1. In this note the Hilbert spaces under consideration are complex, and the operators referred to are bounded, linear operators. If $\mathcal{H}$ is a Hilbert space, then the algebra of all operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$.

It is known (1) that if $\mathcal{H}$ is any Hilbert space, then the class of commutators on $\mathcal{H}$, i.e., the class of all operators that can be written in the form $PQ - QP$ for some $P, Q \in \mathcal{B}(\mathcal{H})$, can be exactly described. A similar problem is that of characterizing all operators on $\mathcal{H}$ that can be written in the form $PAQ - QAP$ for some $P, A, Q \in \mathcal{B}(\mathcal{H})$. If no restrictions are placed on the operators $P, A,$ and $Q,$ it is relatively easy to see that for $\dim \mathcal{H} > 1$, every operator in $\mathcal{B}(\mathcal{H})$ can be written in this form. (A very brief and pretty proof of this fact for infinite-dimensional $\mathcal{H}$ was shown to us by Paul Federbush; it is reproduced in Remark 3.3.)

Since every commutator $PQ - QP$ is automatically a commutator of invertible operators by virtue of the identity

$$PQ - QP = (P + \lambda)(Q + \mu) - (Q + \mu)(P + \lambda),$$

valid for every pair of scalars $\lambda$ and $\mu$, it is natural to ask which operators can be written in the form $PAQ - QAP$ with invertible $P, A,$ and $Q$. This problem is somewhat more difficult, and it is the purpose of this note to furnish the solution by proving the following theorem.

**Theorem.** If $\mathcal{H}$ is a Hilbert space of dimension greater than one, and $T$ is any operator on $\mathcal{H}$, then there exist invertible operators $P, A, Q$ on $\mathcal{H}$ satisfying $T = PAQ - QAP$.

This theorem settles a question posed to us by Olga Taussky-Todd, to whom we are indebted for several interesting conversations.

The proof of the theorem splits naturally into cases depending on the dimension of $\mathcal{H}$. In the finite-dimensional case, the proof depends on the following lemmas and (4, Theorem III).

**Lemma 1.1.** If $T \neq 0$ is an operator on an $n$-dimensional complex Hilbert space $\mathcal{H}$ ($1 < n < \infty$), then there exists an orthonormal basis for $\mathcal{H}$ relative to which the matrix $(\alpha_{ij})_{i,j=1}^{n}$ of $T$ satisfies $\alpha_{11} \neq 0 \neq \alpha_{22}$.

**Proof.** Consider the numerical range (or field of values) $W(T)$ of $T$. If $W(T)$ consists of a single point $|\lambda|$, then $T$ is the scalar operator $T = \lambda I$ and...
the result is obvious. Otherwise, \( W(T) \) contains at least two points, and thus the line segment joining them. Hence, \( W(T) \) contains a number \( \alpha_{11} \neq 0 \) such that \( \alpha_{11} \neq \text{trace } T \). Let \( x_1 \) be a unit vector such that \( (Tx_1, x_1) = \alpha_{11} \), and extend \( \{x_1, \ldots, x_n\} \) for \( \mathcal{S} \). Since \( \alpha_{11} \neq \text{trace } T \), there must be some \( k \) (\( 2 \leq k \leq n \)) such that \( (Tx_k, x_k) = \beta \neq 0 \). If we now interchange \( x_2 \) and \( x_k \), then \( (Tx_2, x_2) = \beta = \alpha_{22} \neq 0 \), and the proof is complete.

**Lemma 1.2.** If \( T \) is an operator on an \( n \)-dimensional Hilbert space \( \mathcal{S} \) (\( 1 < n < \infty \)), then there exist invertible operators \( X \) and \( Y \) on \( \mathcal{S} \) such that determinant \( X = \text{determinant } Y \) and such that \( T = X - Y \).

**Proof.** If \( T = 0 \), the result is clear. We suppose that \( T \neq 0 \), and use the preceding lemma to pick a basis for \( \mathcal{S} \) relative to which the matrix \( (\alpha_{ij}) \) of \( T \) satisfies \( \alpha_{11} \neq 0 \neq \alpha_{22} \). We write

\[
\begin{pmatrix}
\alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\
\alpha_{21} & & & \\
& \ddots & & \\
\alpha_{n1} & \ldots & \ldots & \alpha_{nn}
\end{pmatrix}
= \begin{pmatrix}
\alpha_{11} + d_1 & 0 & & \\
\alpha_{21} & \alpha_{22} + d_2 & & \\
& \ddots & \ddots & \\
\alpha_{n1} & \ldots & \ldots & \alpha_{nn} + d_n
\end{pmatrix}
- \begin{pmatrix}
d_1 & -\alpha_{12} & \ldots & -\alpha_{1n} \\
& \ddots & & \\
0 & \ddots & & \\
& & \ddots & \\
& & & d_n
\end{pmatrix}
\]

where the numbers \( d_1, \ldots, d_n \) are to be determined. Let \( X \) and \( Y \) be the operators having these matrices (relative to the given basis), and note that to complete the proof, it suffices to show that the numbers \( d_i \) can be chosen so that

1. \( d_1d_2 \ldots d_n \neq 0 \) and
2. \( (\alpha_{11} + d_1)(\alpha_{22} + d_2) \ldots (\alpha_{nn} + d_n) = d_1d_2 \ldots d_n \).

This amounts to choosing each \( d_i \neq 0 \) so that (2) is satisfied. If \( n = 2 \), this is equivalent to choosing non-zero numbers \( d_1 \) and \( d_2 \) such that

\[
\alpha_{11}d_2 + \alpha_{22}d_1 + \alpha_{11}\alpha_{22} = 0,
\]

and this is a task that is easily accomplished since \( \alpha_{11} \neq 0 \neq \alpha_{22} \). If \( n > 2 \) we first choose \( d_i, 3 \leq i \leq n \), subject only to the conditions \( d_i \neq 0 \neq d_i + a_{ii} \). Next, we arrange things so that \( \beta = d_2 \ldots d_n \) is unequal to

\[
\gamma = (\alpha_{22} + d_2) \ldots (\alpha_{nn} + d_n).
\]

To this end let \( \xi = d_3 \ldots d_n \) and \( \eta = (\alpha_{33} + d_3) \ldots (\alpha_{nn} + d_n) \). Then we need

\[
d_2\xi = (\alpha_{22} + d_2)\eta, \quad \text{and since } \xi, \eta, \text{ and } \alpha_{22} \text{ are all non-zero, it is easy to see that we can choose } d_2 \text{ so as to satisfy this requirement and also to satisfy } d_2 \neq 0 \neq d_2 + \alpha_{22}. \text{ Assume this done; to complete the proof it then suffices to choose } d_1 \neq 0 \text{ so that }
\]

\[
d_1\beta = (\alpha_{11} + d_1)\gamma,
\]

and since \( \beta \neq \gamma \neq 0 \) and \( \alpha_{11} \neq 0 \), this is possible.
Corollary 1.3. The theorem is true if \( \mathcal{S} \) is finite-dimensional.

Proof. Let \( T \) be an operator on \( \mathcal{S} \). By Lemma 1.2 there exist invertible operators \( X \) and \( Y \) on \( \mathcal{S} \) with equal determinants such that \( T = X - Y \). According to (4, Theorem III), there exist operators \( P, A, \) and \( Q \) such that \( X = PAQ \) and \( Y = QAP \); the invertibility of \( X \) and \( Y \) guarantees that \( P, A, \) and \( Q \) are invertible. Thus \( T = PAQ - QAP \), as desired.

2. The separable case. We turn now to the case in which \( \mathcal{S} \) is a separable, infinite-dimensional, space. According to (1, Theorem 3), an operator \( T \) on \( \mathcal{S} \) is a commutator if it is not of the form \( \lambda + K \) for some non-zero scalar \( \lambda \) and compact operator \( K \). For such a commutator \( T \) there exist operators \( P_1 \) and \( Q_1 \) such that \( T = P_1Q_1 - Q_1P_1 \). Since, as noted before, for any scalar \( \mu \) we also have that
\[
T = (P_1 + \mu)(Q_1 + \mu) - (Q_1 + \mu)(P_1 + \mu),
\]
\( \mu_0 \) can be chosen so that the operators \( P = P_1 + \mu_0 \) and \( Q = Q_1 + \mu_0 \) are invertible. If we then define \( A = 1 \), we have that
\[
T = PAQ - QAP
\]
with invertible \( P, A, \) and \( Q \). Thus, it suffices to prove the theorem for operators \( T \) of the form \( T = \lambda + K \), where \( \lambda \neq 0 \) and \( K \) is compact.

We shall have occasion to write \( T \) as a matrix with operator entries, and in so doing, we observe the usual conventions. If \( \mathcal{S} \) is written as the direct sum
\[
\mathcal{S} = \mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_m,
\]
and if \( E_i \) denotes the projection of \( \mathcal{S} \) onto \( \mathcal{S}_i \), then we write \( T = (T_{ij})_{i,j=1}^m \), where \( T_{ij} \) denotes the linear operator
\[
T_{ij} = E_iTE_j|\mathcal{S}_j.
\]
The following lemma begins our program.

Lemma 2.1. Let \( T \in \mathfrak{L}(\mathcal{S}) \) be of the form \( T = \lambda + K \) for \( \lambda \neq 0 \) and \( K \) compact, and let \( \epsilon > 0 \). Then there exists a finite-dimensional subspace \( \mathfrak{K} \) of \( \mathcal{S} \) such that if \( \mathfrak{Q} \) and \( \mathfrak{M} \) are subspaces satisfying \( \mathfrak{Q} \subset \mathfrak{K}^+ \) and \( \mathfrak{M} \subset \mathfrak{K}^- \), and if \( E \) and \( F \) denote the projections of \( \mathcal{S} \) onto \( \mathfrak{Q} \) and \( \mathfrak{M} \), respectively, then \( ETF, FTE, \) and \( ETE - \lambda E \) all have norm less than \( \epsilon \).

Proof. It is well known that there exist finite-dimensional projections \( P \) with the property that
\[
\|K - PKP\| < \epsilon.
\]
(Indeed, if \( \{P_n\} \) is any sequence of projections converging strongly to the identity operator, then \( \|K - P_nKP_n\| \to 0 \).) Fix any one such projection \( P_n \), choose its range for the subspace \( \mathfrak{K} \), and denote by \( L \) the operator \( K - P_nKP_0 \). Then with \( \mathfrak{Q}, \mathfrak{M}, E, \) and \( F \) as in the statement of the lemma, we have that \( EK = EL \) and \( KE = LE \). Hence, \( EKF, FKE, \) and \( EKE \) all have norm less than \( \epsilon \), and the result follows.
Proposition 2.2. Let $T \in \mathfrak{L}(\mathcal{H})$ be of the form $T = \lambda + K$ for $\lambda \neq 0$ and $K$ compact. Then there exists a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of $\mathcal{H}$ into the direct sum of two infinite-dimensional subspaces such that, if the corresponding matrix for $T$ is
\[
T = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix},
\]
then both $A_1$ and $D_1$ are invertible.

Proof. To begin with, it is a simple matter to obtain via Lemma 2.1 a preliminary resolution $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with respect to which the matrix representation
\[
T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]
has the property that $D$ is invertible. Indeed, we have only to choose for $\varepsilon$ any positive number less than $|\lambda|$, and then choose $\mathcal{H}_2$ to be any infinite-dimensional subspace whose orthocomplement $\mathcal{H}_1$ is infinite-dimensional and contains the subspace $\mathfrak{H}$ of Lemma 2.1. Next, note that if $U$ is a unitary operator on $\mathcal{H}$ with $(U_{ij})_{i,j=1}^{2}$ as its matrix representation relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and if
\[
\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} U_{11}^{*} & U_{12}^{*} \\ U_{21}^{*} & U_{22}^{*} \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix},
\]
then
\[
\begin{pmatrix} U^{*}A_{1}U & U^{*}B_{1}U \\ U^{*}C_{1}U & U^{*}D_{1}U \end{pmatrix}
\]
is the matrix representation for $T$ relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_i = U^{*}(\mathfrak{H}_i)$, $i = 1, 2$. Thus the theorem will be proved if we can find a unitary operator $U$ such that in equation (II), both $A_1$ and $D_1$ are invertible. Now the operator $A$ is a compression of $T$, and therefore is also of the form $A = \lambda + K_1$, where $K_1$ is a compact operator on $\mathfrak{H}_1$. Hence, $A$ is either invertible or has a non-trivial null space. In the former case, the proof is complete; in the latter case, the set of all those vectors $x \in \mathfrak{H}_1$ satisfying $A^kx = 0$ for some positive integer $k$ form a non-trivial finite-dimensional subspace $\mathfrak{N}_1$ of $\mathfrak{H}_1$. Let $\dim(\mathfrak{N}_1) = n$, and define $\mathfrak{N}_2 = \mathfrak{H}_1 \ominus \mathfrak{N}_1$, so that $\mathfrak{H}_1 = \mathfrak{N}_1 \oplus \mathfrak{N}_2$. The subspace $\mathfrak{N}_1$ is invariant under $A$, and if we write $N$ for the nilpotent operator in $\mathfrak{L}(\mathfrak{N}_1)$ defined by $N = A|\mathfrak{N}_1$, then the matrix representation for $A$ relative to the decomposition $\mathfrak{H}_1 = \mathfrak{N}_1 \oplus \mathfrak{N}_2$ has the form
\[
A = \begin{pmatrix} N & A_{12} \\ 0 & A_{22} \end{pmatrix}.
\]
The advantage of this particular dissection of $A$ is that the diagonal entry $A_{22}$ is invertible. To see this, note that $A_{22}$ is of the form $\lambda + K_2$, where $K_2 \in \mathfrak{L}(\mathfrak{N}_2)$ is compact. Thus, it suffices to show that $A_{22}$ has trivial null space.
Suppose, accordingly, that \( A_{22}y = 0 \) with \( y \in \mathcal{H}_2 \). Then \( Ay \in \mathcal{H}_1 \), so that \( A^k(Ay) = A^{k+2}y = 0 \) for some \( k > 0 \). But then \( y \in \mathcal{H}_1 \), and therefore \( y = 0 \).

Let now \( \mathcal{M}_1 \) be an \( n \)-dimensional subspace of \( \mathcal{H}_2 \), the precise determination of which will be made later, and write \( \mathcal{M}_2 = \mathcal{H}_2 \oplus \mathcal{M}_1 \), so that
\[
\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2.
\]
The matrix representation for \( T \) corresponding to this decomposition may be written as
\[
T = \begin{pmatrix}
N & A_{12} & B_{11} & B_{12} \\
0 & A_{22} & B_{21} & B_{22} \\
C_{11} & C_{12} & D_{11} & D_{12} \\
C_{21} & C_{22} & D_{21} & D_{22}
\end{pmatrix}.
\]

We next consider unitary operators \( U(\theta) \) on \( \mathcal{H} \) \((0 < \theta < \pi/2)\) whose matrices relative to this same decomposition of \( \mathcal{H} \) have the form
\[
U(\theta) = \begin{pmatrix}
\cos \theta & 0 & \sin \theta V & 0 \\
0 & 1 & 0 & 0 \\
-\sin \theta V^* & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
where \( V \) is some arbitrary isometry mapping \( \mathcal{M}_1 \) onto \( \mathcal{H}_1 \). A brief calculation shows that in the representation of \( U(\theta)TU^*(\theta) \) as a \( 2 \times 2 \) matrix corresponding to the splitting \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \) (see (II) above), the entries \( A_1 \) and \( D_1 \) are given by
\[
A_1(\theta) = \begin{pmatrix}
\cos^2 \theta N + \sin^2 \theta VD_{11}V^* + \sin \theta \cos \theta (B_{11}V^* + VC_{11}) & \cos \theta A_{12} + \sin \theta VC_{12} \\
\sin \theta B_{21}V^* & A_{22}
\end{pmatrix}
\]
and
\[
D_1(\theta) = \begin{pmatrix}
\sin^2 \theta V^*N V + \cos^2 \theta D_{11} - \sin \theta \cos \theta (C_{11}V + V^*B_{11}) & -\sin \theta V^*B_{12} + \cos \theta D_{12} \\
-\sin \theta C_{21}V + \cos \theta D_{21} & D_{22}
\end{pmatrix}
\]
Thus our task reduces to choosing the subspace \( \mathcal{M}_1 \) and the angle \( \theta \) in such a way that these operators are invertible. To this end, note that the entries of the matrix (III) are all bounded in norm by \( \|T\| \), independently of how the subspace \( \mathcal{M}_1 \) is selected. It follows that \( \|D_1(\theta) - D\| \to 0 \) as \( \theta \to 0 \), and that this convergence is uniform with respect to \( \mathcal{M}_1 \). Since \( D \) is invertible, there exist angles \( \theta > 0 \) so small that \( D(\theta) \) is invertible no matter how \( \mathcal{M}_1 \) is chosen. We choose one such \( \theta_0 \), hold it fixed, and proceed to adjust \( \mathcal{M}_1 \) so as to make \( A_1(\theta_0) \) invertible. That such a choice is possible may be seen most clearly as follows. Let \( D_{11} = \lambda + K_3 \), with \( \lambda \) and \( K_3 \) in \( \mathcal{L}(\mathcal{M}_1) \). (The operator \( K_3 \) depends,
of course, on the choice of \( M_1 \). Also, write \( A_1(\theta_0) = A_0 + \delta(M_1) \), where
\[
A_0 = \begin{pmatrix}
\cos^2 \theta_0 N + \sin^2 \theta_0 \lambda & \cos \theta_0 A_{12} \\
0 & A_{22}
\end{pmatrix}
\]
and
\[
\delta(M_1) = \begin{pmatrix}
\sin^2 \theta_0 V K_3 V^* + \sin \theta_0 \cos \theta_0 (B_{11} V^* + V C_{11}) & \sin \theta_0 V C_{12} \\
\sin \theta_0 B_{21} V^* & 0
\end{pmatrix}
\]
so that \( A_0 \) is independent of the choice of \( M_1 \). Since \( N \) is nilpotent and \( \sin^2 \theta_0 \lambda \) is a non-zero scalar, the entry \( \cos^2 \theta_0 N + \sin^2 \theta_0 \lambda \) of \( A_0 \) is invertible; since \( A_{22} \) is also known to be invertible, it follows that \( A_0 \) is invertible. On the other hand, according to Lemma 2.1, it is possible to choose \( M_1 \) in such a way so as to make \( B_{11}, B_{21}, C_{11}, C_{12}, \) and \( K_3 \) as small in norm as desired. Since \( ||V|| = 1 \), it follows that by appropriate choice of \( M_1 \), \( ||\delta(M_1)|| \) can be made arbitrarily small. Hence \( A_1(\theta_0) \) can be made arbitrarily close to \( A_0 \), and the result follows.

Summary. We have shown that if \( T \) is an arbitrary operator of the form \( \lambda + K \) with \( \lambda \neq 0 \) and \( K \) compact, then \( T \) can be viewed, relative to some decomposition \( \mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2 \) of \( \mathcal{S} \), as a \( 2 \times 2 \) matrix whose diagonal entries are invertible.

If we now identify \( \mathcal{S}_2 \) with \( \mathcal{S}_1 \) via a unitary isomorphism, then \( \mathcal{S} \) is identified with \( \mathcal{S}_1 \oplus \mathcal{S}_1 \), and \( T \) is identified with (is unitarily equivalent to) an operator \( T_1 \in \mathcal{S}(\mathcal{S}_1 \oplus \mathcal{S}_1) \). The advantage of this identification is that \( T_1 \) can be regarded as a \( 2 \times 2 \) matrix all of whose entries act on the same space \( \mathcal{S}_1 \); of course, the diagonal entries of \( T_1 \) remain invertible. The following lemma thus concludes the proof of our theorem in the separable case.

Lemma 2.3. If \( T \) is an operator on \( \mathcal{S}(\mathcal{S} \oplus \mathcal{S}) \) whose \( 2 \times 2 \) matrix over \( \mathcal{S}(\mathcal{S}) \) is
\[
\begin{pmatrix}
T_1 & T_2 \\
T_3 & T_4
\end{pmatrix},
\]
where \( T_1 \) and \( T_4 \) are invertible operators, then there exist invertible operators \( P, A, \) and \( Q \) on \( \mathcal{S} \oplus \mathcal{S} \) such that \( T = PAQ - QAP \).

Proof. We define \( P, A, \) and \( Q \) by writing
\[
P = \begin{pmatrix}
-T_1 & 0 \\
0 & T_4
\end{pmatrix}, \quad A = \begin{pmatrix}
A_1 & 0 \\
0 & A_2
\end{pmatrix}, \quad \text{and} \quad Q = \begin{pmatrix}
Q_1 & 1 \\
0 & Q_2
\end{pmatrix},
\]
where the entries \( A_i, Q_i \), \( i = 1, 2 \), are to be determined. Note that if \( A_i \) and \( Q_i \), \( i = 1, 2 \), are all invertible, then \( P, A, \) and \( Q \) are invertible also. A brief calculation reduces the matrix equation \( PAQ - QAP = T \) to the system of equations
\[
\begin{align*}
Q_1 A_1 T_1 - T_1 A_1 Q_1 &= 0 \\
T_1 A_1 + A_2 T_4 &= -T_3 \\
T_4 Q_1 + Q_2 T_3 &= T_4 \\
T_4 A_2 Q_2 - Q_2 A_2 T_4 &= 0.
\end{align*}
\]
That this system possesses invertible solutions $A_1, A_2, Q_1, Q_2$ when $T_1$ and $T_4$ are themselves both invertible may be seen as follows. If we agree to write

$$A_2 = \alpha Q_2^{-1} \quad \text{and} \quad Q_1 = \beta T_1$$

(where $\alpha$ and $\beta$ denote positive parameters to be determined), then the first and last equations will be automatically satisfied, so that the problem reduces to solving the third equation

$$\beta T_4 T_1 + Q_2 T_1 = T_3$$

for $Q_2$ in such a way as to make $Q_2$ invertible, and then solving the second equation

$$T_1 A_1 + \alpha Q_2^{-1} T_4 = - T_2$$

for $A_1$ in such a way as to make it invertible. Clearly these requirements will be met if $\beta$ is first chosen large enough to ensure the invertibility of $T_3 - \beta T_4 T_1$ and if $\alpha$ is then chosen large enough to make $T_2 + \alpha Q_2^{-1} T_4$ invertible.

3. The non-separable case. In this section we sketch a proof of the theorem in the case that $\dim(\mathfrak{H}) = \mathfrak{N} > \mathfrak{N}_0$. Let $(K)$ denote the maximal proper norm-closed ideal in $\mathfrak{S}(\mathfrak{H})$. According to (1, Theorem 4), the non-commutators on $\mathfrak{H}$ are exactly the operators of the form $\lambda + K$, where $\lambda \neq 0$ and $K \in (K)$. Furthermore, just as above, it suffices to treat the non-commutators. Let $T = \lambda + K$ be such an operator. Then the lemma obtained from Lemma 2.1 above by replacing the phrase “finite-dimensional subspace $\mathfrak{K}$” by “subspace $\mathfrak{K}$ of dimension less than $\mathfrak{N}$” is valid for $T$ and is essentially contained in (1, Lemma 6.1) and (2, Lemma 4.1). Accordingly, in the notation of Lemma 2.1, let $\epsilon = |\lambda|/2$, let $\mathfrak{K}$ be the corresponding subspace of dimension less than $\mathfrak{N}$, and let $\mathfrak{M}$ denote the smallest invariant subspace of $T$ that contains $\mathfrak{K}$. An easy cardinality argument shows that $\mathfrak{M}$ has dimension equal to that of $\mathfrak{K}$. Since $\mathfrak{M}^\perp$ is orthogonal to $\mathfrak{K}$, the compression $Z$ of $T - \lambda$ to $\mathfrak{M}^\perp$ has norm less than $\epsilon = |\lambda|/2$, and it follows that the matrix for $T$ relative to the decomposition $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$ has the form

$$T = \begin{pmatrix} X & Y \\ 0 & Z + \lambda \end{pmatrix}.$$

Since $X \in \mathfrak{S}(\mathfrak{M})$ and $\dim \mathfrak{M} < \mathfrak{N}$, we may assume by transfinite induction that the conclusion of the theorem holds for $X$. To see that the conclusion of the theorem also holds for $Z + \lambda$, write $\mathfrak{M}^\perp = \mathfrak{M}_1 \oplus \mathfrak{M}_2$, where $\dim \mathfrak{M}_1 = \dim \mathfrak{M}_2 = \dim \mathfrak{M}^\perp$. Then the matrix for $Z + \lambda$ relative to this resolution has the form

$$Z + \lambda = \begin{pmatrix} Z_1 + \lambda & Z_2 \\ Z_3 & Z_4 + \lambda \end{pmatrix},$$

and since $\|Z\| < \epsilon = |\lambda|/2, \|Z_2\|, \|Z_4\| < |\lambda|/2$, from which it follows that $Z_1 + \lambda$ and $Z_4 + \lambda$ are invertible. Thus Lemma 2.3, which is easily seen to be

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independent of the dimension of $S$, can be applied to yield the desired conclusion for $Z + \lambda$.

The proof in the non-separable case is completed by the following lemma.

**Lemma 3.1.** Suppose that the conclusion of the theorem holds for operators $X$ and $Z$ on Hilbert spaces $S$ and $\mathcal{K}$, respectively, and let $Y$ be any operator from $\mathcal{K}$ to $S$. Then the conclusion of the theorem also holds for the operator

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$$

on the space $S \oplus \mathcal{K}$.

**Proof.** Choose invertible operators $P_i$, $A_i$, and $Q_i$ ($i = 1, 2$) such that

$$P_1A_1Q_1 - Q_1A_1P_1 = X \quad \text{and} \quad P_2A_2Q_2 - Q_2A_2P_2 = Z.$$

Let $P \in \mathcal{K}(S \oplus \mathcal{K})$ denote the operator

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix},$$

let

$$A_s = \begin{pmatrix} A_1 & 0 \\ 0 & sA_2 \end{pmatrix},$$

where $s$ is a positive real parameter to be determined, and finally, let

$$Q_s(W) = \begin{pmatrix} Q_1 & W \\ 0 & s^{-1}Q_2 \end{pmatrix},$$

where $W$ is an operator from $\mathcal{K}$ to $S$ which also is to be determined. A simple calculation shows that

$$PA_sQ_s(W) - Q_s(W)A_sP = \begin{pmatrix} X & (P_1A_1)W - W(sA_2P_2) \\ 0 & Z \end{pmatrix}$$

so that, to complete the proof, it suffices to solve the equation

(\text{V})

$$(P_1A_1)W - W(sA_2P_2) = Y$$

for $s$ and $W$. Now for fixed $s$, it is well known that this equation possesses a unique solution $W$ provided only that the spectra of $P_1A_1$ and $sA_2P_2$ are disjoint. Furthermore, since $A_sP_2$ is invertible, it is obviously possible to make these spectra disjoint by choosing $s$ sufficiently large.

**Remark 3.2.** The complete story concerning (\text{V}) is as follows: the spectrum of the linear transformation

$$W \rightarrow BW - WC$$

is precisely the set of differences $\beta - \gamma$, where $\beta$ and $\gamma$ run over the spectra of $B$ and $C$, respectively. The usual proof of this fact (see 3) assumes that $B$, $C$, and $W$ are all operators on the same Hilbert space, but the argument can easily
be modified so as to apply to the case in which $B$ and $C$ act on different Hilbert spaces and $W$ is a linear transformation from one Hilbert space to the other.

**Remark 3.3.** A very short construction due to Paul Federbush shows that every operator $T$ on an infinite-dimensional space can be written as $T = PAQ - QAP$ for $P, A, Q$ not invertible. The argument goes as follows. Write $\mathcal{H} = \mathcal{H} \oplus \mathcal{H}^*$, where $\mathcal{H}$ and $\mathcal{H}^*$ are of the same dimension, and let $P (Q)$ be an isometry with range $\mathcal{H} (\mathcal{H}^*)$. If $X$ is an arbitrary operator, then $X = PAQ - QAP$, where $A = P^*XQ^* - Q^*XP^*$. We are also indebted to Federbush for bringing (4, Theorem 3) to our attention.

**References**


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