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MULTIPLICATION OPERATORS ON WEIGHTED SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS

R. K. SINGH AND JASBIR SINGH MANHAS

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Abstract

If V is a system of weights on a completely regular Hausdorff space X and E is a locally convex space, then $CV_0(X, E)$ and $CV_b(X, E)$ are locally convex spaces of vector-valued continuous functions with topologies generated by seminorms which are weighted analogues of the supremum norm. In this paper we characterise multiplication operators on these spaces induced by scalar-valued and vector-valued mappings. Many examples are presented to illustrate the theory.

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1. Introduction

Let X be a non-empty set, let E be a topological algebra and let $T(X, E)$ be a topological vector space of functions from X to E. Let $\theta: X \to \mathbb{C}$ and $\psi: X \to E$ be two mappings. Then scalar multiplication and multiplication in E give rise to two linear transformations M_θ and M_ψ from $T(X, E)$ to the linear space $L(X, E)$ of all functions from X to E, defined as $M_\theta f = \theta \cdot f$ and $M_\psi f = \psi f$, where the product of functions is defined pointwise. In case M_θ and M_ψ take $T(X, E)$ into itself and they are continuous, they are called multiplication operators on $T(X, E)$ induced by θ and ψ respectively. These operators have been the subject matter of study for a long time on different function spaces, especially on L^p-spaces, and they have played a very important role in the study of operators on Hilbert spaces.

In this note we have endeavoured to study multiplication operators on those weighted spaces of scalar-valued and vector-valued continuous functions which come in contact with topological dynamics.

2. Preliminaries

Let X be a completely regular Hausdorff space and E a Hausdorff locally convex topologial vector space over $\mathbb{C}$. Let $cs(E)$ be the set of all continuous seminorms on E. By $C(X, E)$ we mean the collection of all continuous functions from X into E.

A non-negative upper-semicontinuous function on X will be called a weight on X. If V is a set of weights on X such that, given any $x \in X$, there is some $v \in V$ for which $v(x) > 0$, we write $V > 0$. A set V of weights on X is said to be directed upward provided that, for every pair $v_1, v_2 \in V$ and $\lambda > 0$, there exists $v \in V$ so that $\lambda v_i \leq v$ (pointwise on X) for $i = 1, 2$. We hereafter assume that sets of weights are directed upward. A set V of weights on X which additionally satisfies $V > 0$ will be referred to as a system of weights on X.

Now, taking a system V of weights on X and a locally convex space E, we consider the following vector spaces of continuous functions associated with the triple (X, V, E):

$$CV_0(X, E) = \{f \in C(X, E) : vf \text{ vanishes at infinity on } X \text{ for all } v \in V\};$$

$$CV_b(X, E) = \{f \in C(X, E) : vf(X) \text{ is bounded in } E \text{ for all } v \in V\}.$$

Obviously $CV_0(X, E)$, $CV_b(X, E)$ are vector spaces and $CV_0(X, E) \subseteq CV_b(X, E)$. Now, let $v \in V$, $q \in cs(E)$ and $f \in C(X, E)$. If we put

$$P_{v,q}(f) = \operatorname{Sup}\{v(x)q(f(x)) : x \in X\}$$

then $P_{v,q}$ can be regarded as a seminorm on either $CV_b(X, E)$ or $CV_0(X, E)$, and the family $\{P_{v,q} : v \in V, q \in cs(E)\}$ defines the locally convex topology on each of these two spaces.

In case $E = \mathbb{C}$, we will omit E from our notation and write $CV_0(X)$ in place of $CV_0(X, \mathbb{C})$. We also write P_v in place of $P_{v,q}$ for each $v \in V$, where $q(z) = |z|$, $z \in \mathbb{C}$. Moreover, if $E = (E, q)$ is any normed linear space and $v \in V$, we write P_v instead of $P_{v,q}$. We shall denote by $B_{v,q}$ the closed unit ball corresponding to the seminorm $P_{v,q}$. In case $E = (E, q)$, we simply write B_v.

If U and V are two systems of weights on X, we write $U \leq V$ whenever given $u \in U$, there exists $v \in V$ such that $u \leq v$. In this case, we then clearly have that $CV_0(X, E) \subseteq CU_0(X, E)$ and $CV_b(X, E) \subseteq CU_b(X, E)$,

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as well as that the inclusion map is continuous in both instances. If $U \leq V$ and $V \leq U$, then U and V are equivalent systems of weights on X and we denote this by $U \approx V$.

The spaces $CV_0(X)$ and $CV_b(X)$ were first introduced by Nachbin [6] and $CV_0(X, E)$, $CV_b(X, E)$ were subsequently considered in detail by Bierstedt [2] and Prolla [7].

Now, we shall give some examples of these spaces. Let X be a completely regular Hausdorff space. We denote by χ_S, the characteristic function of a subset S of X. We distinguish four systems of weights on X, namely

$U = \{\lambda\chi_K : \lambda > 0,\ K \subset X,\ K \text{ compact}\}$,

$U' = C_c^+(X)$, the set of all positive continuous functions with compact supports,

$V = K^+(X)$, the set of all positive constant functions on X and

$V' = C_0^+(X)$, the set of all positive continuous functions vanishing at infinity. Further, if E is a locally convex space, then we define

$$C_0(X, E) = \{f \in C(X, E) : f \text{ vanishes at infinity on } X\},$$
$$C_b(X, E) = \{f \in C(X, E) : f(X) \text{ is bounded in } E\}.$$

EXAMPLE. Let X be a completely regular Hausdorff space and let E be a locally convex space. Then

(i) $CU_0(X, E) = CU_b(X, E) = (C(X, E), k)$ where k denotes the compact open topology,

(ii) $CU_0'(X, E) = CU_b'(X, E) = (C(X, E), k)$,

(iii) $CV_0(X, E) = (C_0(X, E), u)$ and $CV_b(X, E) = (C_b(X, E), u)$, where u denotes the topology of uniform convergence on X and

(iv) $CV_0'(X, E) = CV_b'(X, E) = (C_b(X, E), \beta_0)$, where β_0 denotes the strict topology.

2. Functions inducing multiplication operators

In developing our characterization of those functions $\theta: X \to \mathbb{C}$ (or $\psi: X \to E$) which induce multiplication operators on weighted spaces of type $CV_0(X)$ and $CV_0(X, E)$ we work under the following modest requirements.

(2.a) X is a completely regular Hausdorff space.

(2.b) E is a locally convex space such that there exists a vector $s \in E$ for which $p(s) \neq 0$, for every $p \in cs(E)$.

(2.c) V is a system of weights on X

(2.d) Corresponding to each $x \in X$, there exists $f_x \in CV_0(X)$ such that $f_x(x) \neq 0$.

In case X happens to be locally compact, (2.d) is automatically satisfied. For a continuous function $\theta: X \to \mathbb{C}$ (or $\psi: X \to E$), the set

$$V|\theta| = \{v|\theta|: v \in V\}$$

(or for every $p \in cs(E)$, $Vp \circ \psi = \{vp \circ \psi: v \in V\}$) is a directed set of weights on X since [9, Theorem 2.2] Summers has shown that the product of two non-negative upper semicontinuous functions is non-negative and upper semicontinuous. In case $\theta: X \to \mathbb{C}$ (or $\psi: X \to E$) is non-zero at each point of X, $V|\theta|$ (or $Vp \circ \psi$) is a system of weights on X.

In the following theorem we characterize multiplication operators on $CV_0(X)$ induced by scalar-valued functions

2.1 THEOREM. *Let $\theta: X \to \mathbb{C}$ be a continuous function. Then $M_\theta: CV_0(X) \to CV_0(X)$ is a multiplication operator if and only if $V|\theta| \leq V$.*

PROOF. First, suppose $V|\theta| \leq V$. Then for every $v \in V$, there exists $u \in V$ such that $v|\theta| \leq u$ (pointwise on X). We show that M_θ is a continuous linear operator on $CV_0(X)$. Clearly M_θ is linear on $CV_0(X)$. In order to prove the continuity of M_θ on $CV_0(X)$, it is enough to show that M_θ is continuous at the origin. For this, suppose $\{f_\alpha\}$ is a net in $CV_0(X)$ such that

$$P_v(f_\alpha) \to 0, \quad \text{for every } v \in V.$$

Now,

$$\begin{aligned} P_v(\theta f_\alpha) &= \text{Sup}\{v(x)|\theta(x)||f_\alpha(x)|: x \in X\} \\ &\leq \text{Sup}\{u(x)|f_\alpha(x)|: x \in X\} \\ &= P_u(f_\alpha) \to 0. \end{aligned}$$

This proves the continuity of M_θ at the origin and hence M_θ is continuous on $CV_0(X)$.

Conversely, suppose M_θ is a continuous linear operator on $CV_0(X)$. We shall show that $V|\theta| \leq V$. Let $v \in V$. Since M_θ is continuous at the origin, there exists $u \in V$ such that $M_\theta(B_u) \subseteq B_v$. We claim that $v|\theta| \leq 2u$. Take $x_0 \in X$ and set $u(x_0) = \varepsilon$. In case $\varepsilon > 0$, $N = \{x \in X: u(x) < 2\varepsilon\}$ is an open neighbourhood of x_0. Thus, according to [6, Lemma 2], there exists $f \in CV_0(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - N) = 0$. Let $g = (2\varepsilon)^{-1} f$. Then clearly $g \in B_u$. Since $M_\theta(B_u) \subseteq B_v$, we have $\theta g \in B_v$ and this yields that

$$v(x)|\theta(x)||g(x)| \leq 1, \quad \text{for every } x \in X.$$

From this it follows that

$$v(x)|\theta(x)||f(x)| \leq 2\varepsilon, \quad \text{for every } x \in X.$$

This implies that

$$v(x_0)|\theta(x_0)| \leq 2u(x_0).$$

Now, suppose $u(x_0) = 0$ and $v(x_0)|\theta(x_0)| > 0$. If we put $\varepsilon = v(x_0)|\theta(x_0)|/2$ and set $N = \{x \in X : u(x) < \varepsilon\}$, then N would be an open neighbourhood of x_0 and we could again find $f \in CV_0(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - N) = 0$. Now let $g = \varepsilon^{-1} f$. Then clearly $g \in B_u$ and therefore $\theta g \in B_v$. Hence

$$v(x)|\theta(x)||g(x)| \leq 1, \quad \text{for every } x \in X.$$

This implies that

$$v(x)|\theta(x)||f(x)| \leq \varepsilon, \quad \text{for every } x \in X.$$

From this it follows that

$$v(x_0)|\theta(x_0)| \leq \frac{v(x_0)|\theta(x_0)|}{2}$$

which is impossible. This proves our claim and hence the proof is complete.

Now, we shall characterise multiplication operators on $CV_0(X, E)$ induced by scalar-valued functions.

2.2 Theorem. *Let $\theta : X \to \mathbb{C}$ be a continuous function. Then*

$$M_\theta : CV_0(X, E) \to CV_0(X, E)$$

is a multiplication operator if and only if $V|\theta| \leq V$.

Proof. First of all, let us suppose $V|\theta| \leq V$. Then for every $v \in V$, there exists $u \in V$ such that $v|\theta| \leq u$ (pointwise on X). We shall show that M_θ is a continuous linear operator on $CV_0(X, E)$. Obviously M_θ is linear on $CV_0(X, E)$. It suffices to show that M_θ is a continuous linear operator at the origin. To prove this, let $\{f_\alpha\}$ be a net in $CV_0(X, E)$ such that for every $v \in V$, $q \in cs(E)$, $P_{v,q}(f_\alpha) \to 0$. Then

$$\begin{aligned} P_{v,q}(\theta f_\alpha) &= \operatorname{Sup}\{v(x)|\theta(x)|q(f_\alpha(x)) : x \in X\} \\ &\leq \operatorname{Sup}\{u(x)q(f_\alpha(x)) : x \in X\} \\ &= P_{u,q}(f_\alpha) \to 0. \end{aligned}$$

This proves the continuity of M_θ at the origin and hence M_θ is a continuous linear operator on $CV_0(X, E)$.

Conversely, suppose $M_\theta : CV_0(X, E) \to CV_0(X, E)$ is a continuous linear operator. Then we shall show that $V|\theta| \leq V$. Let $v \in V$. Since M_θ is continuous at the origin, therefore for every $v \in V$, $p \in cs(E)$, there exists $u \in V$, $q \in cs(E)$ such that $M_\theta(B_{u,q}) \subseteq B_{v,p}$. By our assumption there exists a vector $s \in E$ such that $p(s) \neq 0$, for every $p \in cs(E)$. Let $\alpha = p(s)/q(s)$. Then $\alpha > 0$. We claim that $\alpha v|\theta| \leq 2u$ (pointwise on X). Fix $x_0 \in X$ and set $u(x_0) = \varepsilon$. In case $\varepsilon > 0$, $N = \{x \in X : u(x) < 2\varepsilon\}$ is an open neighbourhood of x_0 and therefore by [6, Lemma 2] there exists $f \in CV_0(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - N) = 0$. Define $g(x) = f(x)s$, for every $x \in X$. Then clearly $g \in CV_0(X, E)$ and for every $p \in cs(E)$, $0 \leq (p \circ g) \leq p(s)$, $(p \circ g)(x_0) = p(s)$ and $(p \circ g)(X - N) = 0$.

Let $h = (2\varepsilon)^{-1} g/q(s)$. Then clearly $h \in B_{u,q}$ and this yields that $\theta \cdot h \in B_{v,p}$. Hence $v(x)|\theta(x)|p(h(x)) \leq 1$, for every $x \in X$. From this, it follows that

$$v(x)|\theta(x)|\frac{1}{q(s)}p(g(x)) \leq 2\varepsilon, \quad \text{for every } x \in X.$$

This implies that

$$v(x_0)|\theta(x_0)|\frac{p(s)}{q(s)} \leq 2u(x_0).$$

Thus

$$\alpha v(x_0)|\theta(x_0)| \leq 2u(x_0).$$

On the other hand, suppose $u(x_0) = 0$ and $\alpha v(x_0)|\theta(x_0)| > 0$. Put $\varepsilon = \alpha v(x_0)|\theta(x_0)|/2$. Then $N = \{x \in X : u(x) < \varepsilon\}$ is an open neighbourhood of x_0 and therefore again by [6, Lemma 2] there exists an $f \in CV_0(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - N) = 0$. Again, define $g(x) = f(x)s$, for every $x \in X$. Then $g \in CV_0(X, E)$ and for every $p \in cs(E)$, $0 \leq (p \circ g) \leq p(s)$, $(p \circ g)(x_0) = p(s)$ and $(p \circ g)(X - N) = 0$. Consider $h = g/\varepsilon q(s)$. Then $h \in B_{u,q}$ and therefore $\theta h \in B_{v,p}$. Hence

$$v(x)|\theta(x)|p(h(x)) \leq 1, \quad \text{for every } x \in X.$$

This implies that

$$v(x)|\theta(x)|\frac{1}{q(s)}p(g(x)) \leq \varepsilon, \quad \text{for every } x \in X.$$

From this it follows that

$$v(x_0)|\theta(x_0)|\frac{p(s)}{q(s)} \leq \frac{p(s)}{q(s)}\frac{v(x_0)|\theta(x_0)|}{2}.$$

Thus $\alpha v(x_0)|\theta(x_0)| \leq \alpha v(x_0)|\theta(x_0)|/2$, which is impossible. Hence our claim is established and the proof is completed.

In order to prove the next theorem, we shall need the following definitions.

Let E be a locally convex algebra with jointly continuous multiplication. It clearly follows that for each $p \in cs(E)$, there exists a $q \in cs(E)$ such that $p(xy) \leq q(x)q(y)$, for every $x, y \in E$. A seminorm p on E is said to be submultiplicative if $p(xy) \leq p(x)p(y)$, for every $x, y \in E$. In [5] Michael defines E to be a locally multiplicatively convex algebra, or in short an lmc algebra, if there exists a base of neighbourhoods of zero consisting of idempotent absolutely convex sets, or equivalently if its topology is defined by a collection of submultiplicative seminorms. Clearly, multiplication in an lmc algebra is always jointly continuous. For more details and examples of lmc algebras we refer to [3] and [5]. Let $\mathscr{P}$ be a family of submultiplicative seminorms inducing the topology of E. Then $\mathscr{P}$ is a subfamily of $cs(E)$. In [10, c.2.3] Zelazako has noted that for any lmc algebra E with unit e, the family $\mathscr{P}$ can be chosen in such a way that $p(e) = 1$, for every $p \in \mathscr{P}$. So we can assume that $\mathscr{P}$ is such a family in this case.

Now, we shall give a characterisation of multiplication operators on $CV_0(X, E)$ induced by vector-valued functions.

2.3 THEOREM. *Let E be a (locally multiplicatively convex) lmc algebra with unit e and let $\psi: X \to E$ be a continuous function. Then*

$$M_\psi: CV_0(X, E) \to CV_0(X, E)$$

is a multiplication operator if and only if $Vp \circ \psi \leq V$, for every $p \in \mathscr{P}$.

PROOF. Suppose $Vp \circ \psi \leq V$, for every $p \in \mathscr{P}$. Then for every $v \in V$, there exists $u \in V$ such that $vp \circ \psi \leq u$ (pointwise on X). We shall show that the mapping $M_\psi: CV_0(X, E) \to C(X, E)$, defined by $M_\psi f = \psi f$, where product is pointwise, is a continuous linear operator on $CV_0(X, E)$. We shall establish the continuity of M_ψ at the origin. For this, let $\{f_\alpha\}$ be a net in $CV_0(X, E)$ such that for every $v \in V$, $q \in \mathscr{P}$, $P_{v,q}(f_\alpha) \to 0$. Then

$$\begin{aligned} P_{v,q}(\psi f_\alpha) &= \operatorname{Sup}\{v(x)q(\psi(x)f_\alpha(x)) : x \in X\} \\ &\leq \operatorname{Sup}\{v(x)q(\psi(x))q(f_\alpha(x)) : x \in X\} \\ &\leq \operatorname{Sup}\{u(x)q(f_\alpha(x)) : x \in X\} \\ &= P_{u,q}(f_\alpha) \to 0. \end{aligned}$$

This proves that M_ψ is continuous at the origin and hence a continuous linear operator on $CV_0(X, E)$.

Conversely, suppose $M_\psi: CV_0(X, E) \to CV_0(X, E)$ is a continuous linear operator. We shall show that $Vp \circ \psi \leq V$, for every $p \in \mathscr{P}$. Let $v \in V$ and $p \in \mathscr{P}$. Since M_ψ is continuous at the origin, there exist $u \in V$ and $q \in \mathscr{P}$ such that $M_\psi(B_{u,q}) \subseteq B_{v,p}$. We claim that $vp \circ \psi \leq 2u$ (pointwise on X). Fix $x_0 \in X$ and set $u(x_0) = \varepsilon$. In case $\varepsilon > 0$, $N = \{x \in X : u(x) < 2\varepsilon\}$ is an open neighbourhood of x_0 and therefore according to [6, Lemma 2] there exists $f \in CV_0(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - N) = 0$.

Define $g(x) = f(x)e$, for every $x \in X$, where e is the unit in E. Then $g \in CV_0(X, E)$ and for every $p \in \mathscr{P}$, $0 \leq (p \circ g) \leq 1$, $(p \circ g)(x_0) = 1$ and

$(p \circ g)(X - N) = 0$. If $h = (2\varepsilon)^{-1}g$, then $h \in B_{u,q}$ and hence $\psi h \in B_{v,p}$. From this it follows that $v(x)p(\psi(x)h(x)) \leq 1$, for every $x \in X$. This implies that

$$v(x)p(\psi(x)f(x)e) \leq 2\varepsilon, \quad \text{for every } x \in X.$$

Thus $v(x_0)p(\psi(x_0)) \leq 2u(x_0)$. On the other hand, suppose $u(x_0) = 0$ and $v(x_0)p(\psi(x_0)) > 0$. Set $\varepsilon = v(x_0)p(\psi(x_0))/2$. Then $N = \{x \in X : u(x) < \varepsilon\}$ is an open neighbourhood of x_0 and therefore again by [6, Lemma 2] there exists $f \in CV_0(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - N) = 0$.

We define $g(x) = f(x)e$, for evey $x \in X$, where e is the unit in E. Then $g \in CV_0(X, E)$ and for every $p \in \mathscr{P}$, $0 \leq (p \circ g) \leq 1$, $(p \circ g)(x_0) = 1$ and $(p \circ g)(X - N) = 0$. Choose $h = \varepsilon^{-1}g$. Then clearly $h \in B_{u,q}$ and therefore $\psi h \in B_{v,p}$. This implies that $v(x)p(\psi(x)h(x)) \leq 1$, for every $x \in X$. From this, it follows that $v(x)p(\psi(x)g(x)) \leq \varepsilon$, for every $x \in X$. Further, we get

$$v(x_0)p(\psi(x_0)f(x_0)e) \leq \frac{v(x_0)p(\psi(x_0))}{2}.$$

Thus $v(x_0)p(\psi(x_0)) \leq v(x_0)p(\psi(x_0))/2$ which is impossible and this establishes our claim. This completes the proof of the theorem.

2.4 Remark. Note that if $\theta: X \to \mathbb{C}$ (or $\psi: X \to E$) is a bounded continuous complex-valued (or vector-valued) function on X, then clearly M_θ (or M_ψ) is a multiplication operator on $CV_0(X)$ (or $CV_0(X, E)$) for any system of weights V.

If V is the system of weights generated by the characteristic functions of compact sets,then it turns out that every continuous map induces a multiplication operator. This we shall establish in the following theorem.

2.5 Theorem. *Let X be a completely regular Hausdorff space and let*

$$V = \{\lambda\chi_K : \lambda > 0 \text{ and } K \subset X,\, K \text{ compact}\}.$$

(i) *Every continuous $\theta: X \to \mathbb{C}$ induces a multiplication operator on $CV_0(X)$ (or $CV_0(X, E)$).*

(ii) *Every continuous $\psi: X \to E$, a locally convex algebra with jointly continuous multiplication, induces a multiplication operator M_ψ on $CV_0(X, E)$*

Proof. (i) In order to prove that M_θ is a continuous linear operator on $CV_0(X)$ (or $CV_0(X, E)$), it is enough to show that for every $v \in V$, there exists $u \in V$ such that $v|\theta| \leq u$ (pointwise on X). Let $v \in V$. Then $v = \lambda\chi_K$, where K is a compact subset of X. Let $m = \operatorname{Sup}\{|\theta(x)| : x \in K\}$ and choose $u = \lambda m\chi_K$. Then $u \in V$. Since $|\theta(x)| \leq m$, for every $x \in K$, we have

$$\lambda\chi_K(x)|\theta(x)| \leq \lambda m\chi_K(x), \qquad x \in K.$$

Hence $v(x)|\theta(x)| \leq u(x)$, for every $x \in K$. If $x \in X - K$, then the above inequality is obviously true. Thus we have shown that $v(x)|\theta(x)| \leq u(x)$, for every $x \in X$, and hence by Theorems 2.1 and 2.2, it follows that M_θ is a multiplication operator on $CV_0(X)$ (or $CV_0(X, E)$).

(ii) In view of Theorem 2.3, it is sufficient to establish the inequality $Vp \circ \psi \leq V$, for every $p \in cs(E)$, that is, for every $v \in V$, there exists $u \in V$ such that $vp \circ \psi \leq u$ (pointwise on X). Let $v \in V$ and $p \in cs(E)$. Then $v = \lambda\chi_K$, K a compact subset of X. Let $m = \operatorname{Sup}\{p(\psi(x)) : x \in K\}$ and choose $u = \lambda m \chi_K$. Then $u \in V$. Since $p(\psi(x)) \leq m$, $x \in K$, we have

$$\lambda\chi_K(x)p(\psi(x)) \leq \lambda m \chi_K(x), \qquad x \in K.$$

This implies that $v(x)p(\psi(x)) \leq u(x)$, for every $x \in K$. If $x \in X - K$, then obviously $v(x)p(\psi(x)) \leq u(x)$. Thus $v(x)p(\psi(x)) \leq u(x)$, for every $x \in X$. This completes the proof of the theorem.

2.6 COROLLARY. *Let X have the discrete topology and $V = \{\lambda\chi_K : \lambda \geq 0, K \subset X, K$ a finite set$\}$. Then every function $\theta: X \to \mathbb{C}$ (or $\psi: X \to E$) induces a multiplication operator M_θ (or M_ψ) on $CV_0(X)$ (or $CV_0(X, E)$).*

2.7 REMARK. (i) In Theorem 2.5, if we replace the system of weights $V = \{\lambda\chi_K : \lambda > 0, K \subset X, K$ compact$\}$ by $U = C_c^+(X)$, the set of all positive continuous functions on X with compact supports, then the new result is also true.

(ii) If X is a locally compact space, then $V = \{\lambda\chi_K : \lambda \geq 0, K \subset X, K$ compact subset of $X\}$ and $U = C_c^+(X)$ are equivalent, and otherwise $V \leq U$.

(iii) In Theorems 2.1–2.3 and 2.5 if we replace $CV_0(X)$ and $CV_0(X, E)$ by $CV_b(X)$ and $CV_b(X, E)$, then all the new results are also true.

(iv) From Theorem 2.5, we noted that if $\theta: X \to \mathbb{C}$ (or $\psi: X \to E$) is an unbounded continuous function, even then θ (or ψ) gives rise to a multiplication operator M_θ (or M_ψ) on $CV_0(X)$ (or $CV_0(X, E)$). For instance, the polynomial functions on $\mathbb{R}$ induce continuous linear operators on $CV_0(\mathbb{R})$, where $V = \{\lambda\chi_K : \lambda > 0, K \subset \mathbb{R}, K$ compact$\}$

Now, we give certain examples of functions which do not induce multiplication operators.

2.8 EXAMPLE. Let $v: \mathbb{N} \to \mathbb{R}^+$ be defined as $v(n) = n$, for every $n \in \mathbb{N}$ and let $V = \{\lambda v : \lambda \geq 0\}$. Then V is a system of weights on $\mathbb{N}$ with discrete topology. Let $\theta: \mathbb{N} \to \mathbb{C}$ be defined as $\theta(n) = n$, for every $n \in \mathbb{N}$. Then $v|\theta|$ is a system of weights on $\mathbb{N}$ and $V|\theta| \not\leq V$. Thus M_θ is not a multiplication operator on $CV_0(\mathbb{N})$. In fact, M_θ is not even an into map. To see this, let $f(n) = 1/n^2$. Then $f \in CV_0(\mathbb{N})$ but $\theta f \notin CV_0(\mathbb{N})$.

2.9 EXAMPLE. Let $\mathbb{N}$ be the set of natural numbers with discrete topology and V be the system of positive constant weights on $\mathbb{N}$. Then $CV_0(\mathbb{N}) = C_0$, the Banach space of all null sequences of complex numbers. Let $\theta: \mathbb{N} \to \mathbb{C}$ be the identity map. Then $V|\theta| \not\leq V$ and M_θ is not a multiplication operator on C_0. Moreover, M_θ is not even an into map. If $f(n) = 1/n$, then $f \in C_0$ but $\theta f \notin C_0$.

2.10 EXAMPLE. Let $\mathbb{R}^+$ be the set of positive reals with usual topology and let $v: \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $v(x) = 1/x$, for every $x \in \mathbb{R}^+$. Let $V = \{\lambda v : \lambda \geq 0\}$ and let $\theta: \mathbb{R}^+ \to \mathbb{C}$ be defined as $\theta(x) = x^2$. Then θ does not induce a multiplication operator M_θ on $CV_0(\mathbb{R}^+)$.

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University of Jammu
Jammu 180001
India