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GORENSTEIN MODEL STRUCTURES AND GENERALIZED DERIVED CATEGORIES

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Abstract In a paper from 2002, Hovey introduced the Gorenstein projective and Gorenstein injective model structures on R-Mod, the category of R-modules, where R is any Gorenstein ring. These two model structures are Quillen equivalent and in fact there is a third equivalent structure we introduce: the Gorenstein flat model structure. The homotopy category with respect to each of these is called the stable module category of R. If such a ring R has finite global dimension, the graded ring $R[x]/(x^2)$ is Gorenstein and the three associated Gorenstein model structures on $R[x]/(x^2)$-Mod, the category of graded $R[x]/(x^2)$-modules, are nothing more than the usual projective, injective and flat model structures on $\mathrm{Ch}(R)$, the category of chain complexes of R-modules. Although these correspondences only recover these model structures on $\mathrm{Ch}(R)$ when R has finite global dimension, we can set $R = \mathbb{Z}$ and use general techniques from model category theory to lift the projective model structure from $\mathrm{Ch}(\mathbb{Z})$ to $\mathrm{Ch}(R)$ for an arbitrary ring R. This shows that homological algebra is a special case of Gorenstein homological algebra. Moreover, this method of constructing and lifting model structures carries through when $\mathbb{Z}[x]/(x^2)$ is replaced by many other graded Gorenstein rings (or Hopf algebras, which lead to monoidal model structures). This gives us a natural way to generalize both chain complexes over a ring R and the derived category of R and we give some examples of such generalizations.

Keywords: Gorenstein ring; derived category; model category; generalized chain complex; stable module category

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1. Introduction

There are two simple ways that homotopy theory occurs in algebra. One is through chain complexes, leading to the derived category of a ring. Another is through quasi-Frobenius rings, or more generally Gorenstein rings, leading to the stable module category of the ring. The purpose of this paper is to point out that these two uses of homotopy theory are closely related. In fact, for a nice enough ring, its derived category is nothing more than the stable module category of some particular graded Gorenstein ring A. The derived

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category of an *arbitrary* ring can then be recovered using a well-known lifting technique from model category theory. Taking A to be another suitably chosen graded Gorenstein ring leads to new generalized derived categories of a ring. We discuss a few of these in this paper but there is room for much more exploration of these generalized derived categories.

In more detail, a (possibly non-commutative) ring R is called *Gorenstein* if R is both left and right Noetherian and has finite injective dimension when thought of as either a left or right R-module. This generalization of the usual notion of commutative Gorenstein rings was introduced by Iwanaga in [**15**] and [**16**], so such rings are sometimes called *Iwanaga–Gorenstein*. For a quasi-Frobenius ring, where projective and injective modules coincide, one gets the stable module category by identifying two maps when their difference factors through a projective. This corresponds to a model structure on R-modules where every module is both cofibrant and fibrant, and the trivial objects are the injective modules. For a Gorenstein ring, the trivial objects are instead the modules of finite injective dimension. But one now has to choose: one can have every module be fibrant, in which case the cofibrant objects are called Gorenstein projective modules, or one can have every module be cofibrant, in which case the fibrant objects are called Gorenstein injective modules. These model structures were introduced in [**14**], though Martsinkovsky [**19**] had also considered related issues. There is also an intermediate module structure that we introduce in this paper, where the cofibrant objects are Gorenstein flat modules and the fibrant objects are cotorsion modules. The homotopy categories of these model structures are all equivalent and the resulting triangulated category is called the *stable module category* of the Gorenstein ring R. Gorenstein projective and injective modules were first introduced by Enochs and Jenda in [**4**] and further information about them and Gorenstein flat modules can be found in [**5**].

Now suppose that K is a commutative Noetherian ring with finite global dimension, like $\mathbb{Z}$. Such a ring is automatically Gorenstein but the stable module category is trivial since every module has finite injective dimension. However, we could fatten K up slightly into a Gorenstein ring with infinite global dimension and then look at its stable module category. The simplest way to do this is to look at $A = K[x]/(x^2)$, which we think of as a graded ring by putting x in degree -1. We can then try to study K by looking at the stable module category of A. But an A-module is just a chain complex over K, and we prove in this paper that the A-modules of finite injective dimension are the exact complexes, and the Gorenstein projective (respectively Gorenstein injective, respectively Gorenstein flat) A-modules are the dg-projective (respectively dg-injective, respectively dg-flat) chain complexes. Thus this approach recovers the two standard model category structures (projective and injective) on chain complexes over K as well as the flat model structure of Gillespie [**9**]. Again, this is when K is commutative Noetherian and has finite global dimension. For more general rings R, it is still possible to recover the projective model structure on chain complexes. To do this one can take $K = \mathbb{Z}$ and lift the Gorenstein projective model structure on $\mathbb{Z}[x]/(x^2)$-modules to $R[x]/(x^2)$-modules, as we explain. We can lift the Gorenstein flat model structure as well, but we do not know whether this lifted model structure is the usual flat model structure.

We can fatten up the ring K in other ways too. Generalizing the above, one might fix an integer $k > 1$ and take $A = K[x]/(x^k)$, again grading it with x in degree -1. In this case an A-module is what we call a k-chain complex, since it is like a normal chain complex but with differentials satisfying $d^k = 0$ instead of $d^2 = 0$. Properties of these generalized chain complexes have been studied by several authors: see, for example, [**1**], [**17**], [**21**] and [**7**]. In analogy to the above situation, we can, for any ring R, lift the Gorenstein projective model structure on $\mathbb{Z}[x]/(x^k)$-modules to a projective model structure on $R[x]/(x^k)$-modules. In fact, we will see that all of the rings $A = K[x]/(x^k)$ are graded Gorenstein K-algebras that are flat over K and we prove in Theorem 4.1 that for such an A the Gorenstein projective model structure on A-Mod always lifts to a 'projective' model structure on $(A \otimes_K R)$-Mod, where R is any K-algebra. Therefore, we may define the *derived category of R with respect to A*, denoted by $\mathcal{D}_A(R)$, to be the associated homotopy category. The question is left unanswered here, but the authors are interested in whether or not these generalized derived categories of a ring R contain information about R other than what is already contained in the usual derived category. For example, is it possible to find rings R and S such that $\mathcal{D}(R) \simeq \mathcal{D}(S)$ and yet $\mathcal{D}_A(R) \not\simeq \mathcal{D}_A(S)$, where A is some (torsion-free) graded Gorenstein ring A?

Finally, we deal with the issue of tensor products. The category of A-modules will have a tensor product if A also happens to be a Hopf algebra over K. In this case one would like the lifted model structure on $(A \otimes_K R)$-Mod to be monoidal, that is, compatible with the tensor product as described in Chapter 4 of [**13**]. For example, $A = \mathbb{Z}[x]/(x^2)$ is a Hopf algebra over $\mathbb{Z}$ and for a ring R the associated tensor product on $(A \otimes_{\mathbb{Z}} R)$-Mod $\cong \mathrm{Ch}(R)$ corresponds exactly to the usual tensor product of chain complexes of R-modules. As we already know from [**13**, Chapter 4], the projective model structure is monoidal with respect to this tensor product. In general, we show in Theorem 5.4 that, when A is a K-projective cocommutative Hopf algebra over K and R is a commutative K-algebra, the projective model structure on $(A \otimes_K R)$-Mod is monoidal. We give examples using $A = \mathbb{Z}[x]/(x^2) \otimes_{\mathbb{Z}} \mathbb{Z}[x]/(x^2)$, and this leads to a monoidal model structure on the category of bicomplexes, and also $A = \mathbb{Z}_p[x]/(x^p)$, which leads to a monoidal model structure on p-chain complexes of modules over $\mathbb{Z}_p$.

The paper is arranged as follows. We start in § 2 by reviewing necessary details about the projective, injective and flat model structures on $\mathrm{Ch}(R)$ and by providing some of the relevant definitions and notation we will need. In § 3 we recall the notion of a Gorenstein ring and give the definition of a graded Gorenstein ring that we will use throughout the rest of the paper. We look at the motivating example of $R[x]/(x^2)$-modules (chain complexes over R) in this section. In particular, we show how the projective (respectively injective, respectively flat) model structure on $\mathrm{Ch}(R)$ is identical to the Gorenstein projective (respectively Gorenstein injective, respectively Gorenstein flat) model structure on $R[x]/(x^2)$-modules when R is a Noetherian ring with finite global dimension. After reading § 3 it will be clear that the usual derived category of a ring R can be thought of as the derived category of R with respect to $A = \mathbb{Z}[x]/(x^2)$. In § 4 we present our method of introducing generalizations of the usual derived category. Letting $A = \mathbb{Z}[x]/(x^k)$ $(k \geqslant 2)$, we describe a model structure on the category of k-chain complexes that is exactly the projective model structure on $\mathrm{Ch}(R)$ when $k = 2$. Finally, § 5 is where we assume that A is a Hopf algebra and prove that the associated projective model categories are monoidal.

2. Preliminaries

Let $\mathcal{A}$ be a bicomplete abelian category. In [**14**] Hovey laid out a correspondence between (nice enough) model structures on $\mathcal{A}$ and cotorsion pairs on $\mathcal{A}$. Essentially, a model structure on $\mathcal{A}$ is two complete cotorsion pairs $(\mathcal{Q}, \mathcal{R} \cap \mathcal{W})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$, where $\mathcal{Q}$ is the class of cofibrant objects, $\mathcal{R}$ is the class of fibrant objects and $\mathcal{W}$ is the class of trivial objects. Before giving the definition of a complete cotorsion pair, we point out that a model structure on $\mathcal{A}$ is determined from the above cotorsion pairs in the following way: the (trivial) cofibrations are the monomorphisms with (trivially) cofibrant cokernel, the (trivial) fibrations are the epimorphisms with (trivially) fibrant kernel and the weak equivalences are the maps that can be factored as a trivial cofibration followed by a trivial fibration.

Definition 2.1. A pair of classes $(\mathcal{F}, \mathcal{C})$ in an abelian category $\mathcal{A}$ is a cotorsion pair if the following conditions hold:

(1) $\mathrm{Ext}^1_{\mathcal{A}}(F, C) = 0$ for all $F \in \mathcal{F}$ and $C \in \mathcal{C}$;

(2) if $\mathrm{Ext}^1_{\mathcal{A}}(F, X) = 0$ for all $F \in \mathcal{F}$, then $X \in \mathcal{C}$;

(3) if $\mathrm{Ext}^1_{\mathcal{A}}(X, C) = 0$ for all $C \in \mathcal{C}$, then $X \in \mathcal{F}$.

Every abelian category $\mathcal{A}$ has the projective cotorsion pair $(\mathcal{P}, \mathcal{A})$ and the injective cotorsion pair $(\mathcal{A}, \mathcal{I})$, where $\mathcal{P}$ is the class of projectives, $\mathcal{I}$ is the class of injectives and $\mathcal{A}$ stands for the class of all objects in $\mathcal{A}$. When $\mathcal{A}$ has a tensor product there ought to be a flat cotorsion pair $(\mathcal{F}, \mathcal{C})$ as well, where here $\mathcal{F}$ is the class of flat (tensor-exact) objects and $\mathcal{C}$ is the class of cotorsion objects. For a proof that $(\mathcal{F}, \mathcal{C})$ is in fact a cotorsion pair when $\mathcal{C}$ is the category of R-modules, see, for example, [**5**]. In fact, [**5**] is also a good reference for both cotorsion pairs and cotorsion modules.

The cotorsion pair is said to have *enough projectives* if for any $X \in \mathcal{A}$ there is a short exact sequence $0 \to C \to F \to X \to 0$, where $C \in \mathcal{C}$ and $F \in \mathcal{F}$. We say it has *enough injectives* if it satisfies the dual statement. If both of these hold, we say the cotorsion pair is *complete*. All of the examples of cotorsion pairs in the last paragraph are complete when the category is R-Mod. The phrases 'enough projectives' and 'enough injectives' are standard in reference to cotorsion pairs and generalize the notion of 'enough projectives' and 'enough injectives' in a *category*. Indeed, $\mathcal{A}$ has enough projectives if and only if the cotorsion pair $(0, \mathcal{A})$ has enough projectives and $\mathcal{A}$ has enough injectives if and only if the cotorsion pair $(\mathcal{A}, \infty)$ has enough injectives.

Let R be a ring. We will always assume without mention that R has an identity, which we denote by 1. We denote the category of chain complexes over R by $\mathrm{Ch}(R)$. The differentials d of our chain complexes lower degree. Given an R-module M, we denote its n-sphere by $S^n(M)$. This is the chain complex with M in degree n and 0 elsewhere. We denote its n-disc by $D^n(M)$. This is the chain complex with M in degree n and $n-1$, $d_n = 1_M$ and 0 elsewhere.

Next we want to recall the projective, injective and flat model structures on $\mathrm{Ch}(R)$. We say a chain complex X is projective (respectively injective, respectively flat, respectively cotorsion) if it is exact and each cycle Z_nX is projective (respectively injective, respectively flat, respectively cotorsion). For projective and injective this is equivalent to the categorical definition. We denote these classes of chain complexes by $\tilde{\mathcal{P}}$, $\tilde{\mathcal{I}}$, $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{C}}$ respectively. A chain complex $X \in \mathrm{Ch}(R)$ is called dg-projective (respectively dg-flat) if each X_n is projective (respectively flat) and every chain map $f : X \to E$ with E an exact (respectively cotorsion) complex is null homotopic. We say a chain complex $Y \in \mathrm{Ch}(R)$ is dg-injective (respectively dg-cotorsion) if each Y_n is injective (respectively cotorsion) and every chain map $f : E \to Y$ with E an exact (respectively flat) complex is null homotopic. We denote the class of all dg-projective complexes (respectively dg-flat, respectively dg-injective, respectively dg-cotorsion) by $dg\tilde{\mathcal{P}}$ (respectively $dg\tilde{\mathcal{F}}$, respectively $dg\tilde{\mathcal{I}}$, respectively $dg\tilde{\mathcal{C}}$).

If we let $\mathcal{A}$ denote the class of all chain complexes and $\mathcal{E}$ the class of exact complexes, then $(dg\tilde{\mathcal{P}}, \mathcal{E} \cap \mathcal{A})$ and $(dg\tilde{\mathcal{P}} \cap \mathcal{E}, \mathcal{A})$ are complete cotorsion pairs. Using Hovey's correspondence theorem, these cotorsion pairs induce the usual projective model structure on $\mathrm{Ch}(R)$. Similarly, there are complete cotorsion pairs $(\mathcal{A}, dg\tilde{\mathcal{I}} \cap \mathcal{E})$ and $(\mathcal{A} \cap \mathcal{E}, dg\tilde{\mathcal{I}})$. These cotorsion pairs correspond to the usual injective model structure on $\mathrm{Ch}(R)$. The reader can find more information on these model structures in § 2.3 of [**13**]. We also have the flat cotorsion pairs $(dg\tilde{\mathcal{F}} \cap \mathcal{E}, dg\tilde{\mathcal{C}})$ and $(dg\tilde{\mathcal{F}}, dg\tilde{\mathcal{C}} \cap \mathcal{E})$, which correspond to the flat model structure constructed in [**9**]. We point out that, for all the above classes, intersecting the 'dg-class' with $\mathcal{E}$ gives the 'tilde-class'. For example, $dg\tilde{\mathcal{P}} \cap \mathcal{E} = \tilde{\mathcal{P}}$.

3. The Gorenstein model structures and Ch(R)

In this section we first gather material on Gorenstein rings and graded Gorenstein rings that is fundamental to the rest of the paper. We then show how the Gorenstein model structures of [**14**, § 8] correspond to the usual projective and injective model structures on $\mathrm{Ch}(R)$ when $\mathrm{gl.dim}(R)$ is finite. Similarly, there is a Gorenstein flat model structure that corresponds to the flat model structure of [**9**] when $\mathrm{gl.dim}(R)$ is finite. We point out that the projective model structure lifts to $\mathrm{Ch}(R)$ for any ring R, so the derived category of R is recoverable from Gorenstein homological algebra.

3.1. Gorenstein rings

A commutative Noetherian ring R is called *Gorenstein* if its injective dimension, $\mathrm{id}(R)$, is finite. Iwanaga generalized this class of rings to the non-commutative setting in [**15**] and [**16**].

Definition 3.1. A ring R is called *Gorenstein* if R is both left and right Noetherian and has finite injective dimension when thought of as either a left or right R-module.

Gorenstein rings are often called *Iwanaga–Gorenstein* to emphasize the possibility that the ring may be non-commutative. These rings generalize quasi-Frobenius rings, which are left and right Noetherian rings that are left and right self-injective. Other examples of non-commutative Gorenstein rings are group rings $R[G]$ of a finite group G over a commutative Gorenstein ring R [**2**].

It can be shown (see, for example, [**6**, Corollary 1.2.3]) that if R is Gorenstein, then the injective dimension of R as a left R-module must coincide with the injective dimension of R as a right R-module. If this number is n, we say that R is *n-Gorenstein*.

The following theorem is due to Iwanaga. We will simply say R-module when we mean left R-module, but everything we do also has right R-module versions.

Theorem 3.2 (Iwanaga). *Suppose R is an n-Gorenstein ring. Then an R-module M has finite injective dimension if and only if M has finite projective dimension if and only if M has finite flat dimension. In this case all these dimensions must be less than or equal to n.*

Due to Iwanaga's Theorem, when R is a Gorenstein ring we will simply say that an R-module M has *finite R-dimension* when $\mathrm{id}(M)$ (and hence $\mathrm{pd}(M)$ and $\mathrm{fd}(M)$) is finite. We denote the class of all R-modules of finite R-dimension by $\mathcal{W}_R$.

Note that if R is left and right Noetherian and $\mathrm{gl.dim}(R) = n$, then R is n-Gorenstein. On the other hand, if R is n-Gorenstein, then either $\mathrm{gl.dim}(R) = n$ or $\mathrm{gl.dim}(R) = \infty$.

3.2. Graded Gorenstein rings and chain complexes

Everything we have said about Gorenstein rings has a graded version. Let A be a graded ring. Our convention is that graded will always mean $\mathbb{Z}$-graded. We denote the category of graded left A-modules by A-Mod and the category of graded right A-modules by Mod-A. Unless otherwise stated, an A-module will refer to a graded left A-module.

We say A is *Gorenstein* if it is left and right Noetherian in the graded sense (so that ascending chains of homogeneous left or right ideals must be finite), and A has finite injective dimension when considered as both a left and right A-module. As mentioned in § 1, because A is Gorenstein, there are two different model structures on A-Mod with the same class of trivial objects $\mathcal{W}_A$. These are the A-modules of finite A-dimension.

We now look at a special case. Let R be a ring and let $A = R[x]/(x^2)$. Then A may be viewed as a graded ring, with a copy of R (generated by 1) in degree 0 and a copy of R (generated by x) in degree -1. One can check that the category A-Mod is isomorphic to the category $\mathrm{Ch}(R)$ of unbounded R-chain complexes, where the differential d corresponds to multiplication by x. Through this isomorphism, the A-module A corresponds to $D^0(R)$. In particular, we have $\mathrm{Ext}^i_A(\cdot\,, A) \cong \mathrm{Ext}^i_{\mathrm{Ch(R)}}(\cdot\,, D^0(R))$. One can use this to give a direct proof of the following proposition. However, it follows from a more general result from [**16**].

Proposition 3.3. *If R is an n-Gorenstein ring, then the graded ring $A = R[x]/(x^2)$ is n-Gorenstein with global dimension ∞.*

So for an n-Gorenstein ring R and $A = R[x]/(x^2)$, the class $\mathcal{W}_A$ of trivial A-modules must correspond to some collection of chain complexes. We would now like to characterize these chain complexes. We will use facts from [**8**] to do this.

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Note that the isomorphism of categories between A-Mod and $\mathrm{Ch}(R)$ automatically preserves injectives and projectives. Since the flat complexes described above (exact complexes with flat cycles) are in fact the direct limits of projective complexes (see [**8**, Theorem 4.1.3]), and of course the flat A-modules are the direct limits of projective A-modules, the isomorphism also takes flat A-modules to flat complexes. Thus the A-modules of finite A-dimension correspond to complexes of finite projective (respectively injective, respectively flat) dimension.

Proposition 3.4. *For any ring R, the class of chain complexes with finite projective (respectively injective, respectively flat) dimension is the class of exact complexes with cycles of bounded projective (respectively injective, respectively flat) dimension. If R is n-Gorenstein, these classes coincide and every exact complex E with cycles of finite R-dimension has $\mathrm{pd}(E) \leqslant n$, $\mathrm{id}(E) \leqslant n$ and $\mathrm{fd}(E) \leqslant n$.*

In particular, if R is left and right Noetherian with $\mathrm{gl.dim}(R) = n$, then $\mathcal{W}_A$ is the class of all exact complexes.

Proof. We refer the reader to [**8**, Theorem 3.1.3] for a proof for the finite injective dimension case. The statement involving $\mathrm{gl.dim}(R) = n$ then follows from Iwanaga's Theorem and Proposition 3.3. □

Note that in the case when R is n-Gorenstein and $\mathrm{gl.dim}(R) = \infty$ there are always exact complexes E with $\mathrm{id}(E) = \mathrm{pd}(E) = \mathrm{fd}(E) = \infty$. For example, if M is any R-module with $\mathrm{id}(M) = \mathrm{pd}(M) = \mathrm{fd}(M) = \infty$, then $D^n(M)$ is an exact complex with a cycle of infinite R-dimension. So, by Proposition 3.4, it must be the case that $D^n(M)$ has infinite A-dimension.

3.3. Gorenstein injective, projective and flat graded $R[x]/(x^2)$-modules

Let A be a graded ring and let X be an A-module. We now give the definitions of Gorenstein projective, Gorenstein injective and Gorenstein flat A-modules. The definitions are direct generalizations of the usual definitions in R-Mod, which can be found in [**5**]. We then show that when $\mathrm{gl.dim}(R) = n$ and $A = R[x]/(x^2)$ a Gorenstein injective (respectively Gorenstein projective, respectively Gorenstein flat) A-module is the same thing as a dg-injective (respectively dg-projective, respectively dg-flat) R-complex.

Definition 3.5. An A-module X is called *Gorenstein injective* if there is an exact sequence

$$\cdots I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

of injective A-modules such that $X = \ker(I^0 \to I^1)$ and such that the sequence remains exact after applying $\mathrm{Hom}_A(I, \cdot)$ for any injective A-module I.

Proposition 3.6. *Suppose that R is a ring and let A be the graded ring $R[x]/(x^2)$. Then every dg-injective chain complex over R is a Gorenstein injective A-module. The converse holds if R is left and right Noetherian and of finite global dimension.*

When R has infinite global dimension, there are Gorenstein injective A-modules that are not dg-injective as chain complexes. These have been studied by Enochs and co-authors: see [**8**, Chapter 3], where it is shown in particular that X is a Goresntein injective chain complex if and only if X_n is Gorenstein injective for all n.

Proof. First suppose that X is a dg-injective chain complex. We want to show that it is a Gorenstein injective A-module. We first take an injective coresolution of X as follows:

$$0 \to X \to I^0 \to I^1 \to I^2 \to \cdots .$$

Note that since X is dg-injective, the kernel at any spot in the sequence is also dg-injective. Next we use the fact that $(\mathcal{E}, dg\tilde{\mathcal{I}})$ is complete to find a short exact sequence $0 \to K \to I_0 \to X \to 0$, where I_0 is exact and K is dg-injective. But I_0 must also be dg-injective since it is an extension of two dg-injective complexes. Therefore, I_0 is an injective complex. Continuing with the same procedure on K we can build an injective resolution of X as follows:

$$\cdots \to I_1 \to I_0 \to X \to 0.$$

Again the kernel at each spot is dg-injective. Pasting this 'left' resolution together with the 'right' coresolution above we get an exact sequence

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

of injective complexes that satisfies the definition of X being a Gorenstein injective A-module. Indeed, since X being dg-injective implies that $\operatorname{Ext}^1(E, X) = 0$ for any exact chain complex E, we certainly have $\operatorname{Ext}^1(I, X) = 0$ for any injective chain complex I. Therefore, applying $\operatorname{Hom}_A(I, \cdot)$ will leave the sequence exact.

Next we let X be a Gorenstein injective A-module and argue that it is a dg-injective R-chain complex when R is both left and right Noetherian and $\operatorname{gl.dim}(R) = n$. Note that by the definition of Gorenstein injective we have $\operatorname{Ext}^i(I, X) = 0$ for all $i > 0$ and injective complexes I. We will be done if we can show that $\operatorname{Ext}^1(E, X) = 0$ for any exact complex E. By Proposition 3.4, $\operatorname{id}(E) \leqslant n$, so there exists a finite injective coresolution

$$0 \to E \to I^0 \to I^1 \to I^2 \to \cdots \to I^n \to 0.$$

By a dimension-shifting argument we see that $\operatorname{Ext}^1(E, X) = \operatorname{Ext}^{n+1}(E, I^n) = 0.$ □

Dualizing the definition of Gorenstein injective gives us the notion of Gorenstein projective.

Definition 3.7. An A-module X is called *Gorenstein projective* if there is an exact sequence

$$\cdots P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective A-modules such that $X = \ker(P^0 \to P^1)$ and such that the sequence remains exact after applying $\operatorname{Hom}_A(\cdot, P)$ for any projective A-module P.

By dualizing the above arguments we get the following proposition.

Proposition 3.8. *Suppose R is a ring and let A be the graded ring $R[x]/(x^2)$. Then every dg-projective chain complex over R is a Gorenstein projective A-module. The converse holds if R is left and right Noetherian and of finite global dimension.*

Again, the converse is not true if R has infinite global dimension (see [**8**, Chapter 3]).

Finally, we present similar results for Gorenstein flat A-modules. As explained in § 3.2, flat A-modules (defined as direct limits of projective A-modules) correspond to R-chain complexes that are exact with flat cycles. However, one would like to describe flatness in terms of a tensor product. We point out that, through the isomorphism $A\text{-Mod} \cong \text{Ch}(R)$, tensoring in A-Mod over R corresponds to the usual tensor product of chain complexes, while tensoring over A corresponds to the modified tensor product $\otimes_{\text{Ch}}$ that appears in [**3**]. The latter tensor product $\otimes_A = \otimes_{\text{Ch}}$ is the correct tensor product for the categorical notion of flatness. Indeed, it was shown in [**3**] that $F \in \text{Ch}(R)$ is flat if and only if the functor $\cdot \otimes_{\text{Ch}} F$ is exact. Therefore, F is a flat complex if and only if $\text{Tor}_1^{\text{Ch}}(X, F) = 0$ for any chain complex X. It is also shown in [**3**] that a chain complex F is dg-flat if and only if $\text{Tor}_1^{\text{Ch}}(E, F) = 0$ for any exact chain complex E. Using these results we can imitate our proof above to characterize Gorenstein flat A-modules when $\text{gl.dim}(R) = n$.

Definition 3.9. An A-module X is called *Gorenstein flat* if there is an exact sequence

$$\cdots F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of flat A-modules such that $X = \ker(F^0 \to F^1)$ and such that the sequence remains exact after applying $I \otimes_A \cdot$ for any injective right A-module I.

Proposition 3.10. *Suppose that R is a ring and let A be the graded ring $R[x]/(x^2)$. Then every dg-flat chain complex over R is a Gorenstein flat A-module. The converse holds if R is left and right Noetherian and of finite global dimension.*

Again, the converse is false when R has infinite global dimension (see [**8**, § 5.4]).

Proof. First suppose that X is a dg-flat chain complex. We want to show that it is a Gorenstein flat A-module. We first take a flat resolution of X as follows:

$$\cdots \to F_2 \to F_1 \to F_0 \to X \to 0.$$

Note that since X is dg-flat, the kernel at any spot in the sequence is also dg-flat. Next we use the fact that $(dg\tilde{\mathcal{F}}, \tilde{\mathcal{C}})$ is complete to find a short exact sequence $0 \to X \to F^0 \to K \to 0$, where F^0 is cotorsion (so exact with cotorsion cycles) and K is dg-flat. But F^0 must also be dg-flat since it is an extension of two dg-flat complexes. Therefore, F^0 is a flat complex since it is both dg-flat and exact. Continuing with the same procedure on K we can build a coresolution of X as follows:

$$0 \to X \to F^0 \to F^1 \to F^2 \to \cdots.$$

Again the kernel at each spot is dg-flat. Pasting this 'right' coresolution together with the 'left' resolution above we get an exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of flat complexes that satisfies the definition of X being a Gorenstein flat A-module. Indeed, since X being dg-flat means that $\mathrm{Tor}_1^{\mathrm{Ch}}(E, X) = 0$ for any exact chain complex E, we certainly have $\mathrm{Tor}_1^{\mathrm{Ch}}(I, X) = 0$ for any injective chain complex I. Therefore, applying $I \otimes_{\mathrm{Ch}} \cdot$ will leave the sequence exact.

Next we let X be a Gorenstein flat A-module and argue that it is a dg-flat chain complex over R. Since $\mathrm{Tor}_i^{\mathrm{Ch}}$ can be computed using flat resolutions, the definition of Gorenstein flat implies that $\mathrm{Tor}_i^{\mathrm{Ch}}(I, X) = 0$ for all $i > 0$ and injective complexes I. We will be done if we can show that $\mathrm{Tor}_1^{\mathrm{Ch}}(E, X) = 0$ for any exact complex E. By Proposition 3.4, $\mathrm{id}(E) \leqslant n$, so there exists a finite injective coresolution

$$0 \to E \to I^0 \to I^1 \to I^2 \to \cdots \to I^n \to 0.$$

But by a dimension-shifting argument one can see that

$$\mathrm{Tor}_1^{\mathrm{Ch}}(E, X) = \mathrm{Tor}_{n+1}^{\mathrm{Ch}}(I^n, X) = 0.$$

□

3.4. The Gorenstein model structures on $R[x]/(x^2)$-Mod

We now present the theorems that construct the usual model structures on $\mathrm{Ch}(R)$ via the corresponding Gorenstein model structures on $R[x]/(x^2)$-Mod.

Theorem 3.11. *Let R be an n-Gorenstein ring. There is then a cofibrantly generated model structure on* $\mathrm{Ch}(R)$ *called the* Gorenstein projective model structure *in which the cofibrations are the monomorphisms with Gorenstein projective cokernel, the fibrations are the epimorphisms and the trivial objects are the exact complexes with cycles of finite R-dimension. Dually, there is a cofibrantly generated model structure on* $\mathrm{Ch}(R)$ *called the* Gorenstein injective model structure *in which the cofibrations are the monomorphisms, the fibrations are the epimorphisms with Gorenstein injective kernel, and the trivial objects are the exact complexes with cycles of finite R-dimension. When* $\mathrm{gl.dim}(R) = n$ *these model structures are the usual projective and injective model structures on* $\mathrm{Ch}(R)$.

Proof. Except for the last sentence, this follows from [**14**, § 8], where we use Proposition 3.4 to identify the trivial objects. For the last sentence, use Proposition 3.4 to identify the trivial objects with the exact complexes and the results from § 3.3 to identify Gorenstein projective A-modules with dg-projective complexes and Gorenstein injective A-modules with dg-injective complexes. □

Given a graded ring A, the class $\mathcal{F}$ of flat graded A-modules forms the left side of a complete cotorsion pair $(\mathcal{F}, \mathcal{C})$. The modules in $\mathcal{C}$ are called (graded) cotorsion modules. Similarly, the class of Gorenstein flat graded A-modules form the left side of a complete cotorsion theory denoted $(\mathcal{GF}, \mathcal{GC})$ and the modules in $\mathcal{GC}$ are called (graded) Gorenstein cotorsion modules. Rather than prove these results we simply refer the reader to [**6**], where much more information on Gorenstein flat modules can be found.

Theorem 3.12. *If A is a Gorenstein ring, there is a model structure on A-Mod in which the cofibrant objects are the Gorenstein flat modules, the fibrant objects are the cotorsion modules and the trivial objects are the modules of finite A-dimension.*

Proof. Using the results of [**14**], the only thing left to check is that $\mathcal{GF} \cap \mathcal{W}_A = \mathcal{F}$ and $\mathcal{C} \cap \mathcal{W}_A = \mathcal{GC}$. It is clear that $\mathcal{F} \subseteq \mathcal{GF} \cap \mathcal{W}_A$. Recall that M is flat if and only if $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is injective as a right A-module. Thus, if $F \in \mathcal{W}_A$, F has finite flat dimension, so F^+ has finite injective dimension. Also, by Theorem 2.2.2 (vii) of [**6**], $F^+ \in \mathcal{GI}$ (as a right A-module) whenever $F \in \mathcal{GF}$. Since we know that $\mathcal{GI} \cap \mathcal{W}_A$ is the class of injective right A-modules, we conclude that $F \in \mathcal{GF} \cap \mathcal{W}_A$ implies that F^+ is injective. But this means that F must be flat. So $\mathcal{F} = \mathcal{GF} \cap \mathcal{W}_A$.

Since $\mathcal{F} \subseteq \mathcal{GF}$ and $\mathcal{GP} \subseteq \mathcal{GF}$, taking the right half of the associated cotorsion theories gives us $\mathcal{GC} \subseteq \mathcal{C} \cap \mathcal{W}_A$. Now suppose that $X \in \mathcal{C} \cap \mathcal{W}_A$. Since $(\mathcal{GF}, \mathcal{GC})$ is complete we can find a short exact sequence $0 \to X \to C \to F \to 0$, where $C \in \mathcal{GC}$ and $F \in \mathcal{GF}$. Since both X and C are in $\mathcal{W}_A$, F is as well. By the last paragraph, we see that $F \in \mathcal{F}$, which makes $0 \to X \to C \to F \to 0$ split and so X is a summand of C. Therefore, $X \in \mathcal{GC}$. □

Corollary 3.13. *Let R be an n-Gorenstein ring. There is then a cofibrantly generated model structure on* Ch(R) *called the* Gorenstein flat model structure *in which the cofibrations are the monomorphisms with Gorenstein flat cokernels, the fibrations are the epimorphisms with cotorsion kernels and the trivial objects are the exact complexes with cycles of finite R-dimensions. When* $\mathrm{gl.dim}(R) = n$ *this model structure coincides with the flat structure on* Ch(R) *constructed in [**9**].*

Unfortunately, a bit of confusion arises at this point with the terminology introduced by Gillespie in [**9**]. What we call a cotorsion complex here is called a dg-cotorsion complex there. Also, when $\mathrm{gl.dim}(R) = n$, what we call a Gorenstein cotorsion complex here is exactly the same as a cotorsion complex in [**9**]. From this perspective it seems like a poor choice of terminology was made in [**9**].

At this point in the paper, we have shown how the usual homological algebra of chain complexes over R is a special case of Gorenstein homological algebra of modules when R is left and right Noetherian and has finite global dimension. But of course most rings R do not have finite global dimension. But one can still recover the projective model structure on Ch(R) in this case. Indeed, with the projective or flat model structure, Ch($\mathbb{Z}$) is a *monoidal* model category. This means that the model structure is compatible with the tensor product in a specific way (see [**13**, Chapter 4] for the precise definition). In such a situation, one expects modules over a monoid in Ch($\mathbb{Z}$), such as a ring R, to inherit a model structure, where the fibrations and weak equivalences of modules over the monoid are just the maps of modules that are fibrations and weak equivalences in the underlying model structure. This is actually only guaranteed to work when the monoid is cofibrant; for it to work for arbitrary monoids, one needs the *monoid axiom* of Schwede and Shipley [**20**]. The monoid axiom was translated into the language of cotorsion pairs in [**14**, Theorem 7.4]. If the class $\mathcal{W}$ of trivial objects is closed under transfinite extensions, and if $X \otimes Y$ is trivial whenever X is trivially cofibrant, then the monoid axiom holds. Exact complexes are of course closed under transfinite extensions, and if X is a flat complex, then one can see (because the cycles of X are flat) that $X \otimes Y$ is still exact. Hence the monoid axiom holds in either the projective or flat model structures on $\mathrm{Ch}(\mathbb{Z})$.

Thus, if R is a ring (or differential graded algebra), we have a model structure on $\mathrm{Ch}(R)$ induced from the projective model structure on $\mathrm{Ch}(\mathbb{Z})$, and this coincides with the usual projective model structure. We therefore recover the derived category of R from Gorenstein homological algebra. We also get a model structure on $\mathrm{Ch}(R)$ induced from the flat model structure on $\mathrm{Ch}(\mathbb{Z})$, but we do not know whether this model structure coincides with the flat model structure on $\mathrm{Ch}(R)$; we suspect that it is the 'degreewise cotorsion' model structure mentioned in [**11**]. And of course we cannot recover the injective model structure on $\mathrm{Ch}(R)$ in this way, because the injective model structure on $\mathrm{Ch}(\mathbb{Z})$ is not monoidal.

4. Generalizing chain complexes and derived categories

Again we let A be a graded Gorenstein ring and A-Mod be the category of (graded) A-modules. In § 3 we saw that if we take $A = \mathbb{Z}[x]/(x^2)$, then the derived category $\mathcal{D}(R)$ of a ring R is easily recovered by lifting the Gorenstein projective model structure on A-Mod to A_R-Mod, where $A_R = A \otimes_{\mathbb{Z}} R \cong R[x]/(x^2)$. Indeed the lifted model structure on A_R-Mod is the usual projective model structure on $\mathrm{Ch}(R)$ and its homotopy category is $\mathcal{D}(R)$. Thus we may regard $\mathcal{D}(R)$ as the derived category of R with respect to $\mathbb{Z}[x]/(x^2)$. It is natural to then wonder what happens if we replace $\mathbb{Z}[x]/(x^2)$ with other graded Gorenstein rings A. What is the resulting derived category $\mathcal{D}_A(R)$?

We first give the construction of $\mathcal{D}_A(R)$. Rather than insist that A be a ring, we allow the possibility that A is a K-algebra, where K is any commutative ring of finite global dimension. This will allow for examples such as the p-chain complexes of Example 5.2.

Theorem 4.1. *Let K be a commutative ring of finite global dimension and suppose that A is a graded n-Gorenstein K-algebra that is flat over K. Then for each K-algebra R there is a natural abelian model structure on $A \otimes_K R$-modules, which corresponds when $R = K$ to the Gorenstein model structure on A-modules. Naturality means that given a map $R \to S$ of K-algebras, the usual induction functor from $A \otimes_K R$-modules to $A \otimes_K S$-modules is a left Quillen functor. In this model structure, the fibrations are the surjections whose kernels each have finite A-dimension.*

The homotopy category of this model structure is $\mathcal{D}_A(R)$. We note that if $A \otimes_K R$ is itself Gorenstein, then we also have the Gorenstein projective model structure on $A \otimes_K R$-modules. For chain complexes, these two model structures agree when R has finite global dimension by Theorem 3.11. We have not been able to prove this for a more general A. The proof of Theorem 4.1 shows that $A \otimes_K R$ itself, and thus every $A \otimes_K R$-module of finite $A \otimes_K R$-dimension, also has finite A-dimension, and thus that the identity functor is a left Quillen functor from the A-induced model structure of Theorem 4.1 to the Gorenstein model structure. But we do not know whether every $A \otimes_K R$-module of finite A-dimension also has finite $A \otimes_K R$-dimension.

In order to prove this theorem, we need a version of Kan's lifting theorem phrased in the language of cotorsion pairs. First we introduce some notation. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. If I is a set of morphisms in $\mathcal{A}$, we let FI denote the set $\{F(i) : i \in I\}$ of morphisms of $\mathcal{B}$. Similarly, if $\mathcal{S}$ is a set of objects in $\mathcal{A}$, then $F\mathcal{S}$ denotes the obvious set of objects in $\mathcal{B}$. If $\mathcal{A}$ is an abelian category with small limits and colimits and if $\mathcal{A}$ has a cofibrantly generated abelian model structure as in [**14**], then we use the following notation: $(\mathcal{Q}, \mathcal{R} \cap \mathcal{W})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ denote the (small) cotorsion pairs associated to the model structure, where $\mathcal{Q}$ is the class of cofibrant objects, $\mathcal{R}$ is the class of fibrant objects and $\mathcal{W}$ is the class of trivial objects. These small cotorsion pairs have generating monomorphisms I and J, respectively, and are cogenerated by sets $\mathcal{S}$ and $\mathcal{T}$, respectively. (We assume that $\mathcal{S}$ and $\mathcal{T}$ contain a generator G as in the definition of small cotorsion pair, rather than writing $\mathcal{S} \cup \{G\}$ and $\mathcal{T} \cup \{G\}$ in what follows.)

Theorem 4.2 (Kan). *Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck categories and let $F : \mathcal{A} \to \mathcal{B}$ be a functor with a right adjoint $U : \mathcal{B} \to \mathcal{A}$. Suppose that $\mathcal{A}$ is a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J. Furthermore, suppose that each of the following is true:*

(i) *each morphism of FJ is a monomorphism;*

(ii) $UF(\mathcal{T}) \subseteq \mathcal{W}$*;*

(iii) *U preserves transfinite compositions;*

(iv) *$\mathcal{W}$ is closed under transfinite extensions.*

There is then a cofibrantly generated model structure on $\mathcal{B}$ with generating cofibrations FI, generating trivial cofibrations FJ and weak equivalences the maps f for which Uf is a weak equivalence in $\mathcal{A}$.

Proof. Use Theorem 11.3.2 of [**12**]. □

We can now prove Theorem 4.1.

Proof of Theorem 4.1. We have the Gorenstein projective model structure on A-modules [**14**, § 8], and the unit $K \to R$ gives us the induction functor F from A-modules to $A \otimes_K R$-modules, whose right adjoint U is the restriction functor. Then U preserves filtered colimits, including transfinite extensions. The class $\mathcal{W}$ consists of modules of finite A-dimension, which are closed under all filtered colimits since there is a uniform bound on the dimension. In this case, J consists of the maps $0 \to s^i A$, so FJ obviously consists of monomorphisms. The set $\mathcal{T}$ can be taken to be A itself (and all its shifts $s^i A$), so we need $UF(A) = A \otimes_K R$ to have finite A-dimension. But R has finite K-dimension, since K has finite global dimension, and a projective resolution of R as a K-module can be tensored with A to give a projective resolution of $A \otimes_K R$ as an A-module, since A is flat over K. □

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4.1. Applications to the category of k-chain complexes

The model structures of Theorem 4.1 are uninteresting when A has finite global dimension. In this case every module is trivial and therefore every map is a weak equivalence. Thus polynomial rings such as $\mathbb{Z}[x]$ are not of particular interest to us. However, a natural generalization of $A = \mathbb{Z}[x]/(x^2)$ is $A = \mathbb{Z}[x]/(x^k)$, where $k > 1$ is fixed. Then A-modules are similar to chain complexes, although the differential d need only satisfy the condition $\mathrm{d}^k = 0$, not necessarily $\mathrm{d}^2 = 0$. In this way the authors were led to studying the notion of a k-chain complex and their interesting homology. However, this notion has already been considered by several authors: we recommend [**1**], [**17**], [**21**] and [**7**] to the interested reader. As an example of Theorem 4.1, we now describe a model structure on the category of k-chain complexes. We will use [**7**] to quote the necessary properties of k-chain complexes.

Let R be a ring with identity. For the rest of this section we assume that $k > 1$ is fixed. We define a *k-chain complex* to be a sequence of R-modules

$$\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$$

such that the composition of any k consecutive differentials d is 0. We let $k\,\mathrm{Ch}(R)$ be the category of k-chain complexes, where a morphism $f : X \to Y$ is defined to be a collection $f_n : X_n \to Y_n$ of R-module homomorphisms such that $d_n f_n = f_{n-1} d_n$. So the category of 2-chain complexes is the usual category of chain complexes. One can see that the category of k-chain complexes is isomorphic to the category of graded modules over the graded ring $R[x]/(x^k)$. One could also consider 1-chain complexes. These just correspond to graded R-modules of course. Note that, for $k > 2$, the category of k-chain complexes (graded $R[x]/(x^k)$-modules) does not have a natural tensor product (unless k is prime and we work over a particular ring such as $R = \mathbb{Z}_k$).

Given an R-module M, we define k-chain complexes $D^i_n(M)$ for all $i = 1, 2, \ldots, k$ as follows. $D^i_n(M)$ consists of M in degrees $n, n-1, n-2, \ldots, n-(i-1)$, all joined by identity maps, and 0 in every other degree. Thus $D^1_n(M)$ is like the usual *n-sphere* $S^n(M)$ and $D^2_n(M)$ is like the usual *n-disc* $D^n(M)$ (except of course that each is technically a k-chain complex when $k > 2$).

Next, for a k-chain complex X there are $k-1$ choices for homology. For $i = 1, 2 \ldots, k$ we define $Z^i_n(X) = \ker(d_{n-(i-1)} \cdots d_{n-1} d_n)$. In particular, we have $Z^1_n(X) = \ker d_n$ and $Z^k_n(X) = X_n$. Next, for $i = 1, 2, \ldots, k$, we define $B^i_n(X) = \mathrm{Im}(d_{n+1} d_{n+2} \cdots d_{n+i})$. In particular, $B^1_n(X) = \mathrm{Im}\, d_{n+1}$ and $B^k_n(X) = 0$. Finally, we define $H^i_n(X) = Z^i_n(X)/B^{k-i}_n(X)$ for $i = 1, 2, \ldots, k-1$.

It is easy to check the following adjointness relationships, which hold for all $i = 1, 2, \ldots, k$:

(1) $k\,\mathrm{Ch}(R)(D^i_n(M), Y) \cong \mathrm{Hom}_R(M, Z^i_n(Y))$;

(2) $k\,\mathrm{Ch}(R)(X, D^i_n(M)) \cong \mathrm{Hom}_R(X_{n-(i-1)}/B^i_{n-(i-1)}, M)$.

It follows from (1) that $D^k_n(P)$ is a projective k-chain complex if and only if P is a projective R-module. Similarly, (2) implies that $D^k_n(I)$ is injective if and only if I is

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injective. Under the correspondence with $R[x]/(x^k)$-modules, $D^k_n(R)$ corresponds to the shifted free module $s^n R[x]/(x^k)$, so every projective module is a summand in a direct sum of copies of $D^k_n(R)$ (where n can vary). The category of k-chain complexes has enough projectives and injectives since it is equivalent to a module category.

Definition 4.3. Let X be a k-chain complex. We say X is *k-exact* (or just *exact*) if $H^i_n(X) = 0$ for each n and all $i = 1, 2, \dots, k-1$.

Lemma 4.4. *Suppose $0 \to X \to Y \to Z \to 0$ is a short exact sequence of k-chain complexes. If any two out of the three are k-exact, then so is the third.*

Proof. Any k-chain complex X has $k-1$ standard chain complexes associated to it. For example, if $k = 5$, then we can form four chain complexes from X by letting the differentials alternate as suggested by the following pairs:

$$(d, d^4), \quad (d^2, d^3), \quad (d^3, d^2), \quad (d^4, d).$$

So, obtaining the long exact sequences associated to each of the possible short exact sequences of chain complexes will give us the result. □

Besides the k-chain complexes $D^i_n(M)$ we defined for $i = 1, 2, \dots, k$, it will also be convenient to set $D^0_n(M) = 0$ for the statement of Theorem 4.5.

Theorem 4.5. *Let $k > 1$ be fixed and let R be a ring. Then the category of k-chain complexes over R has a cofibrantly generated model structure we call the* projective model structure*. The set*

$$I = \{D^i_n(R) \hookrightarrow D^k_{n+(k-i)}(R)\}_{i<k}$$

is a set of generating cofibrations and the set

$$J = \{0 \hookrightarrow D^k_n(R)\}$$

is a set of generating trivial cofibrations. The class J-inj of fibrations is exactly the class of all epimorphisms while the class I-inj of trivial fibrations is the class of all epimorphisms with k-exact kernels. A k-chain map is a weak equivalence if it is an isomorphism on each homology group H^i_n. When R is left and right Noetherian and of finite global dimension, the cofibrations (respectively trivial cofibrations) are the monomorphisms whose cokernel is Gorenstein projective (respectively projective) when viewed as an $R[x]/(x^k)$-module.

It can be shown that the projective (respectively injective) k-chain complexes are precisely the k-exact chain complexes X for which each $Z^i_n X$ is projective (respectively injective). We refer the reader to Corollary 3.3 and Proposition 4.5 of [**7**] for proof. Furthermore, Corollary 4.7 of [**7**] shows that X has finite projective dimension (respectively finite injective dimension) if and only if X is k-exact and there is an upper bound to the set of all $\mathrm{pd}\, Z^i_n X$ (respectively $\mathrm{id}\, Z^i_n X$). It then follows that if R is left and right noetherian with $\mathrm{gl.dim}(R)$ finite, then the class of k-exact complexes coincides with the class of complexes of finite projective dimension (and with the class of complexes with finite injective dimension). Note that, for any R, the graded ring $R[x]/(x^k)$ always has infinite global dimension because any non-k-exact complex has infinite projective dimension.

Proof. Proposition 4.3 of [**7**] tells us that $\mathbb{Z}[x]/(x^k)$ is a graded 1-Gorenstein ring. So taking $A = \mathbb{Z}[x]/(x^k)$ in Theorem 4.1 gives us the desired model structure on $k\,\mathrm{Ch}(R)$. The trivial objects are the complexes of finite A-dimension, which, since $\mathrm{gl.dim}(\mathbb{Z}) = 1$ is finite, are precisely the k-exact complexes. We therefore get the desired characterization of fibrations and trivial fibrations. The long exact sequences in homology used in Lemma 4.4 give us the characterization of weak equivalences as homology isomorphisms.

One can easily check that J serves as a set of generating trivial cofibrations. To check that I will work as a set of generating cofibrations is a little more complicated. For $0 < i < k$ the adjunction $k\,\mathrm{Ch}(R)(D^i_n(M), Y) \cong \mathrm{Hom}_R(M, Z^i_n(Y))$ tells us that a lift in the diagram

$$\begin{array}{ccc} D^i_n(R) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ D^k_{n+(k-i)}(R) & \longrightarrow & 0 \end{array}$$

occurs if and only if $H^i_n(X) = 0$. From this we conclude that $p : X \to 0$ is in I-inj if and only if X is a k-exact complex.

Now let $p : X \to Y$ with $K = \ker p$. From the last paragraph we will be done characterizing I-inj if we can show that p is in I-inj if and only if p is an epimorphism with $K \to 0$ in I-inj. So suppose that p is in I-inj. Since $J \subseteq I$, we know that p is an epimorphism. Furthermore, I-inj is always closed under pullbacks and so $K \to 0$ is in I-inj. On the other hand, suppose p is an epimorphism with $K \to 0$ in I-inj. We seek a lift in the diagram

$$\begin{array}{ccc} D^i_n(R) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ D^k_{n+(k-i)}(R) & \longrightarrow & Y \end{array}$$

for each $i < k$. But if $\mathrm{Ext}^1(D^j_n(R), K) = 0$ for all j, then such a lift exists by Lemma 2.4 of [**10**]. Since K is k-exact one can argue that if

$$0 \to K \to I^0 \to I^1 \to I^2 \to \cdots$$

is an injective coresolution of K, then

$$0 \to Z^j_n K \to Z^j_n(I^0) \to Z^j_n(I^1) \to Z^j_n(I^2) \to \cdots$$

is an injective coresolution of $Z^j_n K$. Thus, for any R-module M, the adjunction $k\,\mathrm{Ch}(R)(D^j_n(M), Y) \cong \mathrm{Hom}_R(M, Z^j_n(Y))$ allows us to prove the isomorphism

$$\mathrm{Ext}(D^j_n(M), Y) \cong \mathrm{Ext}_R(M, Z^j_n(Y))$$

by computing with an injective coresolution of Y. In particular,

$$\mathrm{Ext}^1(D^j_n(R), Y) \cong \mathrm{Ext}^1_R(R, Z^j_n(Y)) = 0.$$

This completes the proof characterizing I-inj.

Finally, when R is left and right Noetherian and $\mathrm{gl.dim}(R) = n$, the graded ring $R[x]/(x^k)$ is n-Gorenstein by Proposition 4.3 of [**7**]. So the objects of finite $R[x]/(x^k)$-dimension are precisely the k-exact complexes. Therefore, in this case, the Gorenstein projective model structure on $R[x]/(x^k)$-modules coincides with the model structure of Theorem 4.1. In particular, the cofibrations (respectively trivial cofibrations) are the monomorphisms whose cokernel is Gorenstein projective (respectively projective) when viewed as an $R[x]/(x^k)$-module. □

5. Generalizing derived categories with tensor products

In practice, most algebraic categories come with a tensor product. In this case one wants a model structure on that category to be *monoidal*, as defined in Chapter 4 of [**13**]. Having a monoidal model structure is fundamental since it gives us a way to easily define and work with a tensor product on the associated homotopy category. In this section we will assume that our K-algebra A is a Hopf algebra over K, since such a structure is essentially the information needed to put a tensor product on A-Mod and hence on the model category of $A \otimes_K R$-modules of Theorem 4.1 when R is a commutative K-algebra.

As a guiding example, let R be a commutative ring and let us again view the category $\mathrm{Ch}(R)$ of chain complexes of R-modules as the category of graded modules over $R[x]/(x^2)$. The ring $R[x]/(x^2)$ is a graded Hopf algebra over R with comultiplication determined by $1 \to 1 \otimes 1$ and $x \to 1 \otimes x + x \otimes 1$. The tensor product on $R[x]/(x^2)$-Mod determined by this comultiplication corresponds exactly to the usual tensor product of chain complexes of R-modules. This tensor product is commutative since the comultiplication on $R[x]/(x^2)$ is cocommutative.

We will show that if K is a commutative ring of finite global dimension and A is a K-projective cocommutative graded Gorenstein Hopf algebra over K, then the projective model structure on $A \otimes_K R$-Mod is monoidal for any commutative K-algebra R. So, for the above example, think of $K = \mathbb{Z}$ and $A = \mathbb{Z}[x]/(x^2)$ and R a commutative ring. We end by looking at some examples of model structures on categories of modules over other Hopf algebras. These include p-chain complexes over the ring of integers mod p and bicomplexes of R-modules.

5.1. The category of modules over a Hopf algebra

We briefly recall the definition of a graded Hopf algebra. We refer the reader to Chapter VI of [**18**] for more details.

Let K be a commutative ring with identity and let A be a graded K-algebra. We say that A is a graded Hopf algebra (over K) if it comes equipped with graded K-algebra homomorphisms $\Delta : A \to A \otimes_K A$ and $\epsilon : A \to K$ (called *comultiplication* and *counit*) making A into a graded coalgebra. We call A *cocommutative* if the comultiplication Δ is cocommutative. Note that graded A-modules M and N may be viewed as graded K-modules, through the map $K \to A$, where K receives the trivial grading. Then the graded tensor product $M \otimes_K N$ naturally inherits an A-module structure. Indeed, $M \otimes_K N$ is an $(A \otimes_K A)$-module and becomes an A-module through the comultiplication $\Delta : A \to A \otimes_K A$.

One can check that the tensor product is commutative (in the graded sense) if we assume that the comultiplication on A is cocommutative. In this case A-Mod is a closed symmetric monoidal category where $\mathrm{Hom}_K(M,N)$ is the graded A-module defined as follows. In degree n, $[\mathrm{Hom}_K(M,N)]_n$ is the graded K-module $\prod_{k\in\mathbb{Z}} \mathrm{Hom}_K(M_k, N_{n+k})$. $\mathrm{Hom}_K(M,N)$ becomes a graded $(A\otimes_K A)$-module as follows. Given homogeneous elements $s,t\in A$, $(f_k)\in[\mathrm{Hom}_K(M,N)]_n$ and $m\in M$, define

$$(s\otimes t)(f_k)(m) = (-1)^{n|t|}s(f_{|m|+|t|}(tm)) \in N_{|s|+n+|m|+|t|}.$$

So, through the comultiplication map $A\to A\otimes_K A$, we see that $\mathrm{Hom}_K(M,N)$ becomes a graded A-module.

Recall that an augmentation of a graded ring A is a homomorphism $\epsilon : A\to K$ of graded rings (where K is trivially graded). The counit $\epsilon : A\to K$ is such an augmentation. Through ϵ we may regard all K-modules as trivial A-modules and in fact we have a fully faithful inclusion functor K-Mod $\to$ A-Mod. In particular, K itself is an A-module that serves as the unit for the monoidal structure. This can be seen by using the compatibility of Δ and ϵ in the coalgebra axioms. As an example, when $A = K[x]/(x^2)$, the functor K-Mod $\to$ A-Mod is the sphere functor $M\to S^0(M)$ from K-modules to chain complexes of K-modules.

Lemma 5.1. *Let K be a commutative ring of finite global dimension and let A be a K-projective cocommutative graded Gorenstein Hopf algebra over K. If P is a Gorenstein projective A-module, then each P_n is K-projective.*

Proof. By the definition of Gorenstein projective, there is an exact sequence

$$0\to P\to Q^1\to Q^2\to\cdots,$$

where each Q^i is a projective A-module. Since A is K-projective, so is each Q^i. Thus, as a K-module, each P_n is an r-fold syzygy for all r. Since K has finite global dimension, P_n must be projective. □

Lemma 5.2. *Let K be a commutative ring of finite global dimension and let A be a K-projective cocommutative graded Gorenstein Hopf algebra over K. Let P be a Gorenstein projective A-module. Then for A-modules M and N we have an isomorphism*

$$\mathrm{Ext}^n_A(M\otimes_K P, N) \cong \mathrm{Ext}^n_A(M, \mathrm{Hom}_K(P,N)).$$

Proof. By Lemma 5.1, $\cdot\otimes_K P$ and $\mathrm{Hom}_K(P,\cdot)$ are each exact endofunctors on A-Mod. (Indeed they are exact as endofunctors of graded K-modules, but the Hopf algebra structure on A just 'extends' the functors to A-Mod.) Being right adjoint to an exact functor, $\mathrm{Hom}_K(P,\cdot)$ preserves injective objects. Since $\mathrm{Hom}_K(P,\cdot)$ itself is also exact it preserves injective coresolutions. The result now follows by computing the Ext groups using an injective coresolution of N and applying the adjoint relationship between $\cdot\otimes_K P$ and $\mathrm{Hom}_K(P,\cdot)$. □

Lemma 5.3. *Let K be a commutative ring of finite global dimension and let A be a K-projective cocommutative graded Gorenstein Hopf algebra over K. If $W \in \mathcal{W}_A$ and $P \in \mathcal{GP}$, then $\operatorname{Hom}_K(P, W)$ and $W \otimes_K P$ are in $\mathcal{W}_A$.*

Proof. We have

$$\operatorname{Ext}^n_A(M \otimes_K P, W) \cong \operatorname{Ext}^n_A(M, \operatorname{Hom}_K(P, W))$$

for an arbitrary A-module M by Lemma 5.2. If W has finite injective dimension, we conclude that $\operatorname{Ext}^n_A(M, \operatorname{Hom}_K(P, W)) = 0$ for all large enough n. But then $\operatorname{Hom}_K(P, W)$ also has finite injective dimension and so is in $\mathcal{W}_A$. The proof that $W \otimes_K P \in \mathcal{W}_A$ is similar, using that W has finite projective dimension. □

Theorem 5.4. *Let K be a commutative ring of finite global dimension and let A be a K-projective cocommutative graded Gorenstein Hopf algebra over K. Then, for any commutative K-algebra R, the model structure of Theorem 4.1 on the category of graded $A \otimes_K R$-modules is monoidal and satisfies the monoid axiom.*

In particular, the derived category $\mathcal{D}_A(R)$ is a symmetric monoidal triangulated category in this case.

Note that we do not know, in general, whether the unit K of the monoidal structure is Gorenstein projective, though it is in our examples. We therefore have to use the complete definition of a monoidal model structure, which allows for a non-cofibrant unit [**13**, Definition 4.2.6].

Proof. Note first that if R is a K-algebra, then R is a monoid in the category of A-modules, where A acts on R through the counit ϵ as usual. Furthermore, an R-module in the symmetric monoidal category of A-modules is the same as an $A \otimes_K R$-module in the usual sense. If R is commutative, the category of $A \otimes_K R$-modules is symmetric monoidal, with the tensor product being the usual tensor product $M \otimes_R N$ over R. The A-action comes from the fact that $A \otimes_K R$ is a Hopf algebra over R. It will therefore mostly suffice to prove this theorem when $R = K$, using [**20**, Theorem 4.1 (2)]. However, the results in [**20**] assume that the unit of the monoidal structure is cofibrant, and this need not be true for us. We will postpone this point to the end of the proof.

We use a more general version of Theorem 7.2 of [**14**] to prove that the Gorenstein projective model structure on A-modules is monoidal. In the current setting this amounts to showing that

(a) every cofibration is pure,

(b) $P \otimes_K Q$ is Gorenstein projective whenever P and Q are Gorenstein projective,

(c) $P \otimes_K Q$ is projective whenever P is projective and Q is Gorenstein projective and

(d) if

$$0 \to F \to QK \to K \to 0$$

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is a short exact sequence, where QK is Gorenstein projective and F has finite A-dimension, and if C is Gorenstein projective, then the kernel of $QK \otimes_K C \to C$ also has finite A-dimension (this condition comes from the unit condition in the definition of a monoidal model category [**13**, Definition 4.2.6 (2)]).

Since Gorenstein projectives are degreewise projective K-modules by Lemma 5.1, part (a) follows from the fact that a monomorphism with projective (or just flat) cokernel is pure. For part (b) we wish to show that $\operatorname{Ext}^1_A(P \otimes_K Q, W) = 0$ when $W \in \mathcal{W}_A$. But this follows immediately from Lemmas 5.2 and 5.3. To prove (c), we use the fact that $\mathcal{P} = \mathcal{GP} \cap \mathcal{W}_A$. If $P \in \mathcal{P}$ and $Q \in \mathcal{GP}$, then $P \otimes_K Q \in \mathcal{GP}$ by part (b). But we also have $P \otimes_K Q \in \mathcal{W}_A$ by Lemma 5.3. So $P \otimes_K Q \in \mathcal{P}$. Part (d) is obvious, since the kernel in question is just $F \otimes_K C$, and Lemma 5.3 guarantees this has finite A-dimension.

Next we use Theorem 7.4 of [**14**] to prove that the model structure satisfies the monoid axiom. This will amount to showing that

(i) $P \otimes_K N$ is projective whenever P is projective and N is any A-module and

(ii) $\mathcal{W}_A$ is closed under transfinite compositions of pure monomorphisms.

To prove (i), let P be projective and let N be arbitrary. Since $(\mathcal{GP}, \mathcal{W}_A)$ is a complete cotorsion pair (see [**14**, Theorem 8.3]), we can find a short exact sequence $0 \to W \to Q \to N \to 0$, where $W \in \mathcal{W}_A$ and $Q \in \mathcal{GP}$. Since $P \otimes_K -$ is exact, the sequence

$$0 \to P \otimes_K W \to P \otimes_K Q \to P \otimes_K N \to 0$$

is exact. By Lemma 5.3 we have $P \otimes_K W \in \mathcal{W}_A$ and by (d) above we have $P \otimes_K Q \in \mathcal{P}$. Since $P \otimes_K W$ and $P \otimes_K Q$ belong to $\mathcal{W}_A$, it follows that $P \otimes_K N$ also belongs to $\mathcal{W}_A$.

Now, from Remark 11.2.3 of [**5**], $(\mathcal{W}_A, \mathcal{GI})$ is a cotorsion pair and the left side of a cotorsion pair is automatically closed under transfinite extensions by Corollary 7.3.5 of [**5**]. Since $\mathcal{W}_A$ is closed under *all* transfinite extensions, condition (ii) is true.

Since it does not follow from [**20**], we still owe the reader a proof of the fact that, if R is a commutative K-algebra, QR is a cofibrant replacement of R in the model structure on $A \otimes_K R$-modules and X is a cofibrant $A \otimes_K R$-module, then the natural map $QR \otimes_R X \to X$ is a weak equivalence. The map $QR \to R$ is a trivial fibration in the model structure, so its kernel F is an $A \otimes_K R$-module with finite A-dimension. The cofibrant A-modules are now cogenerated by some set $\mathcal{S}$ of Gorenstein projective A-modules [**14**, Theorem 8.3], so the cofibrant $A \otimes_K R$-modules will be cogenerated by the set $\mathcal{S} \otimes_K R$. Thus X is a transfinite extension of $A \otimes_K R$-modules of the form $M \otimes_K R$, where M is a Gorenstein projective A-module. Since M is projective as a K-module, it follows that X is projective as an R-module. Thus the surjection $QR \otimes_R X \to X$ will have kernel $F \otimes_R X$. But, since X is a transfinite extension of $A \otimes_K R$-modules of the form $M \otimes_K R$, $F \otimes_R X$ is a transfinite extension of modules of the form $F \otimes_K M$. These modules have finite A-dimension by Lemma 5.3. Now, as we have mentioned above, $\mathcal{W}_A$ is closed under all transfinite extensions, so $F \otimes_R X$ has finite A-dimension as required. $\square$

We now look at some examples of Theorem 5.4.

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5.2. p-chain complexes over $\mathbb{Z}$ mod p

If $A = \mathbb{Z}[x]/(x^2)$, then A is a cocommutative graded Gorenstein Hopf algebra over $\mathbb{Z}$. The comultiplication on A is determined by $1 \to 1 \otimes 1$ and $x \to 1 \otimes x + x \otimes 1$. The counit $A \to \mathbb{Z}$ acts by $1 \to 1$ and $x \to 0$. Applying Theorem 5.4 in this situation recovers the monoidal (projective) model structure on $\mathrm{Ch}(R)$ for commutative rings R.

If $k > 2$, then $\mathbb{Z}[x]/(x^k)$ is not a Hopf algebra over $\mathbb{Z}$, since the comultiplication above will not be a ring homomorphism. In other words, the tensor product of chain complexes does not generalize in any natural way to k-chain complexes when $k > 2$. However, if p is prime and $\mathbb{Z}_p$ denotes the integers mod p, then $\mathbb{Z}_p[x]/(x^p)$ is a cocommutative graded Gorenstein Hopf algebra over $\mathbb{Z}_p$. Theorem 5.4 applies to give us a monoidal model structure on the category of p-chain complexes over $\mathbb{Z}_p$, with the usual formula $\mathrm{d}(x \otimes y) = \mathrm{d}x \otimes y + (-1)^{|x|} x \otimes \mathrm{d}y$.

5.3. Bicomplexes

The proofs in this section hold if we replace 'graded' with 'bigraded'. Then, if we take A to be the bigraded ring $\mathbb{Z}[x,y]/(x^2, y^2, xy+yx)$, with x of degree $(-1,0)$ and y of degree $(0,-1)$, the category of bigraded modules over A is the category of bicomplexes of abelian groups. The notion of 'total degree' $p+q$ is used in place of 'degree' in the sign convention and so $xy + yx = 0$ just means that A is bigraded commutative. Alternatively, we can describe A as the tensor product $\mathbb{Z}[x]/(x^2) \otimes_{\mathbb{Z}} \mathbb{Z}[x]/(x^2)$. Indeed, the tensor product of two graded modules gives a bigraded module (before the summation turns it into a singly graded module). Since the tensor product of two Hopf algebras is again a Hopf algebra we see that $\mathbb{Z}[x,y]/(x^2, y^2, xy+yx)$ is a bigraded Hopf algebra over $\mathbb{Z}$. The comultiplication is determined by $1 \to 1 \otimes 1$, $x \to 1 \otimes x + x \otimes 1$ and $y \to 1 \otimes y + y \otimes 1$. The ring is also cocommutative and the counit sends x and y to 0 and $1 \to 1$.

All that remains is to see that A is Gorenstein as a bigraded ring. That is, A should be left and right Noetherian in the bigraded sense (so that ascending chains of homogeneous left or right ideals must be finite), and A has finite injective dimension as a bigraded A-module. However, this can be proved in the same manner as Proposition 3.3. The analogue of $D^0(\mathbb{Z})$ in that proposition is a bicomplex $J^0(\mathbb{Z})$ with a copy of $\mathbb{Z}$ at each of the four vertices in degrees $(0,0)$, $(0,-1)$, $(-1,0)$, $(-1,-1)$, where the differentials are identities or 0 except the differential from degree $(-1,0)$ to $(-1,-1)$, which is -1. A map $B \to J^0(\mathbb{Z})$ of bicomplexes is completely determined by any homomorphism $B_{-1,-1} \to \mathbb{Z}$.

The comultiplication on A determines a tensor product of bicomplexes similar to the tensor product of complexes. The same formula $\mathrm{d}(x \otimes y) = \mathrm{d}x \otimes y + (-1)^{|x|} x \otimes \mathrm{d}y$ holds for both the vertical and horizontal differentials, with the appropriate interpretation of $|x|$ in each case. Theorem 5.4 tells us that the projective model structure on bicomplexes over a commutative ring R is monoidal with respect to this tensor product.

5.4. $K[G]$-modules

In § 9 of [**14**] Hovey showed that the Gorenstein projective model structure on $K[G]$-modules is monoidal, where K is a principal ideal domain and G is a finite group. Theorem 5.4 generalizes his results to any commutative ring K of finite global dimension. In fact, Theorem 5.4 says that the model structure exists on $R[G]$-modules for *any* commutative ring R, but then the trivial objects are harder to understand since they are the $R[G]$-modules of finite injective dimension as $\mathbb{Z}[G]$-modules.

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