MULTIPLICATION OPERATORS ON WEIGHTED SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS

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Abstract

If \( V \) is a system of weights on a completely regular Hausdorff space \( X \) and \( E \) is a locally convex space, then \( CV_0(X, E) \) and \( CV_p(X, E) \) are locally convex spaces of vector-valued continuous functions with topologies generated by seminorms which are weighted analogues of the supremum norm. In this paper we characterise multiplication operators on these spaces induced by scalar-valued and vector-valued mappings. Many examples are presented to illustrate the theory.


1. Introduction

Let \( X \) be a non-empty set, let \( E \) be a topological algebra and let \( T(X, E) \) be a topological vector space of functions from \( X \) to \( E \). Let \( \theta : X \to \mathbb{C} \) and \( \psi : X \to E \) be two mappings. Then scalar multiplication and multiplication in \( E \) give rise to two linear transformations \( M_\theta \) and \( M_\psi \) from \( T(X, E) \) to the linear space \( L(X, E) \) of all functions from \( X \) to \( E \), defined as \( M_\theta f = \theta \cdot f \) and \( M_\psi f = \psi f \), where the product of functions is defined pointwise. In case \( M_\theta \) and \( M_\psi \) take \( T(X, E) \) into itself and they are continuous, they are called multiplication operators on \( T(X, E) \) induced by \( \theta \) and \( \psi \) respectively. These operators have been the subject matter of study for a long time on different function spaces, especially on \( L^p \)-spaces, and they have played a very important role in the study of operators on Hilbert spaces.

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In this note we have endeavoured to study multiplication operators on those weighted spaces of scalar-valued and vector-valued continuous functions which come in contact with topological dynamics.

2. Preliminaries

Let $X$ be a completely regular Hausdorff space and $E$ a Hausdorff locally convex topological vector space over $\mathbb{C}$. Let $cs(E)$ be the set of all continuous seminorms on $E$. By $C(X, E)$ we mean the collection of all continuous functions from $X$ into $E$.

A non-negative upper-semicontinuous function on $X$ will be called a weight on $X$. If $V$ is a set of weights on $X$ such that, given any $x \in X$, there is some $v \in V$ for which $v(x) > 0$, we write $V > 0$. A set $V$ of weights on $X$ is said to be directed upward provided that, for every pair $v_1, v_2 \in V$ and $\lambda > 0$, there exists $v \in V$ so that $\lambda v_i \leq v$ (pointwise on $X$) for $i = 1, 2$. We hereafter assume that sets of weights are directed upward. A set $V$ of weights on $X$ which additionally satisfies $V > 0$ will be referred to as a system of weights on $X$.

Now, taking a system $V$ of weights on $X$ and a locally convex space $E$, we consider the following vector spaces of continuous functions associated with the triple $(X, V, E)$:

$$CV_0(X, E) = \{f \in C(X, E) : v f \text{ vanishes at infinity on } X \text{ for all } v \in V\};$$

$$CV_b(X, E) = \{f \in C(X, E) : v f(X) \text{ is bounded in } E \text{ for all } v \in V\}.$$ 

Obviously $CV_0(X, E)$, $CV_b(X, E)$ are vector spaces and $CV_0(X, E) \subseteq CV_b(X, E)$. Now, let $v \in V$, $q \in cs(E)$ and $f \in C(X, E)$. If we put

$$P_{v,q}(f) = \text{Sup}\{v(x)q(f(x)) : x \in X\}$$

then $P_{v,q}$ can be regarded as a seminorm on either $CV_b(X, E)$ or $CV_0(X, E)$, and the family $\{P_{v,q} : v \in V, q \in cs(E)\}$ defines the locally convex topology on each of these two spaces.

In case $E = \mathbb{C}$, we will omit $E$ from our notation and write $CV_0(X)$ in place of $CV_0(X, \mathbb{C})$. We also write $P_v$ in place of $P_{v,q}$ for each $v \in V$, where $q(z) = |z|$, $z \in \mathbb{C}$. Moreover, if $E = (E, q)$ is any normed linear space and $v \in V$, we write $P_v$ instead of $P_{v,q}$. We shall denote by $B_{v,q}$ the closed unit ball corresponding to the seminorm $P_{v,q}$. In case $E = (E, q)$, we simply write $B_v$.

If $U$ and $V$ are two systems of weights on $X$, we write $U \leq V$ whenever given $u \in U$, there exists $v \in V$ such that $u \leq v$. In this case, we then clearly have that $CV_0(X, E) \subseteq CU_0(X, E)$ and $CV_b(X, E) \subseteq CU_b(X, E)$,
as well as that the inclusion map is continuous in both instances. If \( U \leq V \) and \( V \leq U \), then \( U \) and \( V \) are equivalent systems of weights on \( X \) and we denote this by \( U \approx V \).

The spaces \( CV_0(X) \) and \( CV_b(X) \) were first introduced by Nachbin [6] and \( CV_0(X, E) \), \( CV_b(X, E) \) were subsequently considered in detail by Bierstedt [2] and Prolla [7].

Now, we shall give some examples of these spaces. Let \( X \) be a completely regular Hausdorff space. We denote by \( \chi_S \), the characteristic function of a subset \( S \) of \( X \). We distinguish four systems of weights on \( X \), namely

- \( U = \{ \lambda \chi_K : \lambda > 0, K \subset X, K \text{ compact} \} \),
- \( U' = C^+_c(X) \), the set of all positive continuous functions with compact supports,
- \( V = K^+(X) \), the set of all positive constant functions on \( X \) and
- \( V' = C^+_b(X) \), the set of all positive continuous functions vanishing at infinity. Further, if \( E \) is a locally convex space, then we define

\[
C_0(X, E) = \{ f \in C(X, E) : f \text{ vanishes at infinity on } X \},
\]
\[
C_b(X, E) = \{ f \in C(X, E) : f(X) \text{ is bounded in } E \}.
\]

**Example.** Let \( X \) be a completely regular Hausdorff space and let \( E \) be a locally convex space. Then

- (i) \( CU_0(X, E) = CU_b(X, E) = (C(X, E), k) \) where \( k \) denotes the compact open topology,
- (ii) \( CU'_0(X, E) = CU'_b(X, E) = (C(X, E), k) \),
- (iii) \( CV'_0(X, E) = (C_0(X, E), u) \) and \( CV_b(X, E) = (C_b(X, E), u) \), where \( u \) denotes the topology of uniform convergence on \( X \) and
- (iv) \( CV'_0(X, E) = CV'_b(X, E) = (C_b(X, E), \beta_0) \), where \( \beta_0 \) denotes the strict topology.

### 2. Functions inducing multiplication operators

In developing our characterization of those functions \( \theta : X \to \mathbb{C} \) (or \( \psi : X \to E \)) which induce multiplication operators on weighted spaces of type \( CV_0(X) \) and \( CV_0(X, E) \) we work under the following modest requirements.

- (2.a) \( X \) is a completely regular Hausdorff space.
- (2.b) \( E \) is a locally convex space such that there exists a vector \( s \in E \) for which \( p(s) \neq 0 \), for every \( p \in cs(E) \).
- (2.c) \( V \) is a system of weights on \( X \)
- (2.d) Corresponding to each \( x \in X \), there exists \( f_x \in CV_0(X) \) such that \( f_x(x) \neq 0 \).
In case $X$ happens to be locally compact, (2.d) is automatically satisfied. For a continuous function $\theta: X \to \mathbb{C}$ (or $\psi: X \to E$), the set

$$V|\theta|=\{v|\theta|: v \in V\}$$

(or for every $p \in cs(E)$, $Vp \circ \psi = \{vp \circ \psi: v \in V\}$) is a directed set of weights on $X$ since [9, Theorem 2.2] Summers has shown that the product of two non-negative upper semicontinuous functions is non-negative and upper semicontinuous. In case $\theta: X \to \mathbb{C}$ (or $\psi: X \to E$) is non-zero at each point of $X$, $V|\theta|$ (or $Vp \circ \psi$) is a system of weights on $X$.

In the following theorem we characterize multiplication operators on $CV_0(X)$ induced by scalar-valued functions.

2.1 Theorem. Let $\theta: X \to \mathbb{C}$ be a continuous function. Then $M_\theta: CV_0(X) \to CV_0(X)$ is a multiplication operator if and only if $V|\theta| \leq V$.

Proof. First, suppose $V|\theta| \leq V$. Then for every $v \in V$, there exists $u \in V$ such that $v|\theta| \leq u$ (pointwise on $X$). We show that $M_\theta$ is a continuous linear operator on $CV_0(X)$. Clearly $M_\theta$ is linear on $CV_0(X)$. In order to prove the continuity of $M_\theta$ on $CV_0(X)$, it is enough to show that $M_\theta$ is continuous at the origin. For this, suppose $\{f_\alpha\}$ is a net in $CV_0(X)$ such that

$$P_v(f_\alpha) \to 0, \quad \text{for every } v \in V.$$

Now,

$$P_v(\theta f_\alpha) = \text{Sup}\{v(x)|\theta(x)||f_\alpha(x)|: x \in X\} \leq \text{Sup}\{u(x)|f_\alpha(x)|: x \in X\} = P_u(f_\alpha) \to 0.$$

This proves the continuity of $M_\theta$ at the origin and hence $M_\theta$ is continuous on $CV_0(X)$.

Conversely, suppose $M_\theta$ is a continuous linear operator on $CV_0(X)$. We shall show that $V|\theta| \leq V$. Let $v \in V$. Since $M_\theta$ is continuous at the origin, there exists $u \in V$ such that $M_\theta(B_u) \subseteq B_v$. We claim that $v|\theta| \leq 2u$. Take $x_0 \in X$ and set $u(x_0) = \varepsilon$. In case $\varepsilon > 0$, $N = \{x \in X: u(x) < 2\varepsilon\}$ is an open neighbourhood of $x_0$. Thus, according to [6, Lemma 2], there exists $f \in CV_0(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - N) = 0$. Let $g = (2\varepsilon)^{-1}f$. Then clearly $g \in B_u$. Since $M_\theta(B_u) \subseteq B_v$, we have $\theta g \in B_v$ and this yields that

$$v(x)|\theta(x)||g(x)| \leq 1, \quad \text{for every } x \in X.$$

From this it follows that

$$v(x)|\theta(x)||f(x)| \leq 2\varepsilon, \quad \text{for every } x \in X.$$
This implies that
\[ v(x_0)|\theta(x_0)| \leq 2u(x_0). \]
Now, suppose \( u(x_0) = 0 \) and \( v(x_0)|\theta(x_0)| > 0 \). If we put \( \varepsilon = v(x_0)|\theta(x_0)|/2 \) and set \( N = \{x \in X: u(x) < \varepsilon\} \), then \( N \) would be an open neighbourhood of \( x_0 \) and we could again find \( f \in CV_0(X) \) such that \( 0 \leq f \leq 1, f(x_0) = 1 \) and \( f(X - N) = 0 \). Now let \( g = \varepsilon^{-1}f \). Then clearly \( g \in B_u \) and therefore \( \theta g \in B_v \). Hence
\[ v(x)|\theta(x)||g(x)| \leq 1, \quad \text{for every } x \in X. \]
This implies that
\[ v(x)|\theta(x)||f(x)| \leq \varepsilon, \quad \text{for every } x \in X. \]
From this it follows that
\[ v(x_0)|\theta(x_0)| \leq \frac{v(x_0)|\theta(x_0)|}{2} \]
which is impossible. This proves our claim and hence the proof is complete.

Now, we shall characterise multiplication operators on \( CV_0(X, E) \) induced by scalar-valued functions.

2.2 Theorem. Let \( \theta: X \rightarrow \mathbb{C} \) be a continuous function. Then
\[ M_\theta: CV_0(X, E) \rightarrow CV_0(X, E) \]
is a multiplication operator if and only if \( V|\theta| \leq V. \)

Proof. First of all, let us suppose \( V|\theta| \leq V \). Then for every \( v \in V \), there exists \( u \in V \) such that \( v|\theta| \leq u \) (pointwise on \( X \)). We shall show that \( M_\theta \) is a continuous linear operator on \( CV_0(X, E) \). Obviously \( M_\theta \) is linear on \( CV_0(X, E) \). It suffices to show that \( M_\theta \) is a continuous linear operator at the origin. To prove this, let \( \{f_\alpha\} \) be a net in \( CV_0(X, E) \) such that for every \( v \in V, q \in cs(E), P_{v,q}(f_\alpha) \rightarrow 0 \). Then
\[ P_{v,q}(\theta f_\alpha) = \operatorname{Sup}\{v(x)|\theta(x)||f_\alpha(x)|: x \in X\} \leq \operatorname{Sup}\{u(x)q(f_\alpha(x)): x \in X\} = P_{u,q}(f_\alpha) \rightarrow 0. \]
This proves the continuity of \( M_\theta \) at the origin and hence \( M_\theta \) is a continuous linear operator on \( CV_0(X, E) \).

Conversely, suppose \( M_\theta: CV_0(X, E) \rightarrow CV_0(X, E) \) is a continuous linear operator. Then we shall show that \( V|\theta| \leq V \). Let \( v \in V \). Since \( M_\theta \) is continuous at the origin, therefore for every \( v \in V, p \in cs(E) \), there exists \( u \in V, q \in cs(E) \) such that \( M_\theta(B_{u,q}) \subseteq B_{v,p} \). By our assumption
there exists a vector \( s \in E \) such that \( p(s) \neq 0 \), for every \( p \in \text{cs}(E) \). Let \( \alpha = p(s)/q(s) \). Then \( \alpha > 0 \). We claim that \( \alpha v|\theta| \leq 2u \) (pointwise on \( X \)).

Fix \( x_0 \in X \) and set \( u(x_0) = \varepsilon \). In case \( \varepsilon > 0 \), \( N = \{ x \in X : u(x) < 2\varepsilon \} \) is an open neighbourhood of \( x_0 \) and therefore by [6, Lemma 2] there exists \( f \in CV_0(X) \) such that \( 0 \leq f \leq 1 \), \( f(x_0) = 1 \) and \( f(X - N) = 0 \). Define \( g(x) = f(x)s \), for every \( x \in X \). Then clearly \( g \in CV_0(X, E) \) and for every \( p \in \text{cs}(E) \), \( 0 \leq (p \circ g) \leq p(s) \), \( (p \circ g)(x_0) = p(s) \) and \( (p \circ g)(X - N) = 0 \).

Let \( h = (2\varepsilon)^{-1} g/q(s) \). Then clearly \( h \in B_{u, q} \) and this yields that \( \theta \cdot h \in B_{v, p} \). Hence \( v(x)|\theta(x)|p(h(x)) \leq 1 \), for every \( x \in X \). From this, it follows that

\[
v(x)|\theta(x)|\frac{p(s)}{q(s)} \leq 2u(x_0).
\]

This implies that

\[
\alpha v(x_0)|\theta(x_0)| \leq 2u(x_0). 
\]

On the other hand, suppose \( u(x_0) = 0 \) and \( \alpha v(x_0)|\theta(x_0)| > 0 \). Put \( \varepsilon = \alpha v(x_0)|\theta(x_0)|/2 \). Then \( N = \{ x \in X : u(x) < \varepsilon \} \) is an open neighbourhood of \( x_0 \) and therefore again by [6, Lemma 2] there exists \( f \in CV_0(X) \) such that \( 0 \leq f \leq 1 \), \( f(x_0) = 1 \) and \( f(X - N) = 0 \). Again, define \( g(x) = f(x)s \), for every \( x \in X \). Then \( g \in CV_0(X, E) \) and for every \( p \in \text{cs}(E) \), \( 0 \leq (p \circ g) \leq p(s) \), \( (p \circ g)(x_0) = p(s) \) and \( (p \circ g)(X - N) = 0 \). Consider \( h = g/eq(s) \). Then \( h \in B_{u, q} \) and therefore \( \theta h \in B_{v, p} \). Hence

\[
v(x)|\theta(x)|p(h(x)) \leq 1, \quad \text{for every } x \in X.
\]

This implies that

\[
v(x)|\theta(x)|\frac{1}{q(s)}p(g(x)) \leq \varepsilon, \quad \text{for every } x \in X.
\]

From this it follows that

\[
v(x_0)|\theta(x_0)|\frac{p(s)}{q(s)} \leq \frac{p(s)}{q(s)} \frac{v(x_0)|\theta(x_0)|}{2}.
\]

Thus \( \alpha v(x_0)|\theta(x_0)| \leq \alpha v(x_0)|\theta(x_0)|/2 \), which is impossible. Hence our claim is established and the proof is completed.

In order to prove the next theorem, we shall need the following definitions.

Let \( E \) be a locally convex algebra with jointly continuous multiplication. It clearly follows that for each \( p \in \text{cs}(E) \), there exists a \( q \in \text{cs}(E) \) such that \( p(xy) \leq q(x)q(y) \), for every \( x, y \in E \). A seminorm \( p \) on \( E \) is said to be submultiplicative if \( p(xy) \leq p(x)p(y) \), for every \( x, y \in E \). In [5] Michael defines \( E \) to be a locally multiplicatively convex algebra, or in short
an lmC algebra, if there exists a base of neighbourhoods of zero consisting of idempotent absolutely convex sets, or equivalently if its topology is defined by a collection of submultiplicative seminorms. Clearly, multiplication in an lmC algebra is always jointly continuous. For more details and examples of lmC algebras we refer to [3] and [5]. Let \( \mathcal{P} \) be a family of submultiplicative seminorms inducing the topology of \( E \). Then \( \mathcal{P} \) is a subfamily of \( cs(E) \). In [10, c.2.3] Zelazako has noted that for any lmC algebra \( E \) with unit \( e \), the family \( \mathcal{P} \) can be chosen in such a way that \( p(e) = 1 \), for every \( p \in \mathcal{P} \). So we can assume that \( \mathcal{P} \) is such a family in this case.

Now, we shall give a characterisation of multiplication operators on \( CV_0(X, E) \) induced by vector-valued functions.

2.3 Theorem. Let \( E \) be a (locally multiplicatively convex) lmC algebra with unit \( e \) and let \( \psi : X \to E \) be a continuous function. Then

\[
M_\psi : CV_0(X, E) \to CV_0(X, E)
\]

is a multiplication operator if and only if \( Vp \circ \psi \leq V \), for every \( p \in \mathcal{P} \).

Proof. Suppose \( Vp \circ \psi \leq V \), for every \( p \in \mathcal{P} \). Then for every \( v \in V \), there exists \( u \in V \) such that \( vp \circ \psi \leq u \) (pointwise on \( X \)). We shall show that the mapping \( M_\psi : CV_0(X, E) \to C(X, E) \), defined by \( M_\psi f = \psi f \), where product is pointwise, is a continuous linear operator on \( CV_0(X, E) \). We shall establish the continuity of \( M_\psi \) at the origin. For this, let \( \{f_\alpha\} \) be a net in \( CV_0(X, E) \) such that for every \( v \in V \), \( q \in \mathcal{P} \), \( P_{v, q}(f_\alpha) \to 0 \). Then

\[
P_{v, q}(\psi f_\alpha) = \sup\{v(x)q(\psi(x)f_\alpha(x)) : x \in X\} \\
\leq \sup\{v(x)q(\psi(x))q(f_\alpha(x)) : x \in X\} \\
\leq \sup\{u(x)q(f_\alpha(x)) : x \in X\} \\
= P_{u, q}(f_\alpha) \to 0.
\]

This proves that \( M_\psi \) is continuous at the origin and hence a continuous linear operator on \( CV_0(X, E) \).

Conversely, suppose \( M_\psi : CV_0(X, E) \to CV_0(X, E) \) is a continuous linear operator. We shall show that \( Vp \circ \psi \leq V \), for every \( p \in \mathcal{P} \). Let \( v \in V \) and \( p \in \mathcal{P} \). Since \( M_\psi \) is continuous at the origin, there exist \( u \in V \) and \( q \in \mathcal{P} \) such that \( M_\psi(B_{u, q}) \subseteq B_{v, p} \). We claim that \( vp \circ \psi \leq 2u \) (pointwise on \( X \)). Fix \( x_0 \in X \) and set \( u(x_0) = \varepsilon \). In case \( \varepsilon > 0 \), \( N = \{x \in X : u(x) < 2\varepsilon\} \) is an open neighbourhood of \( x_0 \) and therefore according to [6, Lemma 2] there exists \( f \in CV_0(X) \) such that \( 0 \leq f \leq 1 \), \( f(x_0) = 1 \) and \( f(X - N) = 0 \).

Define \( g(x) = f(x)e \), for every \( x \in X \), where \( e \) is the unit in \( E \). Then \( g \in CV_0(X, E) \) and for every \( p \in \mathcal{P} \), \( 0 \leq (p \circ g) \leq 1 \), \( (p \circ g)(x_0) = 1 \) and
\((p \circ g)(X - N) = 0\). If \(h = (2\varepsilon)^{-1}g\), then \(h \in B_{u,q}\) and hence \(\psi h \in B_{v,p}\). From this it follows that \(v(x)p(\psi(x)h(x)) \leq 1\), for every \(x \in X\). This implies that
\[
v(x)p(\psi(x)f(x)e) \leq 2\varepsilon, \quad \text{for every } x \in X.
\]
Thus \(v(x_0)p(\psi(x_0)) \leq 2u(x_0)\). On the other hand, suppose \(u(x_0) = 0\) and \(v(x_0)p(\psi(x_0)) > 0\). Set \(\varepsilon = v(x_0)p(\psi(x_0))/2\). Then \(N = \{x \in X: u(x) < \varepsilon\}\) is an open neighbourhood of \(x_0\) and therefore again by [6, Lemma 2] there exists \(f \in CV_0(X)\) such that \(0 \leq f \leq 1\), \(f(x_0) = 1\) and \(f(X - N) = 0\).

We define \(g(x) = f(x)e\), for every \(x \in X\), where \(e\) is the unit in \(E\). Then \(g \in CV_0(X, E)\) and for every \(p \in \mathcal{P}\), \(0 \leq (p \circ g) \leq 1\), \((p \circ g)(x_0) = 1\) and \((p \circ g)(X - N) = 0\). Choose \(h = \varepsilon^{-1}g\). Then clearly \(h \in B_{u,q}\) and therefore \(\psi h \in B_{v,p}\). This implies that \(v(x)p(\psi(x)h(x)) \leq 1\), for every \(x \in X\). From this, it follows that \(v(x)p(\psi(x)g(x)) \leq \varepsilon\), for every \(x \in X\). Further, we get
\[
v(x_0)p(\psi(x_0)f(x_0)e) \leq \frac{v(x_0)p(\psi(x_0))}{2}.
\]
Thus \(v(x_0)p(\psi(x_0)) \leq v(x_0)p(\psi(x_0))/2\) which is impossible and this establishes our claim. This completes the proof of the theorem.

2.4 REMARK. Note that if \(\theta: X \to \mathbb{C}\) (or \(\psi: X \to E\)) is a bounded continuous complex-valued (or vector-valued) function on \(X\), then clearly \(M_\theta\) (or \(M_\psi\)) is a multiplication operator on \(CV_0(X)\) (or \(CV_0(X, E)\)) for any system of weights \(V\).

If \(V\) is the system of weights generated by the characteristic functions of compact sets, then it turns out that every continuous map induces a multiplication operator. This we shall establish in the following theorem.

2.5 THEOREM. Let \(X\) be a completely regular Hausdorff space and let
\[V = \{\lambda\chi_K: \lambda > 0 \text{ and } K \subset X, K \text{ compact}\}.
\]

(i) Every continuous \(\theta: X \to \mathbb{C}\) induces a multiplication operator on \(CV_0(X)\) (or \(CV_0(X, E)\)).

(ii) Every continuous \(\psi: X \to E\), a locally convex algebra with jointly continuous multiplication, induces a multiplication operator \(M_\psi\) on \(CV_0(X, E)\).

PROOF. (i) In order to prove that \(M_\theta\) is a continuous linear operator on \(CV_0(X)\) (or \(CV_0(X, E)\)), it is enough to show that for every \(v \in V\), there exists \(u \in V\) such that \(v|\theta| \leq u\) (pointwise on \(X\)). Let \(v \in V\). Then \(v = \lambda\chi_K\), where \(K\) is a compact subset of \(X\). Let \(m = \operatorname{Sup}\{|\theta(x)|: x \in K\}\) and choose \(u = \lambda m\chi_K\). Then \(u \in V\). Since \(|\theta(x)| \leq m\), for every \(x \in K\), we have
\[
\lambda\chi_K(x)|\theta(x)| \leq \lambda m\chi_K(x), \quad x \in K.
\]
Hence \( v(x)\theta(x) \leq u(x) \), for every \( x \in K \). If \( x \in X - K \), then the above inequality is obviously true. Thus we have shown that \( v(x)\theta(x) \leq u(x) \), for every \( x \in X \), and hence by Theorems 2.1 and 2.2, it follows that \( M_\theta \) is a multiplication operator on \( CV_0(X) \) (or \( CV_0(X, E) \)).

(ii) In view of Theorem 2.3, it is sufficient to establish the inequality \( Vp \circ \psi \leq V \), for every \( p \in cs(E) \), that is, for every \( v \in V \), there exists \( u \in V \) such that \( v \circ \psi \leq u \) (pointwise on \( X \)). Let \( v \in V \) and \( p \in cs(E) \). Then \( v = \lambda x_K \), \( K \) a compact subset of \( X \). Let \( m = \sup \{ p(\psi(x)) : x \in K \} \) and choose \( u = \lambda m x_K \). Then \( u \in V \). Since \( p(\psi(x)) \leq m, x \in K \), we have

\[
\lambda x_K(x)p(\psi(x)) \leq \lambda m x_K(x), \quad x \in K.
\]

This implies that \( v(x)p(\psi(x)) \leq u(x), \) for every \( x \in K \). If \( x \in X - K \), then obviously \( v(x)p(\psi(x)) \leq u(x) \). Thus \( v(x)p(\psi(x)) \leq u(x), \) for every \( x \in X \). This completes the proof of the theorem.

2.6 Corollary. Let \( X \) have the discrete topology and \( V = \{ \lambda x_K : \lambda \geq 0, \ K \subset X, \ K \) a finite set \( \} \). Then every function \( \theta : X \rightarrow \mathbb{C} \) (or \( \psi : X \rightarrow E \)) induces a multiplication operator \( M_\theta \) (or \( M_\psi \)) on \( CV_0(X) \) (or \( CV_0(X, E) \)).

2.7 Remark. (i) In Theorem 2.5, if we replace the system of weights \( V = \{ \lambda x_K : \lambda > 0, \ K \subset X, \ K \) compact \( \} \) by \( U = C_c^+(X) \), the set of all positive continuous functions on \( X \) with compact supports, then the new result is also true.

(ii) If \( X \) is a locally compact space, then \( V = \{ \lambda x_K : \lambda \geq 0, \ K \subset X, \ K \) compact subset of \( X \} \) and \( U = C_c^+(X) \) are equivalent, and otherwise \( V \leq U \).

(iii) In Theorems 2.1–2.3 and 2.5 if we replace \( CV_0(X) \) and \( CV_0(X, E) \) by \( CV_b(X) \) and \( CV_b(X, E) \), then all the new results are also true.

(iv) From Theorem 2.5, we noted that if \( \theta : X \rightarrow \mathbb{C} \) (or \( \psi : X \rightarrow E \)) is an unbounded continuous function, even then \( \theta \) (or \( \psi \)) gives rise to a multiplication operator \( M_\theta \) (or \( M_\psi \)) on \( CV_0(X) \) (or \( CV_0(X, E) \)). For instance, the polynomial functions on \( \mathbb{R} \) induce continuous linear operators on \( CV_0(\mathbb{R}) \), where \( V = \{ \lambda x_K : \lambda > 0, \ K \subset \mathbb{R}, \ K \) compact \( \} \).

Now, we give certain examples of functions which do not induce multiplication operators.

2.8 Example. Let \( u : \mathbb{N} \rightarrow \mathbb{R}^+ \) be defined as \( u(n) = n \), for every \( n \in \mathbb{N} \) and let \( V = \{ \lambda u : \lambda \geq 0 \} \). Then \( V \) is a system of weights on \( \mathbb{N} \) with discrete topology. Let \( \theta : \mathbb{N} \rightarrow \mathbb{C} \) be defined as \( \theta(n) = n \), for every \( n \in \mathbb{N} \). Then \( \theta | \theta | \) is a system of weights on \( \mathbb{N} \) and \( V | \theta | \not\in V \). Thus \( M_\theta \) is not a multiplication operator on \( CV_0(\mathbb{N}) \). In fact, \( M_\theta \) is not even an into map. To see this, let \( f(n) = 1/n^2 \). Then \( f \in CV_0(\mathbb{N}) \) but \( \theta f \not\in CV_0(\mathbb{N}) \).
2.9 Example. Let $\mathbb{N}$ be the set of natural numbers with discrete topology and $V$ be the system of positive constant weights on $\mathbb{N}$. Then $CV_0(\mathbb{N}) = C_0$, the Banach space of all null sequences of complex numbers. Let $\theta: \mathbb{N} \rightarrow \mathbb{C}$ be the identity map. Then $V|\theta| \notin V$ and $M_\theta$ is not a multiplication operator on $C_0$. Moreover, $M_\theta$ is not even an onto map. If $f(n) = 1/n$, then $f \in C_0$ but $\theta f \notin C_0$.

2.10 Example. Let $\mathbb{R}^+$ be the set of positive reals with usual topology and let $v: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be defined as $v(x) = 1/x$, for every $x \in \mathbb{R}^+$. Let $V = \{\lambda v: \lambda \geq 0\}$ and let $\theta: \mathbb{R}^+ \rightarrow \mathbb{C}$ be defined as $\theta(x) = x^2$. Then $\theta$ does not induce a multiplication operator $M_\theta$ on $CV_0(\mathbb{R}^+)$.  

References


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