_{1\leq i\leq n}\frac{n+1-i}{i}}=\frac{n+1}{n}$$

(see [**4**, p. 15]) we conclude that the upper bound $(n+1)/n$ is best possible. □

Remark. Inequality (2.1) states that the sequence $n\mapsto\frac{1}{n^{r+1}}\sum_{i=1}^{n}\left(\frac{n+1-i}{i}\right)^{r}$ ($n=1,2,\ldots$) is strictly decreasing for every $r>0$. This is a counterpart of Bennett's result that $n\mapsto\lambda_n(r)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{n+1-i}{i}\right)^{r}$ ($n=1,2,\ldots$) is strictly increasing for every $r>0$. We pointed out that $\lambda_n(r)$ converges to $\pi r/\sin(\pi r)$ as $n\to\infty$ if $r\in(0,1)$. However, if $r\geq 1$, then $\lambda_n(r)$ is divergent as $n\to\infty$. More precisely we show that $\lambda_n(1)$ is asymptotic to $\log(n)$, and, if $r>1$, then $\lambda_n(r)$ is asymptotic to $\zeta(r)n^{r-1}$. Since $\sum_{i=1}^{n}\frac{1}{i}\sim\log(n)$, we conclude from

$$\frac{\lambda_n(1)}{\log(n)}=\frac{n+1}{n}\frac{1}{\log(n)}\sum_{i=1}^{n}\frac{1}{i}-\frac{1}{\log(n)}$$

that

$$\lambda_n(1)\sim\log(n).$$

Let $r>1$; then we have

$$\frac{\lambda_n(r)}{n^{r-1}}=1+\sum_{i=2}^{n}\left[\frac{1}{i}\left(1-\frac{i-1}{n}\right)\right]^r=1+\sum_{i=2}^{n}i^{-r}\sum_{\nu=0}^{\infty}\binom{r}{\nu}(-1)^{\nu}\left(\frac{i-1}{n}\right)^{\nu}$$

$$=\sum_{i=1}^{n}i^{-r}+\sum_{i=2}^{n}i^{-r}\sum_{\nu=1}^{\infty}\binom{r}{\nu}(-1)^{\nu}\left(\frac{i-1}{n}\right)^{\nu}.$$

Setting

$$x_n(r)=\sum_{i=2}^{n}i^{-r}\sum_{\nu=1}^{\infty}\binom{r}{\nu}(-1)^{\nu}\left(\frac{i-1}{n}\right)^{\nu}$$

we get

$$|x_n(r)|\leq\sum_{\nu=1}^{\infty}\binom{r}{\nu}\frac{1}{n}\sum_{i=2}^{n}(i-1)i^{-r}\left(\frac{i-1}{n}\right)^{\nu-1}$$

$$\leq\sum_{\nu=1}^{\infty}\binom{r}{\nu}\frac{1}{n}\sum_{i=2}^{n}(i-1)i^{-r}=(2^r-1)\frac{1}{n}\sum_{i=2}^{n}(i-1)i^{-r}.$$

Since $r>1$ we conclude from Cauchy's limit theorem that $\frac{1}{n}\sum_{i=2}^{n}(i-1)i^{-r}\to 0$ as $n\to\infty$. This implies $\lambda_n(r)\sim\zeta(r)n^{r-1}$.

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