

A NEW CONSTRUCTION FOR POOLING DESIGNS

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Abstract

Pooling designs are a very helpful tool for reducing the number of tests for DNA library screening. A disjunct matrix is usually used to represent the pooling design. In this paper, we construct a new family of disjunct matrices and prove that it has a good row to column ratio and error-tolerant property.

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1. Introduction

The basic problem of group testing is to identify the set of defective items in a large population of items. Group testing algorithms can roughly be divided into two categories: combinatorial group testing (CGT) and probabilistic group testing (PGT). In CGT, it is often assumed that the number of positives among n items is equal to or at most d for some given positive integer d . In PGT, we fix some probability p of having a positive. Group testing strategies can also be either *adaptive* or *nonadaptive*. A group testing algorithm is nonadaptive if all tests must be specified without knowing the outcomes of other tests. A nonadaptive testing algorithm is useful in many areas such as DNA library screening. A pooling design based on clone library screenings is an experimental strategy to find clones with special nucleotide strings; it is also an algorithm of combinatorial group testing. A group testing algorithm is *error tolerant* if it can detect some errors in test outcomes.

A binary incidence matrix, sometimes called a disjunct matrix, with a row corresponding to an experiment and a column corresponding to a clone, is usually used to represent the pooling design. Kautz and Singleton [6] were first to propose the concept of a d -disjunct matrix. Macula [7] proposed a novel way of constructing d -disjunct matrices based on the containment relation of subsets in a finite set. As a generalization of Macula's construction, Zhao [10] constructed a family of disjunct matrices and discussed its error-tolerant property.

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However, when there are errors in the test outcomes, the design no longer works. To deal with this case, Macula [8] proposed a d^e -disjunct matrix which is a mathematical model of error-tolerance design. D'yachkov *et al.* [2] proved that a d^e -disjunct matrix can detect $e - 1$ errors and correct $\lfloor (e - 1)/2 \rfloor$ errors. D'yachkov *et al.* [3] discussed the error-tolerant property of Macula's construction. Ngo and Du [9] proposed a family of d -disjunct matrices based on matchings of the complete graph K_{2m} . Bai *et al.* [1] generalized Ngo and Du's construction, and obtained two families of d^e -disjunct matrices based on the substructures of Johnson graphs and Grassmann graphs. Huang and Weng [5] generalized Ngo and Du's constructions to pooling spaces, and proved that a d^{2e} -disjunct matrix is e -error-correcting in [4].

The rest of this paper is organized as follows. Section 2 presents basic notations and related works. Section 3 proposes a new construction of disjunct matrix based on an n -partite complete graph $G_{m,m,\dots,m}$ and discusses its row to column ratio and error-tolerant property.

2. Preliminaries

In this paper, for any positive integer v we shall use $[v]$ to denote $\{1, 2, \dots, v\}$. Also, given any set X and integer k , $\binom{X}{k}$ denotes the collection of all k -subsets of X .

For a 0–1 matrix M , a row corresponds to a test (pool) and a column corresponds to a clone. If $M_{ij} = 1$ then clone j is contained in pool i . The weight of a row or a column is the number of 1s it has. For $t + 1$ distinct columns of M , namely C_0, C_1, \dots, C_t , if $C_0 \leq C_1 + \dots + C_t$ (the '+' represents Boolean summation: $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1 + 1 = 1$), it is said that C_0 is covered by C_1, \dots, C_t .

DEFINITION 2.1 [6]. We say M is d -disjunct if the union of any d columns does not contain another column.

LEMMA 2.2 [9]. *The matrix M is d -disjunct if and only if for any set of $d + 1$ distinct columns $C_{j_0}, C_{j_1}, \dots, C_{j_d}$ with one column (say, C_{j_0}) designated, C_{j_0} has a 1 in some row where all C_{j_k} , $1 \leq k \leq d$, contain 0s.*

Let $S(\bar{d}, n)$ denote the set of all subsets of n items (or columns) with size at most d , called the set of *samples*. For $s \in S(\bar{d}, n)$, let $P(s)$ denote the union of all columns corresponding to s , that is, $P(s) = \bigcup_{i \in s} C_i$. A pooling design is e -error-detecting (correcting) if it can detect (correct) up to e errors in test outcomes. In other words, if a design is e -error-detecting then the test outcome vectors form a v -dimensional binary code with minimum Hamming distance at least $e + 1$. Similarly, if a design is e -error-correcting then the test outcome vectors form a v -dimensional binary code with minimum Hamming distance at least $2e + 1$. The following remarks are simple to see, and will be useful later on.

REMARK 2.3 [9]. Suppose that M has the property that for any $s, s' \in S(\bar{d}, n)$, $s \neq s'$, $P(s)$ and $P(s')$ viewed as vectors have Hamming distance k or greater. In other words, $|P(s) \oplus P(s')| \geq k$ where \oplus denotes the symmetric difference. Then M is $(k - 1)$ -error-detecting and $\lfloor (k - 1)/2 \rfloor$ -error-correcting.

DEFINITION 2.4 [8]. We say M is d^e -disjunct if given any $d + 1$ distinct columns with one designated, there are $e + 1$ rows with a 1 in the designated column and 0 in each of the other d columns.

Obviously, a d^e -disjunct matrix with $e = 0$ is said to be d -disjunct. For a d^e -disjunct matrix, the smaller the row to column ratio, the better the design; and the larger e is, the better the design is. So the basic problem of pooling designs is to construct a disjunct matrix such that its row to column ratio is small and e is large.

In the following, we give some related work about constructions of disjunct matrices over graphs.

Macula [7] proposed a novel way of constructing a family of d -disjunct matrices of order $\binom{n}{d} \times \binom{n}{k}$ with row weight $\binom{n-d}{k-d}$ and column weight $\binom{k}{d}$.

DEFINITION 2.5 [7]. For positive integers $1 \leq d < k < n$, let $M(d, k, n)$ be the binary matrix with row (respectively, column) indexed by $\binom{[n]}{d}$ (respectively, $\binom{[n]}{k}$) such that $M(A, B) = 1$ if and only if $A \subseteq B$ and 0 otherwise.

Ngo and Du [9] constructed a $g(m, d) \times g(m, k)$ d -disjunct matrix $M(m, k, d)$ with row weight $g(m-d, k-d)$ and column weight $\binom{k}{d}$, where $g(m, l) = \binom{2m}{2l} (2l)! / 2^l l!$. Furthermore, $M(m, m, d)$ is d^d -disjunct and can detect d errors and correct $\lfloor d/2 \rfloor$ errors. A matching of size l (that is, it has l edges) is called an l -matching and the matrix of Ngo and Du is constructed as follows.

DEFINITION 2.6 [9]. For positive integers $1 \leq d < k \leq m$, let $M(m, k, d)$ be the 0–1 matrix whose rows are indexed by the set of all d -matchings on K_{2m} , and whose columns are indexed by the set of all k -matchings on K_{2m} . All matchings are to be ordered lexicographically. Then $M(m, k, d)$ has a 1 in row i and column j if and only if the i th d -matching is contained in the j th k -matching.

Zhao [10] generalized Macula's construction and constructed a $\binom{n}{d} m^d \times \binom{n}{k} m^k$ d -disjunct matrix with row weight $\binom{n-d}{k-d} m^{k-d}$ and column weight $\binom{k}{d}$. Let G denote the n -partite complete graph $G_{m, m, \dots, m}$ and G_k denote the set of all complete subgraphs of G on k vertices.

DEFINITION 2.7 [10]. For positive integers $1 \leq d < k < n$, let $M(d, k, n; m)$ be the binary matrix with row (respectively, column) indexed by G_d (respectively, G_k) such that $M(D, K) = 1$ if and only if $D \subseteq K$ and 0 otherwise.

3. Main results

The research summarized in the previous section stimulated us to construct a new family of disjunct matrices based on the complete subgraphs of a multipartite complete graph.

Let G denote the n -partite complete graph $G_{m, m, \dots, m}$ and K_n denote a complete subgraph of G on n vertices. Recall that two graphs are disjoint if they have no vertices in common. Let H_l denote a set of l pairwise disjoint complete subgraphs of G on n vertices.

DEFINITION 3.1. For positive integers $1 \leq d < k \leq m$, let $M(d, k, m; n)$ be the binary matrix whose rows (respectively, columns) are indexed by the set of all H_d (respectively, H_k). Then $M(d, k, m; n)$ has a 1 in row i and column j if and only if the i th H_d is contained in the j th H_k .

THEOREM 3.2. Let $h(m, l) = \binom{m}{l} (l!)^{n-1}$. Then $M(d, k, m; n)$ is an $h(m, d) \times h(m, k)$ d -disjunct matrix with row weight $h(m - d, k - d)$ and column weight $\binom{k}{d}$.

PROOF. It is easy to see that $h(m, l)$ is the number of all distinct H_l of G . Thus, $M(d, k, m; n)$ is an $h(m, d) \times h(m, k)$ matrix with row weight $h(m - d, k - d)$ and column weight $\binom{k}{d}$.

To show that $M(d, k, m; n)$ is d -disjunct, we recall Lemma 2.2. Consider $d + 1$ distinct columns $C_{j_0}, C_{j_1}, \dots, C_{j_d}$ of $M(d, k, m; n)$. Since these $d + 1$ columns are indexed by $d + 1$ distinct H_k , for each $i \in [d]$ there exists a K_n^i of G such that $K_n^i \in C_{j_0} \setminus C_{j_i}$. Hence, there exists a $H'_d \subseteq C_{j_0}$ which contains all K_n^i s. If $|\{K_n^i : i \in [d]\}| < d$, we simply add more K_n in C_{j_0} to $\{K_n^i : i \in [d]\}$ to form H'_d . Furthermore, since $H'_d \not\subseteq C_{j_i}$ for all $i \in [d]$, C_{j_0} has a 1 in row H'_d where all other C_{j_i} contain 0. \square

Obviously, when $n = 1$, $M(d, k, m; n)$ is Macula’s construction. When $n \geq 2$, compared with Macula’s construction,

$$\frac{h(m, d)}{h(m, k)} \left| \frac{\binom{mn}{d}}{\binom{mn}{k}} \right| = \frac{(mn - d)(mn - d - 1) \cdots (mn - k + 1)}{(m - d)^n (m - d - 1)^n \cdots (m - k + 1)^n} < 1.$$

Compared with Ngo and Du’s construction,

$$\frac{h(m, d)}{h(m, k)} \left| \frac{g(mn/2, d)}{g(mn/2, k)} \right| = \frac{(mn - 2d)(mn - 2d - 1) \cdots (mn - 2k + 1)}{(2m - 2d)^n (2m - 2d - 2)^n \cdots (2m - 2k + 2)^n} < 1.$$

Compared with Zhao’s construction,

$$\frac{h(m, d)}{h(m, k)} \left| \frac{\binom{n}{d} m^d}{\binom{n}{k} m^k} \right| = \frac{m^{k-d}}{(m - d)^n (m - d - 1)^n \cdots (m - k + 1)^n} < 1.$$

Thus the row to column ratio of $M(d, k, m; n)$ is much smaller than that of the disjunct matrices in [7, 9, 10].

THEOREM 3.3. Let $1 \leq s \leq d < k \leq m$ and $e = \binom{k-s}{k-d} - 1$, Then $M(d, k, m; n)$ is s^e -disjunct.

PROOF. Let $C_{j_0}, C_{j_1}, \dots, C_{j_s}$ be any $s + 1$ distinct columns of $M(d, k, m; n)$. For each $i \in [s]$, there exist a $K_n^i \in C_{j_0} \setminus C_{j_i}$. Let $J = \{K_n^1, K_n^2, \dots, K_n^s\}$. Then $|J| \leq s$ and J is a subset of C_{j_0} , which is not a subset of C_{j_i} for each $i \in [s]$. If $|J| = j$, the number of d -subsets of C_{j_0} containing J is $\binom{k-j}{d-j} = \binom{k-j}{k-d}$. Since $\binom{k-j}{k-d} \geq \binom{k-s}{k-d}$ whenever $j \leq s$, the number of d -subsets of C_{j_0} that are not subsets of C_{j_i} is at least $\binom{k-s}{k-d}$. Therefore $M(d, k, m; n)$ is an s^e -disjunct matrix. \square

An s^e -disjunct matrix is called *fully s^e -disjunct* if it is not $d^{e'}$ -disjunct whenever $d > s$ or $e' > e$. D'yachkov *et al.* [3] discussed the error-correcting property of Macula's construction.

THEOREM 3.4 [3]. *Suppose that $1 \leq s \leq d < k < n$ and $e = e(s) = \binom{k-s}{k-d} - 1$. Then $M(d, k, n)$ is fully s^e -disjunct.*

For a binary matrix M of order $N \times T$, let $B(D)$ denote the Boolean sum of those columns indexed by elements of $D \subseteq [T]$, and let $d_H(B(D), B(D'))$ denote the Hamming distance between $B(D)$ and $B(D')$ where D and D' are two distinct subsets of $[T]$. Let

$$e_s = \min_{|D|=|D'|=s} d_H(B(D), B(D')).$$

The larger the parameter e_s , the better its error-correcting capacity.

D'yachkov *et al.* [2] gave lower bounds of e_s for a fully s^e -disjunct matrix.

THEOREM 3.5 [2]. *Let M be a fully s^e -disjunct matrix. Then $e_s \geq 2(e + 1)$.*

THEOREM 3.6. *Let $1 \leq s \leq d < k \leq m$. Then $M(d, k, m; n)$ is a fully s^e -disjunct matrix with*

$$e = \binom{k-s}{k-d} - 1, \quad e_s = 2 \binom{k-s}{k-d}.$$

PROOF. Note that the maximum size of E can be obtained in Theorem 3.3, which implies that $M(d, k, m; n)$ is fully s^e -disjunct.

By Theorem 3.5, $e_s \geq 2 \binom{k-s}{k-d}$, so we only need to prove $e_s \leq 2 \binom{k-s}{k-d}$.

For all $i, j \in [k + 1]$, $i \neq j$, $K_n^i \cap K_n^j = \emptyset$. Suppose that $Q = \{K_n^1, K_n^2, \dots, K_n^k\}$ and $J = \{K_n^1, K_n^2, \dots, K_n^{k+1}\} = \{K_1, K_2, \dots, K_{k+1}\}$. Let

$$D_0 = \{\widehat{K}_1, \widehat{K}_2, \dots, \widehat{K}_{s-1}, \widehat{K}_{k+1}\}, \quad D'_0 = \{\widehat{K}_1, \widehat{K}_2, \dots, \widehat{K}_{s-1}, \widehat{K}_k\},$$

where $\widehat{K}_i = J - \{K_i\}$. Then

$$\left| \left\{ R \mid R \in \binom{Q}{d}, R \not\subseteq \widehat{K}_1, \widehat{K}_2, \dots, \widehat{K}_{s-1}, \widehat{K}_k \right\} \right| = \binom{k-s}{d-s} = \binom{k-s}{k-d}.$$

By symmetry, we have that $d_H(B(D_0), B(D'_0)) = 2 \binom{k-s}{k-d}$, so $e_s \leq 2 \binom{k-s}{k-d}$. □

DEFINITION 3.7. Let $C_{j_0}, C_{j_1}, C_{j_2}, \dots, C_{j_d}$ denote any $d + 1$ distinct columns of $M(d, k, m; n)$. An H_d is said to be *private for C_{j_0} with respect to C_{j_1}, \dots, C_{j_d}* if $H_d \subseteq C_{j_0} \setminus \bigcup_{i \in [d]} C_{j_i}$. Let $p(C_{j_0}; C_{j_1}, \dots, C_{j_d})$ denote the number of private H_d of C_{j_0} with respect to C_{j_1}, \dots, C_{j_d} .

LEMMA 3.8 [9]. *Given integers $m > d \geq 1$ and any labeled simple graph G with $|V(G)| = m$ and $|E(G)| = d$, then the number of vertex covers of size d (or d -covers, for short) of G is at least $d + 1$.*

THEOREM 3.9. For any $d + 1$ distinct columns $C_{j_0}, C_{j_1}, \dots, C_{j_d}$ of $M(d, m, m; n)$, then $p(C_{j_0}; C_{j_1}, \dots, C_{j_d}) \geq d + 1$.

PROOF. Through the construction of $M(d, k, m; n)$, we know that when $k = m$, $|C_{j_0} \setminus C_{j_i}| \geq 2$ for each $i \in [d]$.

For each $i \in [d]$, choose arbitrarily $E_i \subseteq C_{j_0} \setminus C_{j_i}$ so that $|E_i| = 2$. Suppose that $C_{j_0} = \{K_n^1, K_n^2, \dots, K_n^m\}$ and each $K_n^t, t \in [m]$ is viewed as a vertex. Let G be the graph with $V(G) = C_{j_0}, E(G) = \{E_1, E_2, \dots, E_d\}$. Then G is a simple graph with m vertices and at most d edges. Also, $|E(G)| \leq d$ because the E_i are not necessarily distinct. For arbitrary i , any d -subset R of C_{j_0} such that $R \cap E_i \neq \emptyset$ is a private H_d of C_{j_0} with respect to C_{j_1}, \dots, C_{j_d} . Note that R is nothing but a d -cover of G . To show that $p(C_{j_0}; C_{j_1}, \dots, C_{j_d}) \geq d + 1$, we shall show that the number of d -covers of G is at least $d + 1$. Since adding more edges into G can only decrease the number of d -covers, we can safely assume that G has exactly d edges and apply Lemma 3.8. \square

So when $k = m$, $M(d, k, m; n)$ is d^e -disjunct ($e = d$). According to [9], we also have the following theorem.

THEOREM 3.10. Given integers $m > d \geq 1$:

- (i) $M(d, m, m; n)$ is d -error-detecting and $\lfloor d/2 \rfloor$ -error-correcting;
- (ii) if the number of positives is known to be exactly d , then $M(d, m, m; n)$ is $(2d + 1)$ -error-detecting and d -error-correcting.

PROOF. For any $s, s' \in S(\bar{d}, n), s \neq s'$, we can assume without loss of generality that there exists $C_{j_0} \in s \setminus s'$. Theorem 3.9 implies that $|P(s) \oplus P(s')| \geq d + 1$, hence Remark 2.3 shows (i). If the number of positives is exactly d , we need only consider $|s| = |s'| = d$; hence there exist $C_{j_0} \in s \setminus s'$ and $C'_{j_0} \in s' \setminus s$. This time, Theorem 3.9 implies $|P(s) \oplus P(s')| \geq 2d + 2$. Again, Remark 2.3 yields (ii). \square

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