AN ATOMIC DECOMPOSITION FOR HARDY SPACES ASSOCIATED TO SCHRÖDINGER OPERATORS

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Abstract

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$ where $V$ is a nonnegative function in the space $L^1_{\text{loc}}(\mathbb{R}^n)$ of locally integrable functions on $\mathbb{R}^n$. In this paper we provide an atomic decomposition for the Hardy space $H^1_L(\mathbb{R}^n)$ associated to $L$ in terms of the maximal function characterization. We then adapt our argument to give an atomic decomposition for the Hardy space $H^1_L(\mathbb{R}^n \times \mathbb{R}^n)$ on product domains.

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1. Introduction

Let $V$ be a locally integrable nonnegative function on $\mathbb{R}^n$ (where $n \geq 1$), which is not identically zero. We define the form $Q$ by

$$Q(u, v) := \int_{\mathbb{R}^n} \nabla u \nabla v \, dx + \int_{\mathbb{R}^n} V uv \, dx$$

with domain

$$\mathcal{D}(Q) := \left\{ u \in W^{1,2}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} V |u|^2 \, dx < \infty \right\}.$$

The space $W^{1,2}(\mathbb{R}^n)$ which appears in the formula above is the Sobolev space consisting of those $L^2$ functions on $\mathbb{R}^n$ whose gradients are also square integrable. It is well known that this symmetric form is closed. We recall that it was shown by Simon [25] that this form coincides with the minimal closure of the form given by the same expression but defined on $C_0^\infty(\mathbb{R}^n)$, the space of $C^\infty$ functions with compact support.

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Let $L$ denote the self-adjoint operator associated with $Q$. The domain of $L$, written $\mathcal{D}(L)$, is defined to be the set of all $u \in \mathcal{D}(Q)$ for which there exists $v \in L^2$ such that

$$Q(u, \varphi) = \int_{\mathbb{R}^n} v \bar{\varphi} \, dx \quad \forall \varphi \in \mathcal{D}(Q).$$

Formally, we write $L = -\Delta + V$ as a Schrödinger operator with potential $V$. Since $V$ is a locally integrable nonnegative function on $\mathbb{R}^n$, the Feynman–Kac formula implies that the kernel $p_t(x, y)$ of the semigroup $e^{-tL}$ satisfies the estimate

$$0 \leq p_t(x, y) \leq (4\pi t)^{-n/2} e^{-|x-y|^2/(4t)}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$ (see [24, p. 195]).

Given a function $f \in L^2(\mathbb{R}^n)$, we consider the following nontangential maximal function associated with the Poisson semigroup generated by the operator $L$:

$$f^*_L(x) := \sup_{|y-x| < t} |e^{-t\sqrt{L}} f(y)| \quad \forall x \in \mathbb{R}^n.$$ 

The space $H^1_{L}(\mathbb{R}^n)$ is defined to be the completion of $L^2(\mathbb{R}^n)$ in the norm given by the $L^1$ norm of this maximal function, that is,

$$\|f\|_{H^1_{L}(\mathbb{R}^n)} := \|f^*_L\|_{L^1(\mathbb{R}^n)}.$$ 

See, for example, [1–3, 12, 17, 18] for the properties of $H^1_{L}(\mathbb{R}^n)$.

Note that if $L = -\Delta$, then the space $H^1_{L}(\mathbb{R}^n)$ is the classical Hardy space $H^1(\mathbb{R}^n)$ which is a natural substitute for $L^1(\mathbb{R}^n)$. Recall that the development of the theory of the classical Hardy spaces in $\mathbb{R}^n$ was initiated by Stein and Weiss [26] and was originally tied closely to the theory of harmonic functions. Real variable methods were introduced into this subject in the seminal paper of Fefferman and Stein [15], the evolution of whose ideas led eventually to a characterization of Hardy spaces via the so-called ‘atomic decomposition’ obtained by Coifman [7] when $n = 1$ and in higher dimensions by Latter [21].

An atomic decomposition for $H^1_{L}(\mathbb{R}^n)$ was given in [17] by combining the area $S$-function and the finite speed propagation property for the wave equation. Following [17], a function $a \in L^2(\mathbb{R}^n)$ is called a $(1, 2)$-atom associated to the operator $L$ if there exists a function $b \in \mathcal{D}(L)$, the domain of an operator $L$ and a ball $B$ of $\mathbb{R}^n$ such that

$$a = Lb;$$

$$\text{supp } L^k b \subseteq B;$$

$$\|(r_B^2 L)^k b\|_{L^2(\mathbb{R}^n)} \leq r_B^2 |B|^{-1/2},$$

where $k = 0, 1$ and $r_B$ denotes the radius of the ball $B$.

The aim of this paper is to get an atomic decomposition directly from the fact that $f^*_L \in L^1(\mathbb{R}^n)$ and then to provide a new proof of the atomic decomposition for $H^1_{L}(\mathbb{R}^n)$. Our first main result is the following theorem.
**Theorem 1.1.** Let \( L = -\Delta + V \) where \( V \in L^1_{\text{loc}}(\mathbb{R}^n) \) is a nonnegative function on \( \mathbb{R}^n \). Let \( f \in H^1_L(\mathbb{R}^n) \). Then there exist \((1, 2)\)-atoms \( a_j \) and real numbers \( \lambda_j \) for \( j = 1, 2, 3, \ldots \) such that

\[
f = \sum_{j=1}^{\infty} \lambda_j a_j \tag{1.2}
\]

in \( H^1_L(\mathbb{R}^n) \). Furthermore, matters can be arranged so that the sequence \( \lambda_j \) satisfies the inequality

\[
\sum_{j=1}^{\infty} |\lambda_j| \leq C \| f \|_{H^1_L(\mathbb{R}^n)}
\]

for some positive constant \( C \), which may depend on \( n \).

Conversely, any function \( f \) which is written in the form of (1.2), where the \( a_j \) are \((1, 2)\)-atoms, satisfies the inequality

\[
\| f \|_{H^1_L(\mathbb{R}^n)} \leq C \sum_{j=1}^{\infty} |\lambda_j|.
\]

We mention that the localized version of the atomic decomposition for \( H^1_L(\mathbb{R}^n) \) when \( L = -\Delta + V \) was given in [13], by using the properties of local Hardy spaces (see [16]), under the assumption that \( V \) was a fixed nonnegative function on \( \mathbb{R}^n \) belonging to the reverse Hölder class \( B^q \) for some \( q > 1 \). That is, there exists a positive constant \( C \), possibly depending on \( q \) and \( V \), such that the reverse Hölder inequality

\[
\left( \frac{1}{|B|} \int_B V^q \, dx \right)^{1/q} \leq C \left( \frac{1}{|B|} \int_B V \, dx \right)
\]

holds for every ball \( B \) in \( \mathbb{R}^n \).

Let us now turn to the Hardy space on product domains. We note that the usual space \( H^1(\mathbb{R}^n \times \mathbb{R}^n) \) on the product domain is now well understood (see, for instance, [4, 5, 14]). In this paper we shall be concerned with the space \( H^1_L(\mathbb{R}^n \times \mathbb{R}^n) \) associated to the Schrödinger operator \( L \) (see [11] for more properties).

For any \((x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n\) and \( f \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \), define

\[
f^*_L(x_1, x_2) = \sup_{|y_1-x_1|<t_1 \atop |y_2-x_2|<t_2} |e^{-t_1 \sqrt{L}} \otimes e^{-t_2 \sqrt{L}} f(y_1, y_2)|
\]

where

\[
e^{-t_1 \sqrt{L}} \otimes e^{-t_2 \sqrt{L}} f(y_1, y_2) := \int_{\mathbb{R}^n \times \mathbb{R}^n} p_{t_1}(y_1, z_1) p_{t_2}(y_2, z_2) f(z_1, z_2) \, dz_1 \, dz_2.
\]

The space \( H^1_L(\mathbb{R}^n \times \mathbb{R}^n) \) is defined to be the completion of \( L^2(\mathbb{R}^n \times \mathbb{R}^n) \) in the norm given by

\[
\| f \|_{H^1_L(\mathbb{R}^n \times \mathbb{R}^n)} := \| f^*_L \|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)}.
\]

For our purposes, a product \((1, 2)\)-atom is a function \( a \) on \( \mathbb{R}^{2n} \), together with an associated open set \( \Omega \) of finite measure satisfying the following two properties.
First, the function \( a \) can be further decomposed into the form

\[
a = \sum_{\Omega \in m(\Omega)} a_{\Omega}
\]

where for each \( \Omega \in m(\Omega) \) there exists a function \( b_{\Omega} \) such that

\[
a_{\Omega} = (L^i \otimes L^j) b_{\Omega}, \quad i, j = 0, 1,
\]

where \( 10R \) denotes the rectangle with the same center as \( R \) and 10 times the side lengths.

Second,

\[
\|a\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \leq |\Omega|^{-1/2}
\]

and

\[
\sum_{\Omega \in m(\Omega)} \sum_{i,j=1}^{\Omega} \ell(I)^{4i-4} \ell(J)^{4j-4} \| (L^i \otimes L^j) b_{\Omega} \|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}^2 \leq |\Omega|^{-1}
\]

where \( R = I \times J \) denotes the dyadic rectangle of \( \mathbb{R}^n \times \mathbb{R}^n \) whose side lengths are \( \ell(I) \) and \( \ell(J) \), \( 10R \) denotes the set \( \{10x \mid x \in R\} \) and \( m(\Omega) \) denotes the set of maximal dyadic subrectangles of \( \Omega \) (see Section 4 below).

The second main result of this paper is the following theorem.

**Theorem 1.2.** Let \( L = -\Delta + V \) where \( V \in L^1_{\text{loc}}(\mathbb{R}^n) \) is a nonnegative function on \( \mathbb{R}^n \). Let \( f \in H^1_L(\mathbb{R}^n \times \mathbb{R}^n) \). Then there exist product \((1, 2)\)-atoms \( a_j \) and numbers \( \lambda_j \), where \( j = 0, 1, 2, \ldots \), such that

\[
f = \sum_{j=1}^{\infty} \lambda_j a_j \tag{1.3}
\]

in \( H^1_L(\mathbb{R}^n \times \mathbb{R}^n) \), and the sequence \( \lambda_j \) satisfies the condition that

\[
\sum_{j=1}^{\infty} |\lambda_j| \leq C \| f \|_{H^1_L(\mathbb{R}^n \times \mathbb{R}^n)}.
\]

Conversely, for any decomposition of \( f \) of the form in (1.3),

\[
\| f \|_{H^1_L(\mathbb{R}^n \times \mathbb{R}^n)} \leq C \sum_{j=1}^{\infty} |\lambda_j|.
\]

The organisation of this paper is as follows. In Section 2 we introduce some notation and preliminary lemmas. Our main results, Theorems 1.1 and 1.2, are proved in Sections 3 and 4. The main contribution of this paper is to combine the Calderón reproducing formula, the finite propagation speed property and the methods of Wilson [27] to obtain an atomic decomposition of Hardy spaces and then to verify the required \( L^2 \) norm estimates of the atoms by using square function estimates.

Throughout this paper, the letters \( C \) and \( c \) denote (possibly different) constants that are independent of the essential variables.

### 2. Preliminaries

Recall that if \( L \) is a nonnegative, self-adjoint operator on \( L^2(\mathbb{R}^n) \) and \( E_L(\lambda) \) denotes a spectral decomposition associated with \( L \), then for every bounded Borel...
function $F : [0, \infty) \to \mathbb{C}$ one defines the operator
\[ F(L) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \]
by the formula
\[ F(L) := \int_0^\infty F(\lambda) \, dE_L(\lambda). \]  \hfill (2.1)

In particular, the operator $\cos(t\sqrt{L})$ is well defined on $L^2(\mathbb{R}^n)$. Moreover, it follows from [9, Theorem 3] (see also [6]) that the integral kernel $K_{\cos(t\sqrt{L})}$ of $\cos(t\sqrt{L})$ satisfies
\[ \text{supp } K_{\cos(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x - y| \leq t\}. \]  \hfill (2.2)

By the Fourier inversion formula, whenever $F$ is an even bounded Borel function with Fourier transform $\hat{F}$ in $L^1(\mathbb{R})$, we can write $F(\sqrt{L})$ in terms of $\cos(t\sqrt{L})$. Specifically, using (2.1), we have
\[ F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{F}(t) \cos(t\sqrt{L}) \, dt \]
which, when combined with (2.2), gives us that
\[ K_{F(\sqrt{L})} = (2\pi)^{-1} \int_{|t| \geq |x-y|} \hat{F}(t)K_{\cos(t\sqrt{L})} \, dt. \]

**Lemma 2.1.** Let $\varphi \in C_0^\infty(\mathbb{R})$ be an even function such that $\text{supp } \varphi \subseteq [-1, 1]$. Let $\Phi$ denote the Fourier transform of $\varphi$. Then for every $\kappa = 0, 1, 2, \ldots$ and for every $t > 0$ the kernel $K_{(t^2L)^\kappa \Phi(t\sqrt{L})}$ of $(t^2L)^\kappa \Phi(t\sqrt{L})$ satisfies the condition
\[ \text{supp } K_{(t^2L)^\kappa \Phi(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x - y| \leq t\}. \]

**Proof.** We refer the reader to [17, Lemma 3.5] for the proof. \hfill \Box

In this paper we use $\mathbb{R}^{n+1}_+$ to denote the upper half space of $\mathbb{R}^{n+1}$. In the following lemma we shall assume that $\varphi \in C_0^1(\mathbb{R}^n)$ is nonnegative, radial and nonincreasing. We also assume that $\varphi = 1$ on $B(0, 1/2)$, supp $\varphi \subseteq B(0, 1)$ and $\int \varphi(x) \, dx = 1$. We sometimes use capital letters to denote points of $\mathbb{R}^{n+1}_+$ (for example, $X = (x, t)$), and set
\[ u(x, t) = e^{-t\sqrt{L}}f(x), \]
\[ \nabla_X u(X) = (\nabla_x u, \partial_t u) \]
\[ |\nabla_X u|^2 = |\nabla_x u|^2 + |\partial_t u|^2. \]

**Lemma 2.2.** For every $f, g \in L^2(\mathbb{R}^n)$,
\[ \int \int_{\mathbb{R}^{n+1}} |t \nabla_X u(x, t)|^2 |\varphi \ast g(x)|^2 \, dx \, dt \leq \int_{\mathbb{R}^n} |f(x)|^2 |g(x)|^2 \, dx + \int \int_{\mathbb{R}^{n+1}} |u(x, t)|^2 |\psi \ast g(x)|^2 \, dx \, dt \]  \hfill (2.3)
where $\psi$ is a vector-valued function with the same support as $\varphi$ and mean value 0.
Proof. The proof of Lemma 2.2 can be obtained by making minor modifications to the proof of [23, Lemma 3.1] in the case where \( L = -\Delta \) is the Laplace operator on \( \mathbb{R}^n \).

For the sake of completeness and for the reader’s convenience we give a brief sketch of the proof of this lemma.

Write \( \nabla_X^2 = \nabla_X \nabla_X \). Since \( u = e^{-t\sqrt{L}} f \), we have

\[
\nabla_X^2 u^2 = (\partial_t^2 + \Delta) u^2 = 2|\nabla_X u|^2 + 2 V u^2.
\]

This, together with the condition that \( V \geq 0 \), gives

\[
2 \int_{\mathbb{R}^{n+1}} \frac{|t \nabla_X u|}{t} |\varphi_t * g|^2 \, dx \, dt
= \int_{\mathbb{R}^{n+1}} \nabla_X^2 u^2 |\varphi_t * g|^2 \, dx \, dt - 2 \int_{\mathbb{R}^{n+1}} V u^2 |\varphi_t * g|^2 \, dx \, dt
\leq \int_{\mathbb{R}^{n+1}} \nabla_X^2 u^2 |\varphi_t * g|^2 \, dx \, dt.
\]

After an integration by parts we obtain

\[
\int_{\mathbb{R}^{n+1}} \nabla_X^2 u^2 |\varphi_t * g|^2 \, dx \, dt
= -\int_{\mathbb{R}^{n+1}} \nabla_X u^2 \nabla_X (t (\varphi_t * g)^2) \, dx \, dt
\leq -2 \int_{\mathbb{R}^{n+1}} (2 u \nabla_X u (\varphi_t * g) t \nabla_X (\varphi_t * g) + u \partial_t u |\varphi_t * g|^2) \, dx \, dt.
\]

Note that the conditions \( u(x, 0) \in L^2(\mathbb{R}^n) \) or \( f \in L^2(\mathbb{R}^n) \) are sufficient to ensure that the boundary terms ‘at \( \infty \)’ for this integration by parts vanish, as does the boundary term for \( t = 0 \).

We use a further integration by parts to obtain

\[
2 \int_{\mathbb{R}^{n+1}} u \partial_t u (\varphi_t * g)^2 \, dx \, dt
= -\lim_{t \to 0} \int_{\mathbb{R}^n} u^2 (\varphi_t * g)^2 \, dx \, dt - 2 \int_{\mathbb{R}^{n+1}} u^2 (\varphi_t * g) (\partial_t (\varphi_t * g)) \, dx \, dt
\]

\[
= -\int_{\mathbb{R}^n} f^2 g^2 \, dx - 2 \int_{\mathbb{R}^{n+1}} u^2 (\varphi_t * g) (\partial_t (\varphi_t * g)) \, dx \, dt.
\]

When combined with (2.4), integration by parts and the Cauchy–Schwarz inequality, this gives (2.3) provided that

\[
|\psi_t * f|^2 = 9 |(t \nabla_X \varphi_t) * g|^2 + 9 |\vec{\rho}_t * g|^2.
\]

Here \( \vec{\rho} = (x_1 \varphi, \ldots, x_n \varphi) \). For the details, we refer the reader to [23, (3.8)]. This completes our proof. □
Finally for $s > 0$ we define the set of measurable functions
\[
\mathcal{F}(s) := \left\{ \psi : \mathbb{C} \to \mathbb{C} \mid |\psi(z)| \leq C \frac{|z|^s}{(1 + |z|^{2s})} \right\}.
\]
Then for any nonzero function $\psi \in \mathcal{F}(s)$,
\[
\left\{ \int_0^\infty |\psi(t)|^2 \frac{dt}{t} \right\}^{1/2} < \infty.
\]
We write $\psi_t(z) = \psi(tz)$. It follows from spectral theory (see [10]) that, if $f \in L^2(\mathbb{R}^n)$, then
\[
\left\{ \int_0^\infty \|\psi(t\sqrt{L})f\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} \right\}^{1/2} = \left\{ \int_0^\infty \langle \bar{\psi}(t\sqrt{L})\psi(t\sqrt{L})f, f \rangle \frac{dt}{t} \right\}^{1/2}
\]
\[
= \left\{ \int_0^\infty |\psi|^2(t\sqrt{L}) \frac{dt}{t} f, f \right\}^{1/2} = \kappa \|f\|_{L^2(\mathbb{R}^n)},
\]
where $\kappa = \left\{ \int_0^\infty |\psi(t)|^2 \frac{dt}{t} \right\}^{1/2}$.

3. Proof of Theorem 1.1

We shall use $\mathcal{M}$ to denote the Hardy–Littlewood maximal function with respect to the balls of $\mathbb{R}^n$. We use the notation
\[
\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1} \mid |x - y| < t\}
\]
to denote the standard cone (of aperture 1) with vertex $x \in \mathbb{R}^n$.

For any closed subset $F$ of $\mathbb{R}^n$, we denote by $\mathcal{R}(F)$ the union of all cones with vertices in $F$, that is,
\[
\mathcal{R}(F) = \bigcup_{x \in F} \Gamma(x).
\]
If $O$ is an open subset of $\mathbb{R}^n$, then the ‘tent’ over $O$, denoted by $\widehat{O}$, is defined to be
\[
\widehat{O} = \{\mathcal{R}(cF)\}.
\]

**Proof of Theorem 1.1.** Let $f \in H^1_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. We shall prove that $f$ has an atomic decomposition as in (1.2). We start with a suitable version of the Calderón reproducing formula.

Let $\phi$ and $\Phi$ be as in Lemma 2.1 and set $\Psi(x) := x^4 \Phi(x)$ for all $x \in \mathbb{R}$. By the $L^2$-functional calculus (see [22]) for every $f \in L^2(\mathbb{R}^n)$ we can write
\[
f = c_\Psi \int_0^\infty \Psi(t\sqrt{L})t\sqrt{L}e^{-t\sqrt{L}}f \frac{dt}{t}
\]
\[
= \lim_{N \to \infty} c_\Psi \int_{1/N}^N \Psi(t\sqrt{L})t\sqrt{L}e^{-t\sqrt{L}}f \frac{dt}{t}
\]
with the integral converging in $L^2(\mathbb{R}^n)$. 

For \( i \in \mathbb{Z} \) we define the sets
\[
O_i := \{ x \in \mathbb{R}^n \mid f_L^n(x) > 2^i \}
\]
and consider
\[
O_i^* := \{ x \in \mathbb{R}^n \mid M(\chi_{O_i})(x) > 2^{-(n+1)} \}.
\]
Then \( O_i \subseteq O_i^* \) and \( |O_i^*| \leq C|O_i| \) for every \( i \in \mathbb{Z} \).

Now let \( \{ Q_i^j \} \) be a Whitney decomposition of \( O_i^* \) and let \( \hat{O}_i^* \) be a tent region, that is,
\[
\hat{O}_i^* := \{ (x, t) \in \mathbb{R}^n \times (0, \infty) \mid \text{dist}(x, cO_i^*) \geq t \}.
\]

For every \( i, j \in \mathbb{Z} \) we define
\[
T_j^i = (Q_j^i \times (0, +\infty)) \cap \hat{O}_i^* \setminus \hat{O}_{i+1}^*,
\]
and \( \lambda_i^j = 2^{j}|Q_j^i| \). Using formula (3.1), we write
\[
f = \sum_{j, i} c_j \int_0^\infty \Psi(t\sqrt{L})(\chi_{T_j^i}t\sqrt{L}e^{-i\sqrt{t}f}) \frac{dt}{t} = \sum_{j, i} \lambda_i^j a_j^i
\]
where \( a_j^i = L\tilde{b}_j^i \) and
\[
\tilde{b}_j^i = (\lambda_i^j)^{-1} c_j \int_0^\infty t^4 L\Phi(t\sqrt{L})(\chi_{T_j^i}t\sqrt{L}e^{-i\sqrt{t}f}) \frac{dt}{t}.
\]

We claim that, up to normalization by a multiplicative constant, the \( a_j^i \) are (1, 2)-atoms. Once this claim is established, we shall have
\[
\sum_{j, i} |\lambda_i^j| = \sum_{j, i} 2^j|Q_j^i| \leq C \sum_{i} 2^j|O_i^*| \leq C \sum_{i} 2^j|O_i| \leq C\|f\|_{H^1(L_W)} ,
\]
as desired.

Let us now prove the claim. We shall show that for every \( j, i \in \mathbb{Z} \), the function \( C^{-1}a_j^i \) is a (1, 2)-atom associated with the cube \( 10\sqrt{n}Q_j^i \) for some constant \( C \) (independent of \( i \) and \( j \)). Observe that if \( (x, t) \in T_j^i \), then \( B(x, t) \in O_i^* \). This, together with the fact that \( Q_j^i \) is the Whitney cube of \( O_i^* \), allows us to deduce that
\[
t \leq 6\sqrt{n}e(Q_j^i).
\]

By Lemma 2.1 the integral kernel \( K_{\Phi_t}(\sqrt{L}) \) of the operator \( \Phi_t(\sqrt{L}) \) satisfies the condition that
\[
\text{supp } K_{\Phi_t}(\sqrt{L}) \subseteq \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x - y| \leq t \}.
\]
This enables us to deduce that, whenever \( k = 0, 1 \),
\[
supp(L^k b_i^j) \subseteq 10 \sqrt{n} Q_i^j.
\]

To continue, for each cube \( Q_i^j \) we consider some \( h \in L^2(Q_i^j) \) such that \( \|h\|_{L^2(Q_i^j)} = 1 \). Then for \( k = 0, 1 \),
\[
\left| \lambda_i^j \int_{\mathbb{R}^n} (\ell(Q_i^j)^2 L)^k b_i^j(x) h(x) \, dx \right|
\]
\[
= c_\ell \left| \int_{\mathbb{R}^{n+1}} t^k (\ell(Q_i^j)^2 L)^k \Phi(t \sqrt{\ell}) (X_{T_i^j} t \sqrt{\ell} e^{-t \sqrt{\ell}}) f(x) h(x) \frac{dx \, dt}{t} \right|
\]
\[
\leq C \ell(Q_i^j)^2 \int_{\mathbb{R}^{n+1}} |(X_{T_i^j} t \sqrt{\ell} e^{-t \sqrt{\ell}}) f(x) h(x)|^2 \frac{dx \, dt}{t}
\]
\[
\leq C \ell(Q_i^j)^2 \left( \int_{T_i^j} |(t^2 L)^{k+1} \Phi(t \sqrt{\ell}) (h(x))|^2 \frac{dx \, dt}{t} \right)^{1/2}
\]
\[
\times \left( \int_{T_i^j} |(t^2 L)^k \Phi(t \sqrt{\ell}) (h(x))|^2 \frac{dx \, dt}{t} \right)^{1/2}
\]
\[
\leq C \ell(Q_i^j)^2 \left( \int_{T_i^j} |t \sqrt{\ell} e^{-t \sqrt{\ell}} f(x)|^2 \frac{dx \, dt}{t} \right)^{1/2}.
\]

Note that the first inequality is obtained from the fact that \( 0 < t < 6 \sqrt{n} \ell(Q_i^j) \) and the third inequality follows from (2.5).

Therefore, in order to prove our claim, it suffices to show that
\[
\int_{T_i^j} |t \sqrt{\ell} e^{-t \sqrt{\ell}} f(y)|^2 \frac{dy \, dt}{t} \leq C 2^{2j} |Q_i^j|.
\]
(3.2)

Let us show that (3.2) is satisfied. Let \( \varphi \in C_c^0(\mathbb{R}^n) \) be as in Lemma 2.2 and set
\[
F_i^j = 10 \sqrt{n} Q_i^j \setminus O_{i+1}.
\]

We first show that, for all \( (y, t) \in T_i^j \),
\[
|\varphi_t * \chi_{F_i^j}(y)| \geq C.
\]
(3.3)

Indeed, for any \( (y, t) \in T_i^j \), we can obtain
\[
B(y, t) \subseteq 10 \sqrt{n} Q_i^j
\]
and
\[
B(y, t) \cap \mathcal{O}_{i+1}^* \neq \emptyset.
\]

This shows that there exists \( x_0 \in B(y, t) \cap \mathcal{O}_{i+1}^* \) such that
\[
M(\chi_{\mathcal{O}_{i+1}})(x_0) \leq 2^{-(n+1)}.
\]

It then follows that
\[
|B(y, t) \cap O_{i+1}| \leq 2^{-(n+1)} |B(y, t)|.
\]
This implies that
\[
|B(y, t/2) \cap F_{k}^{i}| \geq |B(y, t/2) \cap 10\sqrt{n}Q_{i}^{j}| - |B(y, t/2) \cap O_{i+1}|
\]
\[
\geq |B(y, t/2)| - 2^{-(n+1)}|B(y, t)|
\]
\[
= 2^{-(n+1)}|B(y, t)|
\]
and then, for any \((y, t) \in T_{i}^{j}\),
\[
|\varphi_{i} \ast \chi_{F_{k}^{i}}(y)| = \left| \int \varphi_{i}(y-z)\chi_{F_{k}^{i}}(z) \, dz \right| \geq t^{-n}|B(y, t/2) \cap F_{k}^{i}| \geq C
\]
which proves estimate (3.3).

By Lemma 2.2, we have
\[
\int_{T_{i}^{j}} |t\sqrt{1}e^{-t\sqrt{1}T}f(y)|^{2} \frac{dy \, dt}{t}
\]
\[
\leq C \int_{\mathbb{R}_{i+1}^{n}} |t\nabla e^{-t\sqrt{1}T}f(y)|^{2}|\varphi_{i} \ast \chi_{F_{k}^{i}}(y)|^{2} \frac{dy \, dt}{t}
\]
\[
\leq C \left( \int_{\mathbb{R}_{i+1}^{n}} |e^{-t\sqrt{1}T}f(y)|^{2}|\varphi_{i} \ast \chi_{F_{k}^{i}}(y)|^{2} \frac{dy \, dt}{t} + \int_{\mathbb{R}^{n}} |f(x)|^{2}\chi_{F_{k}^{i}}(x)^{2} \, dx \right)
\]
\[
=: T_{1} + T_{2}.
\]
Observe that if \(\psi_{i} \ast \chi_{F_{k}^{i}}(y) \neq 0\), then \(F_{k}^{i} \cap B(y, t) \neq \emptyset\) and there exists an element
\[
x_{0} \in B(y, t) \cap (10\sqrt{n}Q_{i}^{j}) \cap ^{c}O_{i+1}.
\]
This gives us that
\[
|e^{-t\sqrt{1}T}f(y)| \leq f_{x_{0}^{i}}^{*}(x_{0}) \leq 2^{i+1}.
\]
Hence
\[
T_{1} \leq C 2^{2i+2} \int_{\mathbb{R}_{i+1}^{n}} |\psi_{i} \ast \chi_{F_{k}^{i}}(y)|^{2} \frac{dy \, dt}{t} \leq C 2^{2i}|Q_{i}^{j}|.
\]
Also
\[
T_{2} \leq C 2^{2i+2} \int_{\mathbb{R}^{n}} |\chi_{F_{k}^{i}}(x)|^{2} \, dx \leq C 2^{2i}|Q_{i}^{j}|
\]
and the estimate (3.2) follows readily.

We have shown that, up to normalization by a multiplicative constant, the \(a_{i}^{j}\) are \((1, 2)\)-atoms associated with the ball \(B(x_{i}^{j}, c_{1} \ell(Q_{i}^{j}))\) for some constant \(c_{1}\) where \(x_{i}^{j}\) is the center of the cube \(Q_{i}^{j}\). This proves that \(f\) has an atomic decomposition as in (1.2).

To prove the converse we assume that \(f = \sum_{j} \lambda_{j}a_{j}\) where the \(a_{j}\) are \((1, 2)\)-atoms and \(\sum_{j} |a_{j}| < \infty\). In this case, it was proved in [17, Theorem 7.4] that \(f \in H_{L}^{1}(\mathbb{R}^{n})\). We omit the details here. The proof of Theorem 1.1 is now complete. □

4. Proof of Theorem 1.2

In this section we shall work exclusively with the domain \(\mathbb{R}_{+}^{n+1} \times \mathbb{R}_{+}^{n+1}\) and its distinguished boundary \(\mathbb{R}^{n} \times \mathbb{R}^{n}\). If \(x = (x_{1}, x_{2}) \in \mathbb{R}^{n} \times \mathbb{R}^{n}\), then we shall denote
by $\Gamma(x)$ the product cone $\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2)$, where

$$\Gamma(x_i) = \{(y_i, t_i) \in \mathbb{R}^{n+1}_+ \mid |x_i - y_i| < t_i\}$$

for $i = 1, 2$. If $(x, t) := ((x_1, t_1), (x_2, t_2)) \in \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+$, then we shall write

$$B_{x,t} := B(x_1, t_1) \times B(x_2, t_2)$$

for the product ball.

For any open set $\Omega \subseteq \mathbb{R}^{2n}$, the tent over $\Omega$, denoted by $\hat{\Omega}$, is the set

$$\{(x, t) \in \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+ \mid B_{x,t} \subseteq \Omega\}.$$

Let $m(\Omega)$ denote the set of maximal dyadic subrectangles of $\Omega$. Let $m_1(\Omega)$ denote the subset of those dyadic subrectangles $R = I \times J$ of $\Omega$ that are maximal in the $x_1$ direction. In other words, if $S = I' \times J \supseteq R$ is a dyadic subrectangle of $\Omega$, then $I = I'$. Similarly, define $m_2(\Omega)$ to be the collection of those dyadic subrectangles of $\Omega$ that are maximal in the $x_2$ direction. Let $M_\delta$ denote the strong maximal operator, that is, for any $x \in \mathbb{R}^{2n}$ let

$$M_\delta f(x) = \sup_{t_1 > 0, t_2 > 0} \frac{1}{|B_{x,t}|} \int_{B_{x,t}} |f(y_1, y_2)| \, dy_1 \, dy_2. \tag{4.1}$$

In order to prove Theorem 1.2 we need some auxiliary results. The first one is Journé’s covering lemma (see [20]).

**Lemma 4.1.** Let $\Omega$ be an open subset of $\mathbb{R}^n \times \mathbb{R}^n$ and let $R = I \times J \in m_2(\Omega)$ where $I, J$ are dyadic cubes of $\mathbb{R}^n$. Suppose that $\hat{I}$ is the biggest dyadic cube of $\mathbb{R}^n$ containing $I$ such that $\hat{I} \times J \subseteq \Omega^*$ where

$$\Omega^* = \{x \in \mathbb{R}^{2n} \mid M_\delta \chi_\Omega(x) > 1/2\}.$$

We set $\gamma_1(R) = |\hat{I}|/|I|$ and define $\gamma_2$ similarly. Then for any $\delta > 0$,

$$\sum_{R \in m_2(\Omega)} |R| \gamma_1^{-\delta}(R) \leq c_\delta |\Omega|$$

and

$$\sum_{R \in m_1(\Omega)} |R| \gamma_2^{-\delta}(R) \leq c_\delta |\Omega|$$

where $c_\delta$ is a constant depending only on $\delta$ and not on $\Omega$.

For every $i = 1, 2$ we let $\nabla x_i = (\nabla x_i, \partial_{x_i})$. In the following lemma we assume that $\varphi \in C^1_0(\mathbb{R}^n)$ is nonnegative, radial and nonincreasing. We also assume that $\varphi = 1$ on $B(0, 1/2)$, supp $\varphi \subseteq B(0, 1)$ and $\int \varphi(x) \, dx = 1$.

**Lemma 4.2.** For every $f, g \in L^2(\mathbb{R}^2n)$ and $i = 1, 2$ there exist vector-valued functions $\psi^{(i)} \in C^{\infty}_0(\mathbb{R}^n)$ satisfying the conditions supp $\psi^{(i)} \subseteq B(0, 1)$, $\int_{\mathbb{R}^n} \psi^{(i)}(x) \, dx = 0$ and
such that
\[
\int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} |t_1 \nabla_X e^{-t_1 \sqrt{L}} \otimes t_2 \nabla_X e^{-t_2 \sqrt{L}} f(y_1, y_2)^2 |(\varphi_{t_1} \otimes \varphi_{t_2}) * g(y_1, y_2)|^2 \frac{dy \, dt}{t_1 t_2} \\
\leq \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} |e^{-t_1 \sqrt{L}} \otimes e^{-t_2 \sqrt{L}} f(y_1, y_2)^2 |(\psi^{(1)}_{t_1} \otimes \psi^{(2)}_{t_2}) * g(y_1, y_2)|^2 \frac{dy \, dt}{t_1 t_2} \\
+ \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n}} |e^{-t_1 \sqrt{L}} f(y_1, x_2)^2 |\psi^{(1)}_{t_1} * g(y_1, x_2)|^2 \frac{dy \, dt}{t_1} \\
+ \int_{\mathbb{R}^{n} \times \mathbb{R}^{n+1}} |e^{-t_2 \sqrt{L}} f(x_1, y_2)^2 |\psi^{(2)}_{t_2} * g(x_1, y_2)|^2 \frac{dy \, dt}{t_2} \\
+ \int_{\mathbb{R}^{n} \times \mathbb{R}^{n}} |f(x_1, x_2)^2 |g(x_1, x_2)|^2 \, dx_1 \, dx_2.
\]

**Proof.** Repeated applications of Lemma 2.2 can be used to prove Lemma 4.2.

Finally we state the following lemma whose proof we omit since it is similar to that of [17, Lemma 4.3].

**Lemma 4.3.** Suppose that $T$ is a bounded sublinear operator on $L^2(\mathbb{R}^{2n})$ and that for every product $(1, 2)$-atom $a$ on product domains we have

\[
\|Ta\|_{L^1(\mathbb{R}^{2n})} \leq C
\]

where the constant $C$ is independent of $a$. Then for any decomposition of the form given in (1.3) of $f$ we have

\[
\|Tf\|_{L^1(\mathbb{R}^{2n})} \leq C \sum_{j=1}^{\infty} |A_j|.
\]

**Proof of Theorem 1.2** By condition (1.1) for every $K = 0, 1, \ldots$ there exists a constant $C_K$ such that the kernel $p_{t,K}$ of the operator $(t \sqrt{L})^{2K} e^{-t \sqrt{L}}$ satisfies the condition that

\[
|p_{t,K}(x, y)| \leq C_K \frac{t}{(t + |x - y|)^{n+1}} \quad \forall t > 0
\]

and almost every $x, y \in \mathbb{R}^n$ (see, for instance, [17, Lemma 7.2]).

**Step 1.** Let $f = \sum_j A_j a_j$ where the $a_j$ are product $(1, 2)$-atoms and $\sum_{j=1}^{\infty} |A_j| < \infty$. Recall that the strong maximal operator $M_t$ defined in (4.1) is bounded on $L^2(\mathbb{R}^{2n})$ (see [14]). This, together with condition (1.1), gives us that

\[
\|f_L^2\|_{L^2(\mathbb{R}^{2n})} \leq C \|M_t f\|_{L^2(\mathbb{R}^{2n})} \leq C\|f\|_{L^2(\mathbb{R}^{2n})}.
\]

By Lemma 4.3, it is enough to show that $\|a_L^*\|_{L^1(\mathbb{R}^{2n})} \leq C$ for every product $(1, 2)$-atom $a$, for some constant $C$ which is independent of $a$. 

[12]
Suppose that
\[ a = \sum_{R \in m(\Omega)} a_R = \sum_{R \in m(\Omega)} (L \otimes L) b_R \]
is a product \((1,2)\)-atom supported on some open subset \( \Omega \) of \( \mathbb{R}^{2n} \). For any maximal dyadic subrectangle \( R = I \times J \in m(\Omega) \) let \( \ell(I) \), \( \ell(J) \) be the side-lengths of cubes \( I \) and \( J \) and let \( I' \) be the longest dyadic interval containing \( I \) so that
\[ I' \times J \subseteq \Omega^* = \{ x \in \mathbb{R}^{2n} \mid M_s(\chi_\Omega)(x) > 1/2 \}. \]

Then \( I' \times J \) is in \( m_1(\Omega^*) \). Let \( S \) be the longest dyadic interval so that \( S \supseteq J \) and \( I' \times S \subseteq \Omega^{**} \) where
\[ \Omega^{**} = \{ x \in \mathbb{R}^{2n} \mid M_s(\chi_{\Omega^*})(x) > 1/2 \}. \]

Let \( \tilde{R} \) be the 10-fold dilate of \( I' \times S \) concentric with \( I' \times S \). Clearly, an application of the strong maximal theorem (see [8, 19] for the proof) shows that
\[ \left| \bigcup_{\tilde{R}} \right| \leq c|\Omega^{**}| \leq c|\Omega^*| \leq c|\Omega|. \]

We then have
\[ \int_{\bigcup \tilde{R}} a_L^*(x) \, dx \leq C \left| \bigcup \tilde{R} \right|^{1/2} \| a_L^* \|_{L^2(\mathbb{R}^{2n})} \leq C \left| \bigcup \tilde{R} \right|^{1/2} \| M_s(a) \|_{L^2(\mathbb{R}^{2n})} \leq C|\Omega|^{1/2} |a|_{L^2(\mathbb{R}^{2n})} \leq C|\Omega|^{1/2} |\Omega|^{-1/2} \leq C. \]

We now find an estimate for
\[ \int_{\bigcup \tilde{R}^c} a_L^*(x) \, dx \leq C. \]

We can write
\[ \int_{\bigcup \tilde{R}^c} a_L^*(x) \, dx \leq \sum_{R \in m(\Omega)} \int_{\tilde{R}^c} (a_R)_L^*(x) \, dx \leq \sum_{R \in m(\Omega)} \int_{x_1 \in 10J} (a_R)_L^*(x) \, dx + \sum_{R \in m(\Omega)} \int_{x_2 \notin 10S} (a_R)_L^*(x) \, dx. \]

We only need to calculate the estimate for the first term above since the proof of the estimate for the second term is similar.

Observe that
\[ \sum_{R \in m(\Omega)} \int_{x_1 \in 10J} (a_R)_L^*(x) \, dx = \sum_{R \in m(\Omega)} \left( \int_{x_1 \in 10J} \int_{x_2 \in 10J} + \int_{x_1 \in 10J} \int_{x_2 \notin 10J} \right) (a_R)_L^*(x) \, dx \]
\[ =: E_1 + E_2. \]
By Hölder’s inequality, we have
\[
E_1 \leq \sum_{\Omega \in \mathcal{R}(\Omega)} |J|^{1/2} \int_{x_1 \in 100I} \|(a_R)_{x_1}(x_1, \cdot)\|_{L^2(dx_2)} \, dx_1
\leq C \sum_{\Omega \in \mathcal{R}(\Omega)} |J|^{1/2} \int_{x_1 \in 100I} \left\{ \int_{\mathbb{R}^n} \sup_{|x_1 - y_1| < t_1} |e^{-t_1 \sqrt{2}a_R(y_1, x_2)|^2} \, dx_2 \right\}^{1/2} \, dx_1
\leq C \sum_{\Omega \in \mathcal{R}(\Omega)} |J|^{1/2} \int_{x_1 \in 100I} \left\{ \int_{\mathbb{R}^n} \sup_{|x_1 - y_1| < t_1, t_1 < \ell(I)} |e^{-t_1 \sqrt{2}a_R(y_1, x_2)|^2} \, dx_2 \right\}^{1/2} \, dx_1
+ C \sum_{\Omega \in \mathcal{R}(\Omega)} |J|^{1/2} \int_{x_1 \in 100I} \left\{ \int_{\mathbb{R}^n} \sup_{|x_1 - y_1| < t_1, t_1 \geq \ell(I)} |e^{-t_1 \sqrt{2}a_R(y_1, x_2)|^2} \, dx_2 \right\}^{1/2} \, dx_1
=: E_{11} + E_{12}.
\]

We consider the term $E_{11}$ above. Let $x_I$ denote the center of cube $I$. Note that $x_1 \notin 100I'$ and $|x_1 - y_1| < t_1 < \ell(I)$. It follows from the estimate (4.2) that
\[
|e^{-t_1 \sqrt{2}a_R(\cdot, x_2)}(y_1)| \leq C \int_{\mathbb{R}^n} \frac{t_1}{(t_1 + |y_1 - z_1|)^{n+1}} |a_R(z_1, x_2)| \, dz_1
\leq C \frac{\ell(I)}{|x_1 - x_I|^n+1} \|a_R(\cdot, x_2)\|_{L^1(\mathbb{R}^n)}
\leq C |I|^{1/2} \frac{\ell(I)}{|x_1 - x_I|^n+1} \|a_R(\cdot, x_2)\|_{L^2(\mathbb{R}^n)}
\]
which, in combination with Lemma 4.1, gives us that
\[
E_{11} \leq C \sum_{\Omega \in \mathcal{R}(\Omega)} |J|^{1/2} |I|^{1/2} \left\{ \int_{x_1 \in 100I'} \frac{\ell(I)}{|x_1 - x_I|^n+1} \, dx_1 \right\} \|a_R\|_{L^2(\mathbb{R}^n)}
\leq C \sum_{\Omega \in \mathcal{R}(\Omega)} |R|^{1/2} \|a_R\|_{L^2(\mathbb{R}^n)} \gamma_1(R)^{-1}
\leq C \left( \sum_{\Omega \in \mathcal{R}(\Omega)} \|a_R\|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2} \left( \sum_{\Omega \in \mathcal{R}(\Omega)} |R| \gamma_1(R)^{-2} \right)^{1/2}
\leq C.
\]

For the term $E_{12}$ above, we apply the definition of the product $(1, 2)$-atom to obtain
\[
|e^{-t_1 \sqrt{2}a_R(\cdot, x_2)}(y_1)|
\leq \left( \frac{\ell(I)}{t_1} \right)^2 |t_1|^{2} \|e^{-t_1 \sqrt{2}a_R(\cdot, x_2)}(y_1)|
\leq C \left( \frac{\ell(I)}{t_1} \right)^2 \int_{\mathbb{R}^n} \frac{t_1}{(t_1 + |y_1 - z_1|)^{n+1}} \|e^{-t_1 \sqrt{2}a_R(\cdot, x_2)}(z_1)| \, dz_1.
Note that
\[ x_1 \notin 100I', \; |x_1 - y_1| < t_1, \; \ell(I) \leq t_1, \; z_1 \in I. \]
We can obtain the estimate
\[ t_1 + |y_1 - z_1| \geq |x_1 - x_I|/2 \]
and deduce that
\[ |e^{-t_1 \sqrt{I}} a_R(-, x_2)(y_1)| \leq C \frac{\ell(I)}{|x_1 - x_I|^n+1} \|\ell(I)^{-2} (L^0 \otimes L^1) b_R(-, x_2)\|_{L^2(\mathbb{R}^n)}. \tag{4.4} \]

It follows from (4.4) and Hölder’s inequality that
\[
E_{12} \leq C \sum_{R \in m(\Omega)} |J|^{1/2} |I|^{1/2} \int_{x_1 \notin 100I} \frac{\ell(I)}{|x_1 - x_I|^n+1} \, dx_1 \\
\times \|\ell(I)^{-2} (L^0 \otimes L^1) b_R\|_{L^2(\mathbb{R}^n)} \\
\leq C \sum_{R \in m(\Omega)} |R|^{1/2} \gamma_1(R)^{-1} \|\ell(I)^{-2} (L^0 \otimes L^1) b_R\|_{L^2(\mathbb{R}^n)} \\
\leq C \left( \sum_{R \in m(\Omega)} |R| \gamma_1(R)^{-2} \right)^{1/2} \left( \sum_{R \in m(\Omega)} \ell(I)^{-4} \|L^0 \otimes L^1 b_R\|_{L^2(\mathbb{R}^n)}^2 \right) \\
\leq C
\]
which, together with the estimate of $E_{11}$, gives us that $E_1 \leq C$.

Consider the term $E_2$. We first estimate the maximal function $(a_R)_L^*$. Now
\[
(a_R)_L^*(x) = \sup_{|y_1 - x_2| < t_2} \sup_{|y_1 - x_1| < t_1} |e^{-t_1 \sqrt{I}} \otimes e^{-t_2 \sqrt{I}} a_R(y_1, y_2)| \\
\leq \sup_{|y_2 - x_2| < t_2} \sup_{|y_1 - x_1| < t_1} |e^{-t_1 \sqrt{I}} \otimes e^{-t_2 \sqrt{I}} a_R(y_1, y_2)| \\
+ \sup_{|y_2 - x_2| < t_2} \sup_{|y_1 - x_1| < t_1} \sup_{\ell(J) < t_1} |e^{-t_1 \sqrt{I}} \otimes e^{-t_2 \sqrt{I}} a_R(y_1, y_2)| \\
+ \sup_{|y_2 - x_2| < t_2} \sup_{|y_1 - x_1| < t_1} \sup_{\ell(J) > t_1} |e^{-t_1 \sqrt{I}} \otimes e^{-t_2 \sqrt{I}} a_R(y_1, y_2)| \\
+ \sup_{|y_2 - x_2| < t_2} \sup_{|y_1 - x_1| < t_1} \sup_{\ell(J) = t_1} |e^{-t_1 \sqrt{I}} \otimes e^{-t_2 \sqrt{I}} a_R(y_1, y_2)| \\
=: E_{21} + E_{22} + E_{23} + E_{24}.
\]

We only need to estimate the term $E_{22}$ since the estimates of the remaining terms are similar.
Applying (4.4) with $a_R(\cdot, x_2)$ replaced by $e^{-t_2 \sqrt{L}}a_R(\cdot, y_2)$, we obtain

$$E_{22} \leq C \sup_{|y_2 - x_2| < t_2, I \subset (x, y)} \frac{\ell(I)}{|x_1 - x|^{n+1}} \frac{\ell(J)}{|x_2 - x_J|^{n+1}} \|\ell(I)^{-2}(L^0 \otimes e^{-t_2 \sqrt{L}}L^1)b_R(\cdot, y_2)\|_{L^1(\mathbb{R}^n)}$$

where

$$\leq C \frac{\ell(I)}{|x_1 - x|^{n+1}} \frac{\ell(J)}{|x_2 - x_J|^{n+1}} \|\ell(I)^{-2}L^0 \otimes L^1)b_R(\cdot, y_2)\|_{L^1(\mathbb{R}^n)}.$$

Applying (4.3) with $a_R(\cdot, x_2)$ replaced by $(\ell(I)^{-2}L^0 \otimes L^1)b_R(x_1, \cdot)$, together with Hölder’s inequality, we obtain

$$E_{22} \leq C \frac{\ell(I)}{|x_1 - x|^{n+1}} \frac{\ell(J)}{|x_2 - x_J|^{n+1}} \|\ell(I)^{-2}L^0 \otimes L^1)b_R(\cdot, y_2)\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)}.$$  

A similar argument to that given for $E_{22}$ shows that

$$(a_R)_L^*(x) \leq C \frac{\ell(J)}{|x_2 - x_J|^{n+1}} \frac{\ell(I)}{|x_1 - x_J|^{n+1}} |R|^{1/2} \times \sum_{i,j=0}^{1} \ell(I)^{2i-2} \ell(J)^{2j-2} \|(L^I \otimes L^J)b_R\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}.$$  

Hence

$$E_2 = \sum_{\text{Rem}(\Omega)} \int_{x_1 \notin 10J} \int_{x_2 \notin 10J} (a_R)_L^*(x) \, dx \leq C \sum_{\text{Rem}(\Omega)} |R|^{1/2} \gamma_1(R)^{-1} \sum_{i,j=0}^{1} \ell(I)^{2i-2} \ell(J)^{2j-2} \|(L^I \otimes L^J)b_R\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}.$$  

Applying Lemma 4.1 and the definition of product $(1, 2)$-atom, together with Hölder’s inequality, we obtain the estimate $E_2 \leq C$. We thus obtain the required estimate $\|a_R^*_L\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)} \leq C$ and can deduce that $f \in H_L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

**Step 2.** Let

$$f \in H_L^1(\mathbb{R}^n \times \mathbb{R}^n) \cap L^2(\mathbb{R}^n \times \mathbb{R}^n).$$

We begin with a version of the Calderón reproducing formula. Let $\Psi(x) = x^4 \Phi(x)$ be the function in Lemma 2.1. Since $f \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, applying the $L^2$-functional calculus gives us that

$$f = c_\Psi \int_0^\infty \int_0^\infty \Psi_{t_1}(\sqrt{L})_{t_1} \sqrt{L}e^{-t_1} \sqrt{L} \Psi_{t_2}(\sqrt{L})_{t_2} \sqrt{L}e^{-t_2} \sqrt{L} f \frac{dt}{t_1 t_2}.$$  

(4.5)
For $k = 0, \pm 1, \ldots$ we set
\[ E_k = \{ x \mid f^*_L(x) > 2^k \}, \]
\[ E^*_k = \{ x \mid M_k \chi_{E_k}(x) > 2^{-(2n+1)} \} \]
and
\[ E^{*\ast}_k = \{ x \mid M_k \chi_{E_k}(x) > (4n)^{-n} \}. \]

Then
\[ E_k \subseteq E^*_k \subseteq E^{*\ast}_k \quad \text{and} \quad |E^{*\ast}_k| \leq C|E^*_k| \leq C'|E_k|. \]

We define $T_k := \overline{E^*_k} \setminus \overline{E^{*\ast}_{k+1}}$ and apply formula (4.5) to write
\[
f = \sum_{k \in \mathbb{Z}} \lambda_k a_k
\]
where $\lambda_k = 2^k|E^*_k|$ and
\[
a_k = \lambda_k^{-1} \psi \int_0^\infty \int_0^\infty \Psi_t(\sqrt{L}) \Psi_s(\sqrt{L})(\chi_{T_k} \chi_{1_1} \sqrt{L} e^{-t_1 \sqrt{L}} \otimes t_2 \sqrt{L} e^{-t_2 \sqrt{L}}) f \frac{dt}{t_1 t_2}.
\]

It is clear that
\[
\sum_k |a_k| \leq C \sum_k 2^k|E^*_k| \leq C\|f_L\|_{L^1(\mathbb{R}^{2n})}.
\]

We claim that for each $k \in \mathbb{Z}$ the term $a_k$ is a product $(1, 2)$-atom associated with the open set $E^{*\ast}_k$ for some constant $C$.

Let us prove the claim. First, it follows by Lemma 2.1 that the integral kernel $K_{\psi, \sqrt{L}}$ of the operator $\psi(\sqrt{L})$ has its support contained in
\[
\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x_1 - y_1| \leq t_1, |x_2 - y_2| \leq t_2 \}.
\]

This, together with the definition of $T_k$, shows that $a_k \subseteq E^{*\ast}_k$. Second, for any dyadic rectangle $R = I \times J$ of $\mathbb{R}^n \times \mathbb{R}^n$, we define
\[
R^+ = \left\{ (y_1, y_2, t_1, t_2) \mid y_1 \in I, y_2 \in J, \frac{\ell(I)}{2} < t_1 \leq \ell(I), \frac{\ell(J)}{2} < t_1 \leq \ell(J) \right\}
\]

It can be verified that if $T_k \cap R^+ \neq \emptyset$, then $R \subseteq E^{*\ast}_k$. Applying the definition of $R^+$, we obtain $T_k = \bigcup_R (T_k \cap R^+)$ where the $R$ are all dyadic rectangles of $\mathbb{R}^n \times \mathbb{R}^n$. We can further decompose $a_k$ as follows:
\[
a_k = \sum_{\overline{R} \in m(E^{*\ast}_k)} \sum_{R \subseteq \overline{R}} \lambda_k^{-1}
\times \int_0^\infty \int_0^\infty \Psi_t(\sqrt{L}) \Psi_s(\sqrt{L})(\chi_{T_k} \chi_{1_1} \sqrt{L} e^{-t_1 \sqrt{L}} \otimes t_2 \sqrt{L} e^{-t_2 \sqrt{L}}) f \frac{dt}{t_1 t_2}
\]
\[=: \sum_{\overline{R} \in m(E^{*\ast}_k)} a_k(\overline{R}) = \sum_{\overline{R} \in m(E^{*\ast}_k)} (L \otimes L) b_{k, \overline{R}}
\]
where $m(E^{*\ast}_k)$ denotes the set of all maximal dyadic rectangles of $E^{*\ast}_k$. 
By Lemma 2.1, if $i, j = 0, 1$, then

$$\text{supp}((L^i \otimes L^j)b_{k,R}) \subseteq 2\bar{R}.$$ 

To continue, for each $\bar{R}$ we consider some $h \in L^2(\bar{R})$ such that $\|h\|_{L^2(\bar{R})} = 1$. Then for every $k \in \mathbb{Z}$, we have

$$\|a_{k,\bar{R}}\|_{L^2} = \sup_{\|h\| \leq 1} |\langle a_{k,\bar{R}}, h \rangle|$$

$$\leq C2^{-k}\|E_k^+\|^{-1} \times \left( \sum_{R \subseteq \bar{R}} \int_{T_k \cap R^*} |t_1 \nabla X_1 e^{-t_1 \sqrt{\mathcal{L}}} \otimes t_2 \nabla X_2 e^{-t_2 \sqrt{\mathcal{L}}} f(y_1, y_2)|^2 \frac{dy dt}{t_1 t_2} \right)^{1/2}.$$ 

In order to verify that

$$\sum_{R \in \text{rem}(E_i^*)} \|a_{k,\bar{R}}\|_{L^2}^2 \leq C\|E_k^{**}\|^{-1},$$

it is enough to prove that

$$\int_{T_k} |t_1 \nabla X_1 e^{-t_1 \sqrt{\mathcal{L}}} \otimes t_2 \nabla X_2 e^{-t_2 \sqrt{\mathcal{L}}} f(y_1, y_2)|^2 \frac{dy dt}{t_1 t_2} \leq C2^{2k}\|E_k^+\|.$$ \hspace{1cm} (4.6)

We now prove inequality (4.6). Let $\varphi \in C_0^\infty$ be the function in Lemma 4.2 and set $F_k = E_k^+ \setminus E_{k+1}$. The argument given to prove formula (3.3) shows that

$$|\langle \varphi_{t_1} \otimes \varphi_{t_2}, \chi_F(y) \rangle| \geq C$$

for all $(y, t) \in T_k$. This, together with Lemma 4.2, gives us that

$$\int_{T_k} |t_1 \nabla X_1 e^{-t_1 \sqrt{\mathcal{L}}} \otimes t_2 \nabla X_2 e^{-t_2 \sqrt{\mathcal{L}}} f(y_1, y_2)|^2 \frac{dy dt}{t_1 t_2}$$

$$\leq \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} |t_1 \nabla X_1 e^{-t_1 \sqrt{\mathcal{L}}} \otimes t_2 \nabla X_2 e^{-t_2 \sqrt{\mathcal{L}}} f(y_1, y_2)|^2 |\langle \varphi_{t_1} \otimes \varphi_{t_2}, \chi_{F_k^+}(y) \rangle|^2 \frac{dy dt}{t_1 t_2}$$

$$\leq \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} |e^{-t_1 \sqrt{\mathcal{L}}} \otimes e^{-t_2 \sqrt{\mathcal{L}}} f(y_1, y_2)|^2 |\langle \varphi_{t_1} \otimes \varphi_{t_2}, \chi_{F_k^+}(y) \rangle|^2 \frac{dy dt}{t_1 t_2}$$

$$+ \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} |e^{-t_1 \sqrt{\mathcal{L}}} f(y_1, x_2)|^2 |\varphi_{t_1}^{(1)} \otimes \varphi_{t_2}^{(2)}, \chi_{F_k^+}(y_1, x_2)|^2 \frac{dy_1 dt_1}{t_1} dx_2$$

$$+ \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} |e^{-t_2 \sqrt{\mathcal{L}}} f(x_1, y_2)|^2 |\varphi_{t_2}^{(1)} \otimes \varphi_{t_1}^{(2)}, \chi_{F_k^+}(x_1, y_2)|^2 \frac{dy_2 dt_2}{t_2} dx_1$$

$$+ \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} |f(x_1, x_2)|^2 |\varphi_{t_1}^{(1)} \otimes \varphi_{t_2}^{(2)}, \chi_{F_k^+}(x_1, x_2)|^2 \frac{dy_1 dx_2}{x_2}$$

$$=: I_1 + I_2 + I_3 + I_4.$$

In order to estimate the term $I_1$, we note that if

$$(\varphi_{t_1}^{(1)} \otimes \varphi_{t_2}^{(2)}, \chi_{F_k^+}(y_1, y_2)) \neq 0,$$
then $F_k \cap B_{y,t} \neq \emptyset$. Moreover, there exists 

$$x^0 = (x_1^0, x_2^0) \in B_{y,t} \cap E_k^* \cap eE_{k+1}$$

and we have 

$$|e^{-t_1 \sqrt{L}} \otimes e^{-t_2 \sqrt{L}} f(y_1, y_2)| \leq f_L^*(x^0) \leq 2^{k+1}$$

which gives us that $I_1 \leq C2^{2k}|E_k^*|$. We similarly have 

$$I_2 + I_3 \leq C2^{2k}|E_k^*|.$$ 

We now obtain an estimate for the term $I_4$. It follows from the inequality $f(x_1, x_2) \leq f_L^*(x_1, x_2)$ that 

$$I_4 \leq C2^{2k}|E_k^*|.$$ 

The desired estimate (4.6) follows easily and then 

$$\sum_{R \in m(E_{k+1}^*)} \|a_{k,R}\|^2_{L^2(\mathbb{R}^{2n})} \leq C|E_{k+1}^*|^{-1}.$$ 

A similar argument to the one given above shows that 

$$\sum_{R \in m(E_{k+1}^*)} \sum_{i,j=0}^1 \ell(I)^{4i-4} \ell(J)^{4j-4} \|(L^i \otimes L^j)b_{k,R}\|^2_{L^2(\mathbb{R}^{2n})} \leq C|E_{k+1}^*|^{-1}.$$ 

We have shown that for every $k \in \mathbb{Z}$ the expression $C^{-1}a_k$ is a product $(1,2)$-atom associated with the open set $E_{k+1}^*$ for some constant $C$. This shows that $f$ has an decomposition of the form given in (1.3). The proof of Theorem 1.2 is complete. \[\square\]

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