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THE MOORE–PENROSE INVERSE OF A SUM OF MATRICES

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Abstract

Suppose U and V are $m \times n$ matrices over the complex field. We obtain a representation for the Moore–Penrose inverse of the sum $U + V$. A well-known result of Cline is then derived as a special case of a corollary of this representation.

1. Introduction

If A is an $m \times n$ matrix over the complex field, then the Moore–Penrose inverse of A, which is denoted by A^+, is an $n \times m$ matrix such that

(1.1) $$AA^+A = A$$

(1.2) $$A^+AA^+ = A^+$$

(1.3) $$(AA^+)^* = AA^+$$

(1.4) $$(A^+A)^* = A^+A.$$

Suppose U, V are $m \times n$ matrices. Cline (1965) obtained a formula for $(U + V)^+$ under the assumption $UV^* = 0$. We will derive a formula for $(U + V)^+$ by constructing a new matrix from U, V, J_n, and J_m where

(1.5) $$J_k = (j_{ts})_{k \times k}, \qquad j_{ts} = \begin{cases} 1 & \text{if } s = k - t + 1 \\ 0 & \text{otherwise} \end{cases}$$

and using our result in Hung, Markham (1975). We note at this point that J_k is that permutation matrix with ones on the secondary diagonal and zeroes elsewhere, and $J_k^2 = I$.

Furthermore, we determine necessary and sufficient conditions for

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$(U+V)^+$ to have the form stated in Cline's Theorem 2, p. 106. We then give some simpler forms for certain special cases. Finally we give an example for which $U^*V \neq 0$ but $(U+V)^+$ has the form given in Cline (1965).

Some techniques employed in the proofs of our theorems are similar to techniques used in Pye, Boullion, and Atchison (1973).

2. A formula for $(U+V)^+$

THEOREM 1. *If both U and V are $m \times n$ matrices, then*

$$(U+V)^+ = J_n J^{-1}[(I - T^*K^+E)L^+(J_n - E^*K^+) + T^*K^+]\,(U^* + V^*),$$

where J_n is as defined in (1.5), *and*

$$K = U^*U + V^*V,$$
$$E = (U^*V + V^*U)J_n,$$
$$R = VJ_n - UK^+E,$$
$$S = J_mUJ_n - J_mVK^+E,$$
$$L = R^*R + S^*S,$$
$$T = K^+E(I - L^+L),$$
$$J = I + T^*T.$$

PROOF. Let

$$Q_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -J_m \\ J_m & I \end{pmatrix}, \qquad Q_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -J_n \\ J_n & I \end{pmatrix},$$

and

$$M = \begin{pmatrix} U & VJ_n \\ J_mV & J_mUJ_n \end{pmatrix}.$$

Then,

$$N = Q_1^T M Q_2 = \frac{1}{2}\begin{pmatrix} 2(U+V) & 0 \\ 0 & 2J_m(U-V)J_n \end{pmatrix}$$
$$= \operatorname{diag}((U+V), J_m(U-V)J_n).$$

Thus, $N^+ = \operatorname{diag}((U+V)^+, J_n(U-V)^+J_m)$. Therefore, $M = Q_1NQ_2^T$ implies $M^+ = Q_2N^+Q_1^T$. That is,

$$M^+ = \frac{1}{2}\begin{pmatrix} (U+V)^+ + (U-V)^+ & [(U+V)^+ - (U-V)^+]J_m \\ J_n[(U+V)^+ - (U-V)^+] & J_n[(U+V)^+ + (U-V)^+]J_m \end{pmatrix}.$$

Now, by the theorem in Hung, Markham (1975), we have

$$M^+ = \begin{pmatrix} K^+(U^* - EF) & K^+(V^*J_m - EH) \\ F & H \end{pmatrix},$$

where

$$K = U^*U + V^*V,$$

$$E = (U^*V + V^*U)J_n,$$

$$R = VJ_n - UK^+E,$$

$$S = J_mUJ_n - J_mVK^+E,$$

$$L = R^*R + S^*S,$$

$$T = K^+E(I - L^+L),$$

$$J = I + T^*T,$$

$$F = L^+R^* + J^{-1}T^*K^+(U^* - EL^+R^*),$$

$$H = L^+S^* + J^{-1}T^*K^+(V^*J_m - EL^+S^*).$$

Since the Moore–Penrose inverse is unique, we have

(2.1) $$\tfrac{1}{2}[(U+V)^+ + (U-V)^+] = K^+(U^* - EF),$$

(2.2) $$\tfrac{1}{2}[(U+V)^+ - (U-V)^+]J_m = K^+(V^*J_m - EH),$$

(2.3) $$\tfrac{1}{2}J_n[(U+V)^+ - (U-V)^+] = F,$$

(2.4) $$\tfrac{1}{2}J_n[(U+V)^+ + (U-V)^+]J_m = H.$$

Thus, from (2.1) and (2.2), we get

(2.5) $$(U+V)^+ = K^+(U^* + V^*) - K^+E(F + HJ_m)$$

and from (2.3) and (2.4), we obtain

(2.6) $$(U+V)^+ = J_nF + J_nHJ_m.$$

Now, $S^*J_m + R^* = (J_n - E^*K^+)(U^* + V^*)$ implies

$$J_nF + J_nHJ_m = J_nJ^{-1}[(I - T^*K^+E)L^+(J_n - E^*K^+) + T^*K^+](U^* + V^*).$$

Hence the proof is complete.

Cline (1965) used the condition $UV^* = 0$ and his formula for $(UU^* + VV^*)^+$ to obtain a formula for $(U+V)^+$. From our theorem, we obtain the following corollary, of which Cline's formula is a special case. Cline's condition $UV^* = 0$ becomes $U^*V = 0$ in our corollary, and this implies $(U+V)E = (U+V)(U^*V + V^*U)J_n = 0$ which is a necessary and sufficient condition for $(U+V)^+$ to have the form (3.1) of Cline (1965).

Let $N(A)$ denote the null column space of A. It is well-known that $N(A) \subset N(B)$ if and only if $B = BA^+A$.

COROLLARY 1.1. *$(U + V)^+ = K^+(U^* + V^*)$ if and only if $(U + V)E = 0$, and in this case, we have*

$$K^+(U^* + V^*) = U^+ + [G^{*+} + (I - G^{*+}V)U^+U^{*+}V^*Q^*(I - G^+G)](I - VU^+),$$

where $G = (I - U^*U^{*+})V^*$, and

$$Q = [I + (I - G^+G)VU^+U^{*+}V^*(I - G^+G)]^{-1}.$$

PROOF. Suppose $(U + V)^+ = K^+(U^* + V^*)$. Then, $(U + V)(U + V)^+(U + V) = (U + V)K^+(K + EJ_n)$. This implies $(U + V) = (U + V) + (U + V)K^+EJ_n$ since $N(K) \subset N(U)$ and $N(K) \subset N(V)$. Thus, $(U + V)K^+EJ_n = 0$. Hence, $(U + V)K^+E = 0$ since $J_n^2 = I$. Therefore, $E^*K^+(U^* + V^*) = 0$. Now by assumption $(U + V)^+ = K^+(U^* + V^*)$, we obtain $E^*(U + V)^+ = 0$. This implies $(U + V)E = 0$ since $BA^+ = 0$ if and only if $AB^* = 0$. On the other hand, if $(U + V)E = 0$, then $E^*(U + V)^+ = 0$. Thus, by (2.6) we have $E^*(J_nF + J_nHJ_m) = 0$. This implies $J_nE^*J_n(F + HJ_m) = 0$. Hence, $E(F + HJ_m) = 0$ since $J_nE^*J_n = E$. Therefore, by (2.5), we have $(U + V)^+ = K^+(U^* + V^*)$. This completes the proof.

COROLLARY 1.2. *$(U + V)^+ = J_nL^+(J_n - E^*K^+)(U^* + V^*)$ if and only if $N(L) \subset N((U + V)J_n)$.*

PROOF. Suppose $(U + V)^+ = J_nL^+(J_n - E^*K^+)(U^* + V^*)$. Then,

$$(U + V)^+ = J_nL^+(S^*J_m + R^*) = J_nL^+(LHJ_m + LF) = J_nL^+LJ_n(U + V)^+.$$

This implies $(I - J_nL^+LJ_n)(U + V)^+ = 0$. Thus, $(U + V)(I - J_nL^+LJ_n) = 0$. That is, $(U + V) = (U + V)J_nL^+LJ_n$ or $(U + V)J_n = (U + V)J_nL^+L$. Hence, $N(L) \subset N((U + V)J_n)$.

For the necessity, we reverse the proof for sufficiency. Then the result follows.

COROLLARY 1.3. *$(U + V)^+ = J_nJ^{-1}T^*K^+(U^* + V^*)$ if and only if $L = 0$.*

PROOF. Suppose $(U + V)^+ = J_nJ^{-1}T^*K^+(U^* + V^*)$. Then, by Theorem 1, we can get $J_nJ^{-1}[(I - T^*K^+E)L^+(J_n - E^*K^+) + T^*K^+](U^* + V^*) = J_nJ^{-1}T^*K^+(U^* + V^*)$.

This implies

$$(I - T^*K^+E)L^+(J_n - E^*K^+)(U^* + V^*) = 0. \tag{2.7}$$

Postmultiplying (2.7) by $(U + V)J_n$, we obtain

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$$(2.8) \qquad (I - T^*K^+E)L^+(J_n - E^*K^+)(U^* + V^*)(U + V)J_n = 0.$$

Premultiplying (2.8) by L and using the fact $LT^* = 0$, we have

$$LL^+(J_n - E^*K^+)(K + \hat{E}J_n)J_n = 0.$$

That is,

$$LL^+(J_nK - E^*K^+K + J_nEJ_n - E^*K^+EJ_n)J_n = 0.$$

Thus, $LL^+(J_nKJ_n - E^*K^+E) = 0$ since $N(K) \subset N(E^*)$ and $E^* = J_nEJ_n$. Therefore, $LL^+L = 0$ since $L = J_nKJ_n - E^*K^+E$. This implies $L = 0$. On the other hand, if $L = 0$, then the result follows immediately from Theorem 1.

If we let $P = I + TT^*$, then the formula for

$$\begin{pmatrix} U & VJ_n \\ J_mV & J_mUJ_n \end{pmatrix}$$

can be written as

$$M^+ = \begin{pmatrix} P^{-1}K^+(U^* - EL^+R^*) & P^{-1}K^+(V^*J_m - EL^+S^*) \\ L^+R^* + T^*P^{-1}(U^* - EL^+R^*) & L^+S^* + T^*P^{-1}(V^*J_m - EL^+S^*) \end{pmatrix}.$$

Thus, by the uniqueness of M^+, we have

$$(2.9) \qquad \tfrac{1}{2}((U + V)^+ + (U - V)^+) = P^{-1}K^+(U^* - EL^+R^*)$$

$$(2.10) \qquad \tfrac{1}{2}((U + V)^+ - (U - V)^+)J_m = P^{-1}K^+(V^*J_m - EL^+S^*).$$

Thus, from (2.9) and (2.10), we get

$$(U + V)^+ = P^{-1}K^+(U^* + V^*) - P^{-1}K^+EL^+(R^* + S^*J_m).$$

This implies

$$(U + V)^+ = P^{-1}K^+(I - EL^+(J_n - E^*K^+))(U^* + V^*)$$

since $R^* + S^*J_m = (J_n - E^*K^+)(U^* + V^*)$. Hence we have obtained an alternate representation of $(U + V)^+$.

THEOREM 2. *If both U and V are $m \times n$ matrices, then*

$$(2.11) \qquad (U + V)^+ = P^{-1}K^+(I - EL^+(J_n - E^*K^+))(U^* + V^*).$$

COROLLARY 2.1. $(U + V)^+ = P^{-1}K^+(U^* + V^*)$ if and only if $RE^* = 0$ and $SE^* = 0$.

PROOF. Suppose $(U + V)^+ = P^{-1}K^+(U^* + V^*)$. Then, from (2.11), we have

$$P^{-1}K^+EL^+(J_n - E^*K^+)(U^* + V^*) = 0. \tag{2.12}$$

Premultiplying (2.12) by P and then postmultiplying by $(U+V)J_n$, we get $K^+EL^+(J_n - E^*K^+)(K+EJ_n)J_n = 0$. This implies $EL^+L = 0$ since $N(K) \subset N(E^*)$, $E^* = J_nEJ_n$, and $L = J_nKJ_n - E^*K^+E$, as in the proof of Corollary 1.3. Hence $EL^+ = 0$. This implies $EL = 0$ since $L^+L = LL^+$. Now $L = R^*R + S^*S$, so we have $ER^*RE^* + ES^*SE^* = 0$. This implies $RE^* = 0$ and $SE^* = 0$. On the other hand if $RE^* = 0$ and $SE^* = 0$, then $R^*RE^* + S^*SE^* = 0$, which implies $LE^* = 0$. That is $EL = 0$. Therefore $EL^+ = 0$ and the result follows by (2.11).

Corollary 2.2. $(U+V)^+ = K^+[I - EL^+(J_n - E^*K^+)]\ (U^* + V^*)$ *if and only if* $(U+V)T = 0$.

Proof. Suppose

$$(U+V)^+ = K^+(I - EL^+(J_n - E^*K^+))(U^* + V^*). \tag{2.13}$$

Postmultiplying (2.13) by $(U+V)J_n$ and premultiplying by $(U+V)$, we get

$$(U+V)J_n = (U+V)K^+((K+EJ_n)J_n - EL^+L).$$

Thus $(U+V)K^+E(I - L^+L) = 0$ since $N(K) \subset N(U)$ and $N(K) \subset N(V)$. Hence $(U+V)T = 0$ since $T = K^+E(I - L^+L)$. On the other hand if $(U+V)T = 0$, then

$$T^*(U^* + V^*) = 0. \tag{2.14}$$

Postmultiplying (2.14) by $((U+V)(U+V)^*)^+$, we have $T^*(U+V)^+ = 0$. This implies $TT^*(U+V)^+ = 0$. Therefore, $P(U+V)^+ = (U+V)^+$ since $P = I + TT^*$. From Theorem 2, we can see that

$$P(U+V) = K^+(I - EL^+(J_n - E^*K^+))\ (U^* + V^*).$$

Hence the proof is complete. Furthermore, we have noted the following:

$$(U+V)T = 0 \text{ if and only if } N(L) \subset N((U+V)K^+E)$$

and in particular if L is nonsingular, then $T = 0$ implies $(U+V)T = 0$. Therefore Corollary 2.2 is applicable if L is nonsingular. In case $L = 0$, the conditions of Corollary 2.1 are satisfied.

3. An application

We can use Theorem 1 to determine $(AB)^+$. If we partition

$$A = (A_1, A_2), \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

conformably, then $AB = A_1B_1 + A_2B_2$, so now we can apply the theorem with $U = A_1B_1$, $V = A_2B_2$. For example, suppose

$$A = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{array}\right) = (A_1, A_2),$$

$$B = \left(\begin{array}{ccc} 1 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{array}\right) = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Then

$$AB = A_1B_1 + A_2B_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Hence

$$U^*V \neq 0_{3\times 3} \quad \text{and} \quad V^*U = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $U^*V + V^*U = 0$, which implies $E = 0$. Now clearly $(U + V)E = 0$, and Corollary 1.1 is applicable.

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