Can. J. Math., Vol. XXVI, No. 3, 1974, pp. 551–564

ON ODD FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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1. Introduction. Let V_K denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the unit disc U, satisfy $f'(z) \neq 0$ in U, and map U onto a domain with boundary rotation at most $K\pi$ (for a definition of this concept, see [**9**]). V. Paatero [**9**] showed that $f(z) \in V_K$ if and only if

$$(1.1)\quad f(z) = \int_0^z \exp\left[- \int_0^{2\pi} \log(1 - ze^{-it}) d\mu(t) \right] dz,$$

where $\mu(t)$ is real valued and of bounded variation on $[0, 2\pi]$ with

$$\text{(i)} \int_0^{2\pi} d\mu(t) = 2, \qquad \text{(ii)} \int_0^{2\pi} |d\mu(t)| \leqslant K.$$

V_2 is precisely the class of normalized univalent convex functions and it is known that for $2 \leqq K \leqq 4$, V_K consists only of univalent functions. J. Noonan [**8**] has recently shown that if $f(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_K$, then

$$(1.2)\quad \lim_{n\to\infty} \frac{|b_n|}{n^{K/2} - 1} = \frac{\beta}{\Gamma((K+2)/2)},$$

where

$$\beta = \lim_{r\to 1} (1 - r)^{(K+2)/2} M(r, f').$$

Let W_K denote the class of odd functions in V_K. Let

$$A_{2n+1}(K) = \max_{f \in W_K} |a_{2n+1}|.$$

In this paper we will determine $A_3(K)$ and $A_5(K)$ for all $K \geqq 2$. We will prove a result for W_K analogous to (1.2) and will show that

$$A_{2n+1}(K) < \frac{e^3}{2} (K^2 + K) (\tfrac{2}{3})^{K/4 - 1/2} (2n + 1)^{K/4 - 3/2}.$$

2. Distortion theorems. We begin our study of the class W_K with a theorem relating this class to the class V_K.

Received October 31, 1972 and in revised form, June 26, 1973. This research constitutes part of a Ph.D. thesis written at the University of Maryland under the direction of Professor W. E. Kirwan.

THEOREM 2.1. *$f(z) \in W_K$ if and only if there is a $g(z) \in V_K$ such that $f'(z) = \{g'(z^2)\}^{1/2}$.*

Proof. Suppose that $g(z) \in V_K$. Then $g'(z) \neq 0$ in U and hence if

$$f(z) = \int_0^z f'(z)dz, \quad \text{where } f'(z) = \{g'(z^2)\}^{1/2},$$

then $f(z)$ is single valued and analytic in U. We have

$$1 + \frac{z^2 g''(z^2)}{g'(z^2)} = 1 + \frac{zf''(z)}{f'(z)}.$$

A short calculation shows that (with $\zeta = z^2 = r^2 e^{i\alpha}$)

$$\int_0^{2\pi} \left| \operatorname{Re}\left\{1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right\} \right| d\theta = \int_0^{2\pi} \left| \operatorname{Re}\left\{1 + \frac{\zeta g''(\zeta)}{g'(\zeta)}\right\} \right| d\alpha.$$

Therefore

$$\text{(2.1)} \quad \lim_{|z|\to 1} \int_0^{2\pi} \left| \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \right| d\theta = \lim_{|\zeta|\to 1} \int_0^{2\pi} \left| \operatorname{Re}\left\{1 + \frac{\zeta g''(\zeta)}{g'(\zeta)}\right\} \right| d\alpha$$

and consequently $f(z) \in W_K$ if $g(z) \in V_K$, since $f(z)$ is odd.

Suppose now that $f(z) \in W_K$. Let

$$g(z) = \int_0^z g'(z)dz, \quad \text{where } g'(z^2) = [f'(z)]^2.$$

Then (2.1) holds and thus $g(z) \in V_K$.

COROLLARY 2.2. *$f(z) \in W_K$ if and only if there are two odd starlike functions $s_1(z)$ and $s_2(z)$ such that*

$$f'(z) = \left[\frac{s_1(z)}{z}\right]^{(K+2)/4} \Big/ \left[\frac{s_2(z)}{z}\right]^{(K-2)/4}.$$

Proof. This follows immediately from Theorem 2.1 and a result due to D. Brannan [**2**].

We observe that Corollary 2.2 and the distortion theorem for odd univalent functions easily yield sharp bounds for $|f'(z)|$ and $|f(z)|$.

THEOREM 2.3. *Let $f(z) \in W_6$. Then $f(z)$ is close-to-convex in U.*

Proof. By a result due to Kaplan [**5**], $f(z)$ is close-to-convex in U if and only if for each r with $0 < r < 1$ and each θ_1 and θ_2 with $0 \leqq \theta_1 < \theta_2 \leqq 2\pi$,

$$\text{(2.2)} \quad \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta > -\pi.$$

The conclusion is well-known for convex functions and thus we may assume $f(z)$ is a non-convex function in W_K with $2 < K \leqq 6$. Let $h(r, \theta)$ denote the integrand in (2.2). Since Re $\{1 + zf''(z)/f'(z)\}$ is harmonic in U, the subharmonic function $|\text{Re}\,\{1 + zf''(z)/f'(z)\}|$ cannot be harmonic in $\rho < |z| < 1$. By a result of F. Riesz [**12**] on subharmonic functions,

$$\int_0^{2\pi} |h(r, \theta)| d\theta$$

is a strictly increasing function of r for near 1. Consequently, for any fixed r sufficiently near 1,

$$\int_0^{2\pi} |h(r, \theta)| d\theta < 6\pi.$$

Suppose that for some θ_1, θ_2 with $0 \leqq \theta_1 < \theta_2 < 2\pi$,

$$\int_{\theta_1}^{\theta_2} h(r, \theta) d\theta = -\alpha\pi \qquad (\alpha > 0).$$

We may suppose without loss of generality that $\theta_1 = 0$ since if not we consider $e^{i\theta_1} f(e^{-i\theta_1} z)$. There are three possibilities to consider.

If $\theta_2 = \pi$, then since $f(z)$ is odd,

$$2 \int_0^{\pi} h(r, \theta) d\theta = -\alpha\pi,$$

which is impossible since $\alpha > 0$.

Suppose $\theta_2 < \pi$. Then $[0, \theta_2]$ and $[\pi, \theta_2 + \pi]$ are disjoint subintervals of $[0, 2\pi]$ with

$$-\alpha\pi = \int_0^{\theta_2} h(r, \theta) d\theta = \int_{\pi}^{\theta_2+\pi} h(r, \theta) d\theta.$$

Since

$$\int_0^{2\pi} h(r, \theta) d\theta = 2\pi \quad \text{and} \quad \int_0^{2\pi} |h(r, \theta)| d\theta < 6\pi,$$

it follows that, denoting the union of $[0, \theta_2]$ and $[\pi, \theta_2 + \pi]$ by E,

$$-2\alpha\pi = \int_E h(r, \theta) d\theta > -2\pi$$

and hence $\alpha < 1$.

Finally, if $\theta_2 > \pi$, we have

$$\int_0^{\pi} h(r, \theta) d\theta + \int_{\pi}^{\theta_2} h(r, \theta) d\theta = -\alpha\pi$$

Thus $(0, \theta_2 - \pi]$ and $[\pi, \theta_2]$ are disjoint subintervals of $[0, 2\pi]$ with

$$-\alpha\pi - \pi = \int_0^{\theta_2-\pi} h(r, \theta) d\theta = \int_{\pi}^{\theta_2} h(r, \theta) d\theta.$$

https://doi.org/10.4153/CJM-1974-051-0 Published online by Cambridge University Press

Denoting the union of the two intervals by F, we have

$$-2\alpha\pi - 2\pi = \int_F h(r,\theta)d\theta > -2\pi$$

which is impossible since $\alpha > 0$.

Thus in any case $\alpha < 1$ and hence since r was arbitrary, $f(z)$ is close-to-convex in U.

In order to obtain an estimate on the radius of close-to-convexity of W_K, we need to use the following lemma, the proof of which is implicitly contained in a result due to Goluzin [**3**, p. 533].

LEMMA 2.4. *Let $s(z)$ be an odd starlike function. Then* $|\arg s(z)/z| \leqq \arcsin |z^2|$ *and this result is sharp.*

THEOREM 2.5. *Let $f(z) \in W_K$. Then $f(z)$ is close-to-convex (and hence univalent for $|z| < r_0$, where $r_0 = 1$, if $K \leqq 6$ and $r_0 = [\sin \pi/(K-2)]^{1/2}$ if $K > 6$.*

Proof. By Corollary 2.2, there are two odd starlike functions $s_1(z)$ and $s_2(z)$ such that

$$f'(z) = \left[\frac{s_1(z)}{z}\right]^{(K+2)/4} \Big/ \left[\frac{s_2(z)}{z}\right]^{(K-2)/4}.$$

Then

$$\left|\arg \frac{zf'(z)}{s_1(z)}\right| = \left|\arg\left[\frac{s_1(z)}{z} \Big/ \frac{s_2(z)}{z}\right]^{(K-2)/4}\right|$$

$$\leqslant \frac{K-2}{4} 2 \arcsin r^2 \qquad (z = re^{i\theta}).$$

Now $\operatorname{Re} zf'(z)/s_1(z) > 0$ for $|z| < r$ if and only if $|\arg zf'(z)/s_1(z)| < \pi/2$ for $|z| < r$. Thus $f(z)$ is close-to-convex (relative to the starlike function $s_1(z)$) if

$$\frac{K-2}{2} \arcsin r^2 < \frac{\pi}{2},$$

which gives the result.

3. Sharp coefficient bounds. In this section we will find the values of $A_3(K)$ and $A_5(K)$ for all values of K for all $K \geqq 2$. In addition we will show that $A_{2n+1}(6) = 1$, for all $n \geqq 0$. We will need the following result due to 0. Lehto [**6**].

LEMMA 3.1. *Let* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_K$ *and suppose*

$$g'(z) = \exp\left[-\int_0^{2\pi} \log(1 - ze^{-it})d\mu(t)\right].$$

https://doi.org/10.4153/CJM-1974-051-0 Published online by Cambridge University Press

Then for $n \geqq 2$,

$$b_n = \frac{1}{n(n-1)} \sum_{j=1}^{n-1} j b_j \int_0^{2\pi} e^{-(n-j)i\theta} d\mu(\theta).$$

Lehto conjectured that the coefficients of the function

$$g_0(z) = \frac{1}{K}\left[\left(\frac{1+z}{1-z}\right)^{K/2} - 1\right]$$

solve the problem of determining max $|b_n|$ for the class V_K. Using a method similar to Lehto's, we will see that the function $f_0(z)$ defined by

$$(3.1) \quad f_0(z) = \int_0^z \{g_0'(z^2)\}^{1/2} dz = \int_0^z \frac{(1+z^2)^{(K-2)/4}}{(1-z^2)^{(K+2)/4}} dz$$
$$= z + \frac{K}{6} z^3 + \frac{K^2+4}{40} z^5 + \ldots$$

gives the value of $A_3(K)$ for all $K \geqq 2$, but that $(K^2+4)/40 = A_5(K)$ only for $K = 2$ or $K \geqq 4$. We conjecture that the coefficients of $f_0(z)$ yield $A_{2n+1}(K)$ for all $K \geqq 4$.

THEOREM 3.2. *Let* $f(z) = z + \sum_{n=1}^{\infty} a_{2n+1} z^{2n+1} \in W_K$. *Then*

(3.2) $|a_3| \leqq K/6 \quad 2 \leqq K,$

(3.3) $|a_5| \leqq (K^2+4)/40 \quad 4 \leqq K,$

(3.4) $|a_5| \leqq (14K+4)/20(10-K) \quad 2 \leqq K < 4.$

All of these results are sharp for the indicated ranges of K.

Proof. By Lemma 3.1, we have that

$$|a_3| = \left|\frac{1}{6}\int_0^{2\pi} e^{-2i\theta} d\mu(\theta)\right|,$$

where

$$f'(z) = \exp\left[-\int_0^{2\pi} \log(1 - ze^{-it}) d\mu(t)\right].$$

Since

$$\int_0^{2\pi} |d\mu(t)| \leqslant K,$$

we have $|a_3| \leqq K/6$.

By Lemma 3.1, we have that

$$20a_5 = \int_0^{2\pi} e^{-4i\theta} d\mu(\theta) + \frac{1}{2}\left[\int_0^{2\pi} e^{-2i\theta} d\mu(\theta)\right]^2.$$

We may suppose without loss of generality that a_5 is real and non-negative, since if not we may consider $e^{-i\theta}f(e^{i\theta}z)$, where θ is chosen so that $e^{4i\theta}a_5 \geqq 0$. Thus we have

$$\begin{aligned} 20a_5 &= \int_0^{2\pi} \cos 4\theta d\mu(\theta) + \frac{1}{2}\left[\left(\int_0^{2\pi} \cos 2\theta d\mu(\theta)\right)^2 - \left(\int_0^{2\pi} \sin 2\theta d\mu(\theta)\right)^2\right] \\ &= \int_0^{2\pi} 2\cos^2 2\theta d\mu(\theta) - 2 + \frac{1}{2}\left[\left(\int_0^{2\pi} \cos 2\theta d\mu(\theta)\right)^2 - \left(\int_0^{2\pi} \sin 2\theta d\mu(\theta)\right)^2\right] \\ &\leqslant 2\int_0^{2\pi} \cos^2 2\theta d\mu(\theta) + \frac{1}{2}\left[\int_0^{2\pi} \cos 2\theta d\mu(\theta)\right]^2 - 2. \end{aligned}$$

Let us first suppose that $\mu(\theta)$ is a step function with at most N jumps. If $\mu(\theta)$ has jumps d_j at θ_j $(0 \leqq \theta_j \leqq 2\pi)$, then

(3.5) $$\sum_{j=1}^{N} d_j = 2, \qquad \sum_{j=1}^{N} |d_j| \leqslant K,$$

and

(3.6) $$20a_5 \leqslant 2\sum_{j=1}^{N} \cos^2 2\theta_j d_j + \frac{1}{2}\left[\sum_{j=1}^{N} \cos 2\theta_j d_j\right]^2 - 2.$$

We wish to find a maximum for the right hand side of (3.6), subject to the constraints (3.5). The existence of a maximum is obvious, since we are considering a continuous function of the N variables $\cos 2\theta_j$ defined on a compact subset of E_N.

First we suppose that the maximum value of (3.6) is attained at a point where not all of the $|\cos 2\theta_j| = 1$. By relabeling if necessary, we may assume that $|\cos 2\theta_j| \neq 1$ for $1 \leqq j \leqq r (r \leqq N)$. Then a differentiation of the right hand side of (3.6) with respect to $\cos 2\theta_h$ yields

$$4\cos 2\theta_h d_h + \left[\sum_{j=1}^{N} \cos 2\theta_j d_j\right] d_h = 0 \qquad (1 \leqslant h \leqslant r).$$

Thus $\cos 2\theta_h$ is identically constant, say $\cos 2\theta_h \equiv \cos 2\alpha$ for $1 \leqq h \leqq r$ and

(3.7) $$-4\cos 2\alpha = \sum_{j=1}^{N} \cos 2\theta_j d_j.$$

Substituting in (3.6) we have

(3.8) $$20a_5 \leqslant 2\cos^2 2\alpha\left[\sum_{j=1}^{r} d_j\right] - 2 + 2\left[\sum_{j=r+1}^{N} d_j\right] + 8\cos^2 2\alpha.$$

(In (3.8) we adopt the convention that if $r = N$, $\sum_{j=r+1}^{N} d_j = 0$.) From (3.5) we see that

$$\sum_{j=1}^{r} d_j = 2 - \sum_{j=r+1}^{N} d_j$$

and

$$\sum_{j=r+1}^{N} d_j \leqslant 1 + \frac{K}{2}.$$

It follows that

$$20a_5 \leqslant 2\cos^2 2\alpha\left(2 - \sum_{j=r+1}^{N} d_j\right) + 2\left(\sum_{j=r+1}^{N} d_j\right) + 8\cos^2 2\alpha - 2$$
$$\leqslant K + (10 - K)\cos^2 2\alpha.$$

If $K \geqq 4$, we use the inequality $\cos^2 2\alpha \leqq 1$ to obtain

$$20a_5 \leqq 10 \leqq (K^2 + 4)/2.$$

Let us now suppose $K < 4$. From (3.7) we have

$$-\cos 2\alpha\left(4 + \sum_{j=1}^{r} d_j\right) = \sum_{j=r+1}^{N} \cos 2\theta_j d_j$$

and hence

$$|\cos 2\alpha| = \left|\frac{\sum_{r+1}^{N} \cos 2\theta_j d_j}{4 + \sum_{j=1}^{r} d_j}\right| \leqslant \frac{2 + K}{10 - K}.$$

Thus if $K < 4$,

$$20a_5 \leqslant K + (10 - K)\left(\frac{2 + K}{10 - K}\right)^2$$
$$= \frac{14K + 4}{10 - K}.$$

We observe that $(K + 4)/2 < (14K + 4)/(10 - K)$ if and only if

$$K^3 - 10K^2 + 32K - 32 > 0,$$

which is true for $2 < K < 4$.

It remains to consider the case that each $|\cos 2\theta_j| = 1$ at the maximum. In this case we have from (3.6) that

$$20a_5 \leqslant 2\sum_{j=1}^{N} d_j + \frac{1}{2}\left[\sum_{j=1}^{N} \cos 2\theta_j d_j\right]^2 - 2$$
$$\leqslant 4 + \frac{1}{2}\left[\sum_{j=1}^{N} |d_j|\right]^2 - 2$$
$$= \frac{K^2 + 4}{2}.$$

Since step functions are dense in the family of functions of bounded variation with the constraints (3.5), our results are valid for each function in W_K.

The function

$$f_0(z) = z + \frac{K}{6} z^3 + \frac{K^2+4}{40} z^5 + \ldots$$

of (3.1) shows that (3.2) and (3.3) are sharp. To show that (3.4) is sharp we will construct a function $\mu(t)$ for which equality holds in (3.6) for $2 \leqq K < 4$.

Let $\alpha = \frac{1}{2} \arccos (K+2)/(10-K)$. We define $\mu(\theta)$ on $[0, 2\pi]$ as a step function with jumps $d_j (1 \leqq j \leqq 4)$ of $\frac{1}{4}(1 - K/2)$ at the values $\theta_1 = \alpha$, $\theta_2 = \pi - \alpha$, $\theta_3 = \pi + \alpha$, and $\theta_4 = 2\pi - \alpha$ and jumps $d_j (5 \leqq j \leqq 6)$ of $\frac{1}{2}(1 + K/2)$ at $\theta_5 = \pi/2$ and $\theta_6 = 3\pi/2$. A short calculation using (1.1), Corollary 2.2 and Lemma 3.1 shows that

$$f'(z) = [(1 - z^2 e^{-2i\alpha})(1 - z^2 e^{2i\alpha})]^{(K/2-1)/4}/(1+z^2)^{(1+K/2)/2}$$

and that $a_5 = (14K + 4)/20(10 - K)$.

As a conclusion to this section, we obtain the sharp bounds for the coefficients of a function in W_6.

THEOREM 3.3. *Let* $f(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1} \in W_6$. *Then* $|a_{2n+1}| \leqq 1$, *with equality for* $f(z) = z/(1 - z^2)$.

Proof. By Theorem 2.3, $f(z)$ is close-to-convex in U. The result then follows from a result of C. Pommerenke [**10**].

4. Asymptotic coefficient estimates. We first consider the problem of estimating $||a_{2n+3}| - |a_{2n+1}||$ for a function $f(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1} \in W_K$. K. Lucas [**7**] has shown that if $f(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}$ is univalent,

$$||a_{2n+3}| - |a_{2n+1}|| = O(n^{1-\sqrt{2}}).$$

M. S. Robertson [**13**] has shown that if $g(z) = \sum_{n=1}^{\infty} b_n z^n \in V_K \cap S$, then

$$||b_{n+1}| - |b_n|| \leqslant 2\left(\frac{e}{3}\right)^3 (K^2 + K).$$

We will obtain estimates for $W_K \cap S$ using Robertson's technique.

LEMMA 4.1. *Let* $f(z)$ *be an odd function in* S. *Then if* $|z_1| = |z_2| = r$,

$$\min(|f(z_1)|, |f(z_2)|) \leqslant \frac{2^{1/2} r}{[|z_1^2 - z_2^2|(1 - r^4)]^{1/2}} \tag{4.1}$$

$$\min(|f'(z_1)|, |f'(z_2)|) \leqslant \frac{2^{1/2}(1+r^2)^{1/2}}{|z_1^2 - z_2^2|^{1/2}(1 - r^2)^{3/2}}. \tag{4.2}$$

Proof. Goluzin [**4**], has shown that if $g(z) \in S$,

$$\min(|g(t_1)|, |g(t_2)|) \leqslant \frac{2|t_2|}{|t_1 - t_2|(1 - |t_1|^2)} \qquad (|t_1| = |t_2| < 1). \tag{4.3}$$

If we apply (4.3) to the function $g(z)$ defined by $f(z) = \{g(z^2)\}^{1/2}$, then (4.1) follows. To obtain (4.2) we note that since $f(z)$ is an odd univalent function,

$$\left|\frac{zf'(z)}{f(z)}\right| \leqslant \frac{1+r^2}{1-r^2} \qquad (z = re^{i\theta}).$$

Observing that $|f'(z)| = |f'(z)/f(z)| \cdot |f(z)|$ for $z \neq 0$, we see that

$$\min(|f'(z_1)|, |f'(z_2)|) \leqslant \frac{2^{1/2}r}{[|z_1^2 - z_2^2|(1-r^4)]^{1/2}} \frac{1}{r} \frac{1+r^2}{1-r^2}$$
$$= \frac{2^{1/2}(1+r^2)^{1/2}}{|z_1^2 - z_2^2|^{1/2}(1-r^2)^{3/2}}$$
$$(0 < |z_1| = |z_2| = r < 1).$$

THEOREM 4.2. *Let* $f(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1} \in W_K$ *and suppose* $f(z)$ *is univalent. Then we have*

$$|\,|a_{2n+3}| - |a_{2n+1}|\,| \leqslant \frac{8(K^2+K)e^3 5^{5/2}}{243\sqrt{6}} (2n+1)^{-1/2} \qquad (n \geqslant 1).$$

Proof. Let z_1 be a point on $|z| = r$ where $|f'(z_1)| = M(r, f')$. Then by Lemma 4.1,

$$|z^2 - z_1^2||f'(z)| \leqslant \frac{2^{1/2}(1+r^2)^{1/2}|z^2 - z_1^2|^{1/2}}{(1-r^2)^{3/2}} \qquad (|z| = r).$$

Now

$$(z^2 - z_1^2)f'''(z) = -6a_3 z_1^2 - \sum_{n=1}^{\infty} [(2n+3)(2n+2)a_{2n+3}z_1^2$$
$$- 2n(2n-1)a_{2n+1}](2n+1)z^{2n}.$$

Therefore

$$(2n+1)|(2n+3)(2n+2)a_{2n+3}z_1^2 - 2n(2n-1)a_{2n+1}|$$
$$\leqslant \frac{1}{2\pi r^{2n}} \int_0^{2\pi} |z^2 - z_1^2||f'(z)| \left|\frac{f'''(z)}{f'(z)}\right| d\theta$$
$$\leqslant \frac{[2|z^2 - z_1^2|(1+r^2)]^{1/2}}{(1-r^2)^{3/2}} \cdot \frac{1}{2\pi r^{2n}} \int_0^{2\pi} \left|\frac{f'''(z)}{f'(z)}\right| d\theta$$
$$\leqslant \frac{2\sqrt{2}}{r^{2n}(1-r^2)^{3/2}} \frac{1}{2\pi} \int_0^{2\pi} \left|\frac{f'''(z)}{f'(z)}\right| d\theta.$$

Robertson [13] has shown that if $f(z) \in V_K$, and hence certainly if $f(z) \in W_K$,

$$(4.4) \quad \frac{1}{2\pi} \int_0^{2\pi} \left|\frac{f'''(z)}{f'(z)}\right| d\theta \leqslant \frac{K^2+K}{1-r^2}.$$

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It follows that

$$(2n+1)|(2n+3)(2n+2)a_{2n+3}z_1^2 - 2n(2n-1)a_{2n+1}| \leqslant \frac{2\sqrt{2}(K^2+K)}{r^{2n}(1-r^2)^{5/2}}.$$

Choose $|z_1|^2 = r^2 = 2n(2n-1)/(2n+2)(2n+3)$. Then (4.5) yields

$$(2n+1)(2n)(2n-1)|\,|a_{2n+3}| - |a_{2n+1}|\,| \leqslant \frac{2\sqrt{2}(K^2+K)}{\left[1 - \frac{2n(2n-1)}{(2n+3)(2n+2)}\right]^{5/2}} \left[\frac{(2n+3)(2n+2)}{(2n)(2n-1)}\right]^n.$$

Consequently,

$$|\,|a_{2n+3}| - |a_{2n+1}|\,| \leqslant \frac{2\sqrt{2}(K^2+K)}{(2n+1)(2n)(2n-1)}\left[\frac{(2n+3)(2n+2)}{12n+6}\right]^{5/2} \times \left(1+\frac{1}{n}\right)^n\left(1+\frac{4}{2n-1}\right)^n$$

$$\leqslant \frac{2\sqrt{2}(K^2+K)}{2^{5/2}}\left(\frac{2n+3}{6n+3}\right)^{5/2} e^3 \frac{(2n+2)^2}{(2n)(2n-1)}\left(\frac{2n+2}{2n+1}\right)^{1/2}(2n+1)^{-1/2}$$

$$< \frac{8(K^2+K)e^3 5^{5/2}}{243\sqrt{3}}(2n+1)^{-1/2} \qquad (n \geqslant 1).$$

We now study the asymptotic behavior of $|a_{2n+1}|$ by relating the growth of the coefficients of $f(z)$ to the growth of $M(r, f')$. We will show that

$$\alpha = \lim_{r\to 1}(1-r^2)^{(K+2)/4}M(r,f')$$

exists and that α (and hence $|a_{2n+1}|$ for large n) is maximal for the class W_K only for the function

$$f_0(z) = \int_0^z \frac{(1+z^2)^{(K-2)/4}}{(1-z^2)^{(K+2)/4}}\,dz$$

or its rotations.

THEOREM 4.3. *Let* $f(z)\,W_K$. *Then*

$$\alpha = \lim_{r\to 1}(1-r^2)^{(K+2)/4}M(r,f')$$

exists and $\alpha \leqq 2^{(K-2)/4}$ *with equality if and only if*

$$f(z) = \int_0^z \frac{(1+e^{2i\theta}z^2)^{(K-2)/4}}{(1-e^{2i\theta}z^2)^{(K+2)/4}}\,dz.$$

Further, if $\alpha > 0$, *there are precisely two values of* θ_0 *such that*

$$\lim_{r\to 1} (1 - r^2)^{(K+2)/4}|f'(re^{i\theta_0})| = \alpha.$$

Proof. By Corollary 2.2, there are two odd starlike functions $s_1(z)$ and $s_2(z)$ such that

$$f'(z) = \left[\frac{s_1(z)}{z}\right]^{(K+2)/4} \Big/ \left[\frac{s_2(z)}{z}\right]^{(K-2)/4}.$$

Let $S_1(z) = [s_1(\sqrt{z})]^2$. Pommerenke [**11**] has shown that, unless $S_1(z) = z/(1 - e^{-2i\theta_0}z)^2$,

$$\lim_{r\to 1} (1 - r)^2 M(r, S_1) = 0.$$

Hence if $s_1(z)$ is not of the form $s_1(z) = z/(1 - e^{-2i\theta}z^2)$,

$$\lim_{r\to 1}(1 - r^2)M(r, s_1) = 0$$

and thus since $z/s_2(z)$ is bounded in U

$$\lim_{r\to 1} (1 - r^2)^{(K+2)/4}M(r, f') = 0.$$

Suppose now that

$$\limsup_{r\to 1} (1 - r^2)^{(K+2)/4}M(r, f') > 0.$$

Then $s_1(z)$ is of the form $z/(1 - e^{-2i\theta_0}z^2)$ and we may assume $\theta_0 = 0$. Since $f(z)$ is odd, we may choose a sequence $r_n \to 1$ and a point z_n on $|z| = r_n$ with $\operatorname{Re} z_n \geqq 0$ such that $\lim_{n\to\infty}(1 - r_n^2)^{(K+2)/4}|f'(z_n)| > 0$. We will show that for each such sequence, there is a Stoltz angle A with vertex at $z = 1$ such that z_n eventually lie in A. Suppose not. Let $C > 0$ be given. Then there is a subsequence $\{z_j\}$ with $|1 - z_j| > C(1 - r_j)$ and hence, since $|z/s_2(z)| \leqq 2$, we have for j sufficiently large

$$2^{(K-2)/4} \geqslant \tfrac{1}{2}(1 - r_j^2)^{(K+2)/4} \frac{C^{(K+2)/4}}{|1 - z_j^2|^{(K+2)/4}} \left|\frac{z_j}{s_2(z_j)}\right|^{(K-2)/4}$$

$$= \tfrac{1}{2}C^{(K+2)/4}(1 - r_j^2)^{(K+2)/4}|f'(z_j)|. \tag{4.6}$$

Letting $j \to \infty$ in (4.6) we obtain the inequality

$$2^{(K+2)/4} \geqslant C^{(K+2)/4} \lim_{j\to\infty} (1 - r_j^2)^{(K+2)/4}|f'(z_j)|,$$

which is impossible since $C > 0$ is arbitrary and $\lim_{j\to\infty}(1 - r_j^2)^{(K+2)/4}|f'(z_j)| > 0$. It follows that the points z_n eventually lie interior to some fixed Stoltz angle with vertex at $z = 1$. We recall that since $s_2(z)$ is starlike, [**1**; **11**], $\lim_{r\to 1} r/s_2(r)$

exists and is finite. (Since

$$r\frac{\partial}{\partial r}\log|f(z)| = \operatorname{Re}\frac{zf'(z)}{f(z)} \geqslant 0,$$

$\lim_{r\to 1}|f(r)|$ exists (possibly $=\infty$).) Since the z_n lie interior to some fixed Stoltz angle, we have

$$\lim_{n\to\infty}\frac{r_n}{s_2(r_n)} = \lim_{n\to\infty}\frac{z_n}{s_2(z_n)}\,.$$

It follows that

$$\left|\frac{z_n}{s_2(z_n)}\right|^{(K-2)/4} > \left|\frac{1-r_n^2}{1-z_n^2}\right|^{(K+2)/4}\left|\frac{z_n}{s_2(z_n)}\right|^{(K-2)/4}$$
$$= (1-r_n^2)^{(K+2)/4}|f'(z_n)|$$

and hence

$$\lim_{n\to\infty}\left|\frac{r_n}{s_2(r_n)}\right|^{(K-2)/4} \geqslant \lim_{n\to\infty}(1-r_n^2)^{(K+2)/4}|f'(z_n)|$$
$$\geqslant \lim_{n\to\infty}(1-r_n^2)^{(K+2)/4}|f'(r_n)|$$
$$= \lim_{n\to\infty}\left|\frac{r_n}{s_2(r_n)}\right|^{(K-2)/4}$$

for any sequence r_n so that $\lim_{n\to\infty}(1-r_n^2)^{(K+2)/4}|f'(z_n)| > 0$. Therefore α exists and equals $\lim_{r\to 1}(1-r^2)^{(K+2)/4}|f'(r)|$.

A similar argument shows that

$$\lim_{r\to 1}(1-r^2)^{(K+2)/4}|f'(-r)| = \alpha.$$

We have $\alpha \leqq 2^{(K-2)/4}$ with equality when $\theta_0 = 0$ if and only if $s_2(z) = z/(1-z^2)$ and $s_2(z) = z/(1+z^2)$ and consequently equality holds in general only for rotations of the function

$$f_0(z) = \int_0^z \frac{(1+z^2)^{(K-2)/4}}{(1-z^2)^{(K+2)/4}}\,dz.$$

Remark. Noonan [**8**] has obtained a result similar to Theorem 4.3 for the class V_K using the Hardy-Stein-Spencer equality.

A straightforward modification of Noonan's technique yields the following theorem, whose proof will be omitted.

THEOREM 4.4. *Let* $f(z) = \sum_{n=0}^{\infty} a_{2n+1}z^{2n+1} \in W_K$ *and let*

$$\alpha = \lim_{r\to 1}(1-r^2)^{(K+2)/4}M(r,f').$$

Then

$$\lim_{n\to\infty}\frac{|a_{2n+1}|}{(2n+1)^{K/4-3/2}} = \frac{\alpha}{\Gamma((K+2)/4)}\,.$$

Remark. Theorem 4.4 shows that for any fixed function $f(z) \in W_K$, the moduli of the coefficients of $f(z)$ are eventually less than those of the function

$$f(z) = \int_0^z \frac{(1+z^2)^{(K-2)/4}}{(1-z^2)^{(K+2)/4}}\,dz$$

unless $f(z) = e^{-i\theta}f(e^{i\theta}z)$. This is somewhat surprising since by Theorem 3.2, $f_0(z)$ does not maximize $|a_5|$ for $2 < K < 4$.

To conclude this paper we will study the behavior of $A_{2n+1}(K)$ as $K \to \infty$. The proof is based on a technique due to Robertson [**13**].

THEOREM 4.5. *Let* $A_{2n+1}(K) = \max_{f\in W_K}|a_{2n+1}|$. *Then for* $K \geqq 2$,

$$(4.7)\quad A_{2n+1}(K) < \frac{e^3}{2}(K^2+K)\tfrac{2}{3}^{K/4-1/2}(2n+1)^{K/4-3/2} \qquad (n > 1)$$

$$(4.8)\quad \lim_{K\to\infty} \frac{A_{2n+1}(K)}{(2n+1)^{K/4-3/2}} = 0 \qquad (n = 3, 4, \ldots).$$

Proof. Using the Cauchy integral formula, we see that

$$(2n+1)(2n)(2n-1)a_{2n+1} = \frac{1}{2\pi i}\int_{|z|=r} \frac{f'''(z)}{z^{2n-1}}\,dz$$

and consequently

$$(4.9)\quad (2n+1)(2n)(2n-1)|a_{2n+1}| \leqslant \frac{1}{2\pi r^{2n-2}}\int_0^{2\pi} |f'(re^{i\theta})| \left|\frac{f'''(re^{i\theta})}{f'(re^{i\theta})}\right| d\theta.$$

Corollary 2.2 easily yields

$$|f'(re^{i\theta})| \leqslant \frac{(1+r^2)^{(K-2)/4}}{(1-r^2)^{(K+2)/4}}.$$

Also, by (4.4) we have that for each $f \in W_K$,

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{f'''(re^{i\theta})}{f'(re^{i\theta})}\right| d\theta \leqslant \frac{K^2+K}{1-r^2}.$$

Substituting into (4.9) we obtain

$$(2n+1)(2n)(2n-1)A_{2n+1}(K) \leqslant \frac{(1+r^2)^{(K-2)/4}}{(1-r^2)^{(K+2)/4}} \frac{1}{r^{2n-2}} \frac{K^2+K}{1-r^2}.$$

The choice $r^2 = 1 - 3/(2n+1)$ yields

$$(2n+1)(2n)(2n-1)A_{2n+1}(K)$$
$$\leqslant \frac{K^2+K}{\left(1-\frac{3}{2n+1}\right)^{2n-2}}\left(2-\frac{3}{2n+1}\right)^{(K-2)/4}\left(\frac{2n+1}{3}\right)^{K/4+3/2}$$

and

$$A_{2n+1}(K) \leqslant \frac{e^3}{2}(K^2 + K)(\tfrac{2}{3})^{K/4-1/2}(2n+1)^{K/4-3/2} \qquad (n > 1)$$

This is (4.7). To obtain (4.8) we note that

$$\frac{A_{2n+1}(K)}{(2n+1)^{K/4-3/2}} < \frac{e^3}{3}(K^2 + K)(\tfrac{2}{3})^{K/4-1/2}$$

and consequently (4.8) follows by letting $K \to \infty$.

I would like to thank the referee for his helpful comments.

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