An Interpolation Series for Integral Functions

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1. The Gontcharoff interpolation series

\[ \sum_{n=0}^{\infty} F^{(n)}(a_n) G_n(z), \]  

where

\[ G_0(z) = 1, \quad G_n(z) = \int_{a_0}^{z} \int_{a_1}^{z'} \int_{a_2}^{z''} \ldots \int_{a_{n-1}}^{z^{(n-1)}} dz^{(n)} \quad (n > 0), \]

has been studied in various special cases. For example, if \( a_n = a_0 \) (all \( n \)), (1.0) reduces to the Taylor expansion of \( F(z) \). If \( a_n = (-1)^n \), J. M. Whittaker\(^2\) showed that the series (1.0) converges to \( F(z) \) provided \( F(z) \) is an integral function whose maximum modulus satisfies

\[ \lim_{r \to \infty} \frac{\log M(r)}{r} < \frac{1}{2}\pi, \]

the constant \( \frac{1}{2}\pi \) being the "best possible". In the case \( |a_n| \leq 1 \), I have shown\(^3\) that the series converges to \( F(z) \) provided \( F(z) \) is an integral function whose maximum modulus satisfies

\[ \lim_{r \to \infty} \frac{\log M(r)}{r} < 0.7259, \]

and\(^4\) that while 0.7259 is not the "best possible" constant here, it cannot be replaced by a number as great as 0.7378.

In this paper, I consider a generalisation of Whittaker's result, namely the case in which \( a_n = \omega^n \) where \( |\omega| = 1 \) (arg \( \omega \neq 0 \)), and prove

**Theorem I.** The series

\[ \sum_{n=0}^{\infty} F^{(n)}(\omega^n) P_n(z), \]

where \( |\omega| = 1, \text{ arg } \omega \neq 0, \)

\[ P_0(z) = 1, \quad P_n(z) = \int_{1}^{z} \int_{1}^{z'} \int_{1}^{z''} \ldots \int_{1}^{z^{(n-1)}} dz^{(n)} \quad (n > 0), \]

1 The notation used here differs from that adopted in 6 (Chapter III) in the omission of a factor \( n! \) from \( G_n(z) \).


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converges uniformly to $F(z)$ in any bounded region, provided $F(z)$ is an integral function whose maximum modulus satisfies

$$\lim_{r \to \infty} \frac{\log M(r)}{r} < \rho_1,$$

(1.1)

$\rho_1$ being the modulus of the smallest zero of the integral function $f(z, \omega)$ defined by the power series

$$f(z, \omega) = \sum_{n=0}^{\infty} \omega^{\ln(n+1)} z^n/n!.$$

(1.2)

The constant $\rho_1$ is shown to be the "best possible" in this case, and it is evident that Whittaker's result follows as a special case, since

$$f(z, -1) = \sin z + \cos z.$$

It is possible to sharpen condition (1.1) of Theorem I, and we prove

**Theorem II.** If we define $\omega, P_n(z), \rho_1$ as in Theorem I, the series

$$\sum_{n=0}^{\infty} F^{(n)}(\omega^n) P_n(z)$$

converges uniformly to $F(z)$ in any bounded region, provided $F(z)$ is an integral function satisfying

$$F(z) = O\{\phi^{|z|}\phi(\rho_1 |z|)\},$$

(1.3)

where $\phi(z)$ is a function of $z$ such that $\sum_{k=1}^{\infty} \sqrt{k} \phi(k)$ is absolutely convergent.

2. Let the moduli of the zeros of $f(z, \omega)$ be arranged in a sequence $\rho_n$ in ascending order of magnitude. Differentiating (1.2) we have

$$\frac{\partial f}{\partial z} = \sum_{n=0}^{\infty} \omega^{\ln(n+1)} z^n/n!$$

$$= \sum_{n=0}^{\infty} \omega^{\ln(n+1)} z^n/n!$$

$$= f(\omega z, \omega).$$

(2.1)

Consider

$$g(x, z) = f\left(\frac{xz}{\omega}, \frac{1}{\omega}\right)/f\left(z, \frac{1}{\omega}\right)$$

$$= \sum_{n=0}^{\infty} z^n Q_n(x) \quad (|z| < \rho_1).$$

(2.3)

Since

$$g(x, 0) = 1, \quad g(1, z) = 1,$$

it follows that

$$Q_0(x) = 1 \quad \text{and} \quad Q_n(1) = 0 (n > 0).$$

(2.5)
Now it follows from (2.4), (2.5) that
\[
\frac{\partial g}{\partial x} = \sum_{1}^{n} z^n Q_n'(x) \quad (|z| < \rho_1),
\] (2.6)
and from (2.2), (2.3) that
\[
\frac{\partial g}{\partial x} = z g(x/\omega, z)
\]
\[
= \sum_{0}^{\infty} z^{n+1} Q_n(x/\omega).
\] (2.7)
Hence, using (2.6), (2.7), we have
\[
Q_n'(x) = Q_{n-1}(x/\omega) \quad (n \geq 1).
\] (2.8)
It follows from (2.5), (2.8) that
\[
Q_n(x) = \int_{1}^{z} dx' \int_{1}^{x'/\omega} dx'' \int_{1}^{x''/\omega} dx''' \cdots \int_{1}^{x^{(n-1)}/\omega} dx^{(n)},
\]
or, by the transformations \( \zeta^{(k)} = \omega^{k-1} z^{(k)} \),
\[
\omega^{in(n-1)} Q_n(x) = \int_{1}^{x} d\zeta' \int_{1}^{\zeta'} d\zeta'' \cdots \int_{1}^{\zeta^{(n-1)}} d\zeta^{(n)} = P_n(x).
\] (2.9)
Now, integrating
\[
R_n(z) = \int_{1}^{z} dz' \int_{1}^{z'} dz'' \cdots \int_{1}^{z^{(n-1)}} F^{(n)}(\zeta^{(n)}) d\zeta^{(n)}
\] repeatedly by parts\(^1\), we find
\[
R_n(z) = F(z) - \sum_{r=0}^{n-1} \frac{F^{(r)}(\omega^r)}{r!} P_r(z).
\]
Hence
\[
F(z) = \sum_{r=0}^{n-1} \frac{F^{(r)}(\omega^r)}{r!} P_r(z) + R_n(z).
\] (2.11)
Let \( C, \Gamma \) be the circles \(|z| = \frac{1}{2}\rho_1, |z| = \frac{1}{2}(\rho_1 + \rho_2)\) respectively. From (2.4) we have
\[
Q_n(x) = \frac{1}{2\pi i} \int_{C} \frac{g(x, z)}{z^{n+1}} dz.
\] (2.12)
If \( f(z, 1/\omega) \) has \( p \) zeros (denoted by \( z_1, z_2, \ldots, z_p \)) on \(|z| = \rho_1\), then \( g(x, z) \) has \( p \) poles (at most) between \( C \) and \( \Gamma \), residues \( A_1(x), A_2(x), \ldots, A_p(x) \) respectively, these residues being bounded for \( x \) in any bounded region.

Now \(|f(z, 1/\omega)|\) has no zeros on \( \Gamma \) and thus has a positive minimum on \( \Gamma \) which will be denoted by \( m \). Since \(|f(z, 1/\omega)| = |f(z, 1/\omega)|\), we have, using
\(^1\) See J. M. Whittaker, 6, 39, for a detailed argument of this nature.
where $B(x)$ is bounded for $x$ in any bounded region. Moreover the residue of $g(x, z)/z^{n+1}$ at $z = z_s$ ($s = 1, 2, \ldots, p$) is $A_s(x)/z_s^{n+1}$ and this is of absolute magnitude $|A_s(x)|/\rho_1^{n+1}$. Now

$$
\left| \frac{1}{2\pi i} \int_\Gamma g(x, z) \frac{dz}{z^{n+1}} \right| \leq e^{\Im(x+\rho)\Im(z)} \left( \frac{\rho_1 + \rho_2}{2} \right)^{n+1} \tag{2.13}
$$

and

$$
\left| \frac{1}{2\pi i} \int_\Gamma \frac{g(x, z)}{z^{n+1}} \frac{dz}{z^{n+1}} \right| \leq B(x) \left( \frac{\rho_1 + \rho_2}{2} \right)^{n+1}, \tag{2.14}
$$

is equal to the sum of the residues of $g(x, z)/z^{n+1}$ at the points $z_1, z_2, \ldots, z_p$ and it follows from (2.12), (2.14) that

$$
|Q_n(x)| \leq \sum_{s=1}^{p} \frac{|A_s(x)|}{\rho_1^{n+1}} + B(x) \left( \frac{\rho_1 + \rho_2}{2} \right)^n \tag{2.15}
$$

where $A(x)$ and $B(x)$ are bounded for $x$ in any bounded region. On integrating both sides of the equation from $1$ to $x/\omega$ we can show by induction that, for any integer $L$,

$$
x_L = \sum_{r=0}^{L} \frac{Q_{L-r}(x/\omega)}{r! \omega^{n(r-1)}} \tag{2.16}
$$

and thus

$$
S_{n,k}(\frac{x}{\omega}) = \left[ \omega^{n} \right]^{x/\omega} \int \left[ \omega^{n} \right]^{x/\omega} dx' \ldots \int \left[ \omega^{n} \right]^{x/\omega} dx^{(n)} \int \left[ \omega^{n} \right]^{x/\omega} dx^{(n+1)} \ldots \int \left[ \omega^{n} \right]^{x/\omega} dx^{(k)}

= \frac{1}{\omega^{n(k-n)(k-n+1)}} \int \left[ \omega^{n} \right]^{x/\omega} dx' \ldots \int \left[ \omega^{n} \right]^{x/\omega} dx^{(n-1)} \frac{x^{k-n}}{(k-n)!} dx

= \sum_{r=0}^{k-n} \frac{Q_{k-r}(x/\omega)}{r! \omega^{n(r-1)}} \tag{2.17}
$$

from (2.16), making use of (2.5) and (2.8). Also, from (2.15) and (2.17) it follows that

$$
\left| S_{n,k}(\frac{x}{\omega}) \right| \leq A(x) \sum_{r=0}^{k-n} \frac{1}{r! \rho_1^{k-r+1}} + B(x) \sum_{r=0}^{k-n} \frac{1}{r! \left( \frac{\rho_1 + \rho_2}{2} \right)^{k-r+1}} \tag{2.18}
$$

\[ \leq A(x) \left( e^{\Im(x)} \rho_1^{k+1+1} + B(x) e^{\Im(x)} \rho_2^{k+1+1} \right) \left( \frac{\rho_1 + \rho_2}{2} \right)^k. \]
Again, if we use the transformations (2.9), the formula

\[ \omega^{i(k-1)} S_{n,k}(x) = \int_{1}^{x} \frac{d\xi'}{w} \int_{w'}^{x} \frac{d\xi''}{w''} \ldots \int_{w^{(n-1)}}^{x} \frac{d\xi^{(n)}}{w^{(n)}} \ldots \int_{0}^{w^{(k-1)}} d\xi^{(k)} \quad (2.19) \]

arises from the definition of \( S_{n,k}(x/\omega) \) in (2.17).

3. From (2.10), (2.19), on expanding \( F^{(n)}(z) \) in its Taylor series, we get

\[ R_n(z) = \int_{1}^{z} \frac{dz'}{w'} \int_{w'}^{z} \frac{dz''}{w''} \ldots \int_{w^{(n-1)}}^{z} \frac{dz^{(n)}}{w^{(n)}} \sum_{k=n}^{\infty} \frac{F^{(k)}(0)}{(k-n)!} dz^{(k-n)} \]

and using Stirling’s approximation for \( k! \), if \( F(z) \) satisfies (1.3), we have

\[ F^{(k)}(0) = O\{p_1^k \sqrt{k \phi(k)}\}. \]

Hence, from (2.18) and (3.1), \( R_n(z) \) is less in modulus than the sum of the remainders of two convergent series and thus tends to zero as \( n \) tends to infinity. From (2.11) it then follows that the interpolation series

\[ \sum_{r=0}^{\infty} F^{(r)}(z^n) P_r(z) \quad (3.3) \]

converges uniformly to \( F(z) \) in any bounded region provided \( F(z) \) is an integral function satisfying (1.3). This completes the proof of Theorem II and hence of Theorem I.

Let \( z_1 \), where \( |z_1| = \rho_1 \), be the zero of smallest modulus of \( f(z, 1/\omega) \). That the constant \( \rho_1 \) of Theorems I and II is the “best possible” is seen by taking \( F(z) = f(zz_1, 1/\omega) \) for which the maximum modulus \( M(r) \) clearly satisfies

\[ \lim_{r \to \infty} \frac{\log M(r)}{r} = |z_1| = \rho_1. \]

Then, by (2.2), \( F^{(m)}(z^n) = z_1^n f(z_1, 1/\omega) \) for all \( n \). Thus for this function all the terms of the series (3.3) are identically zero. It should be noted that \( z_1 \) is the zero of smallest modulus of \( f(z, \omega) \).

The numerical value of \( \rho_1 \) has been calculated\(^1\) for \( \arg \omega = -\pi/3, -\pi/3, \pi/3 \pi \)

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\(^1\) See R. P. Boas, 1 and 2; S. S. Macintyre, 3,
and the equivalent in radians of 136°, 137°, the values of \( \rho_1 \) in these cases being approximately \( \cdot 746 \), \( \cdot 7398 \), \( \cdot 7379 \), \( \cdot 7378 \) and \( \cdot 7378 \) respectively.

REFERENCES.


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