ON THE CUBE OF A GRAPH

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The $n$th power $G^n$ of a connected graph $G$ is the graph with the same point set as $G$ and where two points $u$ and $v$ are adjacent in $G^n$ if and only if the distance between $u$ and $v$ in $G$ is at most $n$. The graph $G^2$ is called the square of $G$ while $G^3$ is referred to as the cube of $G$.

It has been conjectured by M.D. Plummer, among others, that the square of every nonseparable (2-connected) graph is hamiltonian; however, it is known (although evidently never published) that the cube of any connected graph (with 3 or more points) is hamiltonian. In this note we prove the stronger result that the cube of any connected graph is hamiltonian-connected, i.e., every two points are joined by some hamiltonian path.

THEOREM. The cube of every connected graph is hamiltonian-connected.

Proof. Let $G$ be an arbitrary connected graph with $p$ points, and let $T$ be a spanning tree of $G$. Clearly, if $T^3$ is hamiltonian-connected, it follows immediately that $G^3$ is hamiltonian-connected.

We proceed by induction on $p$, the result being obvious for small values of $p$.

Assume then for all trees $T$ with fewer than $p$ points that $T^3$ is hamiltonian-connected. Let $u$ and $v$ be any two points of $T$. Since $T$ is a tree, there exists a unique path $P$ between $u$ and $v$. We now consider two cases

Case 1. $u$ and $v$ are adjacent. Let $x$ be the line joining $u$ and $v$, and consider the disconnected forest (two trees) $T - x$ obtained from $T$ by removing $x$. Denote by $T_u$ and $T_v$ the trees containing $u$ and $v$, respectively. By hypothesis $T^3_u$ and $T^3_v$ are hamiltonian-connected. Let $u_1$ be any point of $T_u$ adjacent to $u$ if $T_u$ is non-trivial, and let $u_1 = u$ otherwise; the point $v_1$ in $T_v$ is selected analogously. Note that in $T^3$ the points $u_1$ and $v_1$ are adjacent since the distance between $u_1$ and $v_1$ in $T$ is at most 3.

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Let $P_u$ be a hamiltonian path of $T_u^3$ from $u$ to $u_1$ and $P_v$ a hamiltonian path of $T_v^3$ from $v_1$ to $v$. Thus the path $P_u$ followed by the line $u_1v_1$ and then the path $P_v$ is a hamiltonian path of $T_v^3$ from $u$ to $v$.

**Case 2.** $u$ and $v$ are not adjacent. Let $y = uw$ be the line of $P$ incident with $u$, and consider the forest $T - y$. Again, let $T_u$ denote the tree of $T - y$ containing $u$ and $T_w$ the tree containing $w$.

By hypothesis, there exists a hamiltonian path $P_w$ of $T_w^3$ from $w$ to $v$. Select $u_1$ in $T_u$ as a point adjacent to $u$ (or $u_1 = u$ if $T_u$ is trivial), and let $P_u$ be a hamiltonian path of $T_u^3$ from $u$ to $u_1$.

Because the distance between $u_1$ and $w$ does not exceed 2, $u_1$ and $w$ are adjacent in $T_u^3$ so that the path of $T_u^3$ beginning with $P_u$ and followed by the line $u_1w$ and then the path $P_w$ is hamiltonian.

This completes the proof.

Since every hamiltonian-connected graph $G$ with $p \geq 3$ is hamiltonian we obtain as a corollary the previously mentioned result.

**COROLLARY.** The cube of every connected graph with $p \geq 3$ points is hamiltonian.